BOUNDARY NULL-CONTROLLABILITY OF 1D LINEARIZED COMPRESSIBLE NAVIER-STOKES SYSTEM BY ONE CONTROL FORCE

KUNTAL BHANDARI*, SHIRSHENDU CHOWDHURY†,1, RAJIB DUTTA†,2, AND JITEN KUMBHAKAR†,3

ABSTRACT. In this paper, we prove the boundary null-controllability of the compressible Navier-Stokes equations linearized around a positive constant steady state in a bounded interval when the time is sufficiently large. The novelty of this work is that we consider only one Dirichlet boundary control at one end of the interval acting either on the velocity or density part of the concerned system, where the first-order couplings between transport and heat-type equations arise. Moreover, we establish that the null-controllability results are optimal/sharp concerning the regularity of initial states for the velocity case and with respect to time for the density case.

The proofs of controllability results rely on a new parabolic-hyperbolic joint Ingham-type inequality, a mixed parabolic-hyperbolic moments method, and some complex analytic arguments. To this end, a careful spectral analysis of the associated non-self-adjoint operator is performed, which is involved due to the effect of the boundary conditions.

CONTENTS

1. INTRODUCTION AND MAIN RESULTS

1.1. The system under study. The Navier-Stokes (NS) system for a viscous compressible isentropic fluid in $(0, L)$ is

$$\begin{cases} \rho_t + (\rho u)_x = 0, & \text{in } (0,+\infty)\times(0,L), \\ \rho\big(u_t + uu_x\big) + (p(\rho))_x - \nu u_{xx} = 0, & \text{in } (0,+\infty)\times(0,L), \end{cases} \tag{1.1}$$

where $L > 0$ denotes the finite length of the interval, ρ is the fluid density and u is the velocity. The viscosity of the fluid is denoted by $\nu > 0$ and we assume that the pressure p satisfies the constitutive law $p(\rho) = a\rho^\gamma$ for $a > 0$ and $\gamma \geq 1$. Upon linearization of (1.1) around some constant steady state (Q_0, V_0)

2020 *Mathematics Subject Classification.* Primary: 76N06, 93B05, 93B60, 93C20, 35P10 ; Secondary: 93B07, 35P20.

Key words and phrases. Linearized compressible Navier-Stokes system, boundary null-controllability, spectral analysis, moments method, Ingham-type inequality.

*Institute of Mathematics of the Czech Academy of Sciences, Žitná 25, 11567 Praha 1, Czech Republic; bhandari@math.cas.cz.

†Indian Institute of Science Education and Research Kolkata, Campus road, Mohanpur, West Bengal 741246, India; [1]shirshendu@iiserkol.ac.in ([1]corresponding author), [2]rajib.dutta@iiserkol.ac.in, [3]jk17ip021@iiserkol.ac.in.

(with $Q_0 > 0, V_0 > 0$), we get

$$(1.2)\qquad \begin{cases} \rho_t + V_0\rho_x + Q_0 u_x = 0, & \text{in } (0,+\infty)\times(0,L), \\ u_t - \dfrac{\nu}{Q_0}u_{xx} + V_0 u_x + a\gamma Q_0^{\gamma-2}\rho_x = 0, & \text{in } (0,+\infty)\times(0,L). \end{cases}$$

Now, if we consider the change of variables:

$$\rho(t,x) \to \alpha\rho(\beta t, \delta x), \quad u(t,x) \to u(\beta t, \delta x), \quad \forall (t,x) \in (0,+\infty)\times(0,L),$$

with the choices of $\alpha, \beta, \delta > 0$ as

$$\alpha := \left(a\gamma Q_0^{\gamma-3}\right)^{-1/2}, \quad \beta := \frac{Q_0 V_0^2}{\nu}, \quad \delta := \frac{Q_0 V_0}{\nu},$$

then the system (1.2) reduces to

$$(1.3)\qquad \begin{cases} \rho_t + \rho_x + b u_x = 0, & \text{in } (0,+\infty)\times(0,\delta L), \\ u_t - u_{xx} + u_x + b\rho_x = 0, & \text{in } (0,+\infty)\times(0,\delta L), \end{cases}$$

with $b = \frac{Q_0}{V_0}\left(a\gamma Q_0^{\gamma-3}\right)^{1/2}$.

Let us describe the problems on which we are going to work in the present article. Our goal is to study the boundary controllability properties of the linearized Navier-Stokes system (1.3) at time $T > 0$ with a single control force acting either on the velocity or density component. Here, we must mention that the whole analysis of this paper will be performed in the space domain $(0,1)$, which is mainly for the simplicity of spectral computations. The same can be done in the interval $(0, \delta L)$.

I. Control on velocity: The first problem under consideration is

$$(1.4)\qquad \begin{cases} \rho_t + \rho_x + b u_x = 0, & \text{in } (0,T)\times(0,1), \\ u_t - u_{xx} + u_x + b\rho_x = 0, & \text{in } (0,T)\times(0,1), \\ \rho(t,0) = \rho(t,1), & \text{for } t \in (0,T), \\ u(t,0) = 0, \quad u(t,1) = q(t), & \text{for } t \in (0,T), \\ \rho(0,x) = \rho_0(x), \quad u(0,x) = u_0(x), & \text{in } (0,1), \end{cases}$$

with a Dirichlet control q acting at the right boundary point only through the velocity component u, and (ρ_0, u_0) is the given initial state from some suitable Hilbert space.

II. Control on density: Next, we consider the case when a boundary control p acts on the density part instead of velocity. More precisely, the system under consideration is

$$(1.5)\qquad \begin{cases} \rho_t + \rho_x + b u_x = 0, & \text{in } (0,T)\times(0,1), \\ u_t - u_{xx} + u_x + b\rho_x = 0, & \text{in } (0,T)\times(0,1), \\ \rho(t,0) = \rho(t,1) + p(t), & \text{for } t \in (0,T), \\ u(t,0) = 0, \quad u(t,1) = 0, & \text{for } t \in (0,T), \\ \rho(0,x) = \rho_0(x), \quad u(0,x) = u_0(x), & \text{in } (0,1). \end{cases}$$

The aim is to study the null-controllability of the systems (1.4) and (1.5) at a given time $T > 0$. Moreover, as a consequence of the null-controllability result for the system (1.5), we can also achieve the null-controllability for the following full Dirichlet system when a control h is exerted on the density part, that is

$$(1.6)\qquad \begin{cases} \rho_t + \rho_x + b u_x = 0, & \text{in } (0,T)\times(0,1), \\ u_t - u_{xx} + u_x + b\rho_x = 0, & \text{in } (0,T)\times(0,1), \\ \rho(t,0) = h(t), & \text{for } t \in (0,T), \\ u(t,0) = 0, \quad u(t,1) = 0, & \text{for } t \in (0,T), \\ \rho(0,x) = \rho_0(x), \quad u(0,x) = u_0(x), & \text{in } (0,1). \end{cases}$$

Let us prescribe the notions of null- and approximate controllability for the concerned systems.

Definition 1.1. *Let H be a Hilbert space. We say the system (1.4) (resp. (1.5) and (1.6)) is*

- ***null-controllable*** *at a finite time $T>0$ in H if, for any given initial state $(\rho_0,u_0)\in H$, there exists a control $q\in L^2(0,T)$ (resp. $p,h\in L^2(0,T)$) such that the solution (ρ,u) to (1.4) (resp. (1.5) and (1.6)) can be driven to 0 at the time T, that is,*
$$(\rho(T,x),u(T,x))=(0,0), \quad \textit{for all } x\in(0,1).$$
- ***approximately controllable*** *at a finite time $T>0$ in H if, for any given initial state $(\rho_0,u_0)\in H$, final state $(\rho_T,u_T)\in H$ and given $\varepsilon>0$, there exists a control $q\in L^2(0,T)$ (resp. $p,h\in L^2(0,T)$) such that the solution (ρ,u) to (1.4) (resp. (1.5) and (1.6)) satisfies*
$$\|(\rho(T),u(T))-(\rho_T,u_T)\|_H\le\varepsilon.$$

If the system (1.4) is null-controllable at some time $T>0$ by using a control $q\in L^2(0,T)$ acting only on the velocity part, then we have the following compatibility condition (obtained by integrating the first equation of (1.4)):
$$\int_0^1\rho_0(x)dx=b\int_0^T q(t)dt.$$
We also get a similar compatibility condition for the density case (that is, for system (1.5)), given by
$$\int_0^1\rho_0(x)dx=-\int_0^T p(t)dt.$$
To avoid these constraints, we shall work on the Hilbert space $\dot{L}^2(0,1)\times L^2(0,1)$, where
$$\dot{L}^2(0,1):=\left\{f\in L^2(0,1)\ :\ \int_0^1 f dx=0\right\}.$$

1.2. **Functional setting.** For any $s>0$, we introduce the following Sobolev space
$$H^s_\sharp(0,L):=\{\varphi\in H^s(0,L)\ :\ \varphi(0)=\varphi(L)\}$$
and denote $(H^s_\sharp(0,L))'$ as the dual space of $H^s_\sharp(0,L)$ with respect to the pivot space $L^2(0,L)$. We also denote, for any $s>0$, $H^{-s}(0,L)$ and $(\dot{H}^s_\sharp(0,L))'$ as the dual spaces of $H^s_0(0,L)$ and $\dot{H}^s_\sharp(0,L)$ with respect to the pivot spaces $L^2(0,L)$ and $\dot{L}^2(0,L)$ respectively. We note here that, although the trace $\varphi(0)$ or $\varphi(L)$ is meaningful only for $s>\frac12$, we still keep the same notation for $s\le\frac12$ to simplify the presentation.

Let us now write the underlying operator associated with the control systems (1.4) or (1.5), given by
$$A=\begin{pmatrix}-\partial_x & -b\partial_x\\ -b\partial_x & \partial_{xx}-\partial_x\end{pmatrix}, \tag{1.7}$$
with its domain
$$D(A)=\Big\{\Phi=(\xi,\eta)\in H^1(0,1)\times H^2(0,1)\ :\ \xi(0)=\xi(1),\quad \eta(0)=\eta(1)=0\Big\}. \tag{1.8}$$
The adjoint of the operator A has the following formal expression
$$A^*=\begin{pmatrix}\partial_x & b\partial_x\\ b\partial_x & \partial_{xx}+\partial_x\end{pmatrix}, \tag{1.9}$$
also with the same domain $D(A^*)=D(A)$, given by (1.8). Note that the operator A is non-self-adjoint in nature.

Notations: Throughout the paper, $C,C_i>0$ for $i\in\mathbb{N}^*$, denote the generic constants that may vary from line to line and may depend on T.

1.3. **Main results.** This section is devoted to announce the main results of the present work.

Theorem 1.2 (Control on velocity)**.** *Let $T>1$ and $b>0$ such that $b^4+8b^2+5<4\pi^2$. Then, there exists a countable set $\mathcal{N}$ such that for chosen $b\notin\mathcal{N}$ and any given $(\rho_0,u_0)\in\dot{H}^{\frac12}_\sharp(0,1)\times L^2(0,1)$, there exists a Dirichlet boundary control $q\in L^2(0,T)$ acting on the velocity component such that the system (1.4) is null-controllable at time T, that is*
$$\rho(T,x)=u(T,x)=0,\quad \forall x\in(0,1). \tag{1.10}$$
Moreover, if $0\le s<\frac12$, the system (1.4) fails to satisfy the null-controllability criterion (1.10) in the space $\dot{H}^s_\sharp(0,1)\times L^2(0,1)$ for any given time $T>0$ and $b>0$.

Theorem 1.3 (Control on density)**.** *Let $T > 1$ and $b > 0$ such that $b^4 + 8b^2 + 5 < 4\pi^2$. Then, for any given initial state $(\rho_0, u_0) \in \dot{L}^2(0,1) \times L^2(0,1)$, there exists a boundary control $p \in L^2(0,T)$ acting through the density component such that the system (1.5) is null-controllable at time T, that is*

$$\rho(T,x) = u(T,x) = 0, \quad \forall x \in (0,1). \tag{1.11}$$

Remark 1.4. *We must mention here that the restrictions on b appear in the above results because of the difficulty in proving that roots of the auxiliary equation (which comes from the differential equation satisfied by the eigenfunctions of A^*) are distinct. Moreover, the set $\mathcal{N}$ appears while proving that all the observation terms are non-zero in the case when a control acts only on the velocity part; see Section 4 for details.*

We also have the lack of null-controllability result for the system (1.5) when $T < 1$. Precisely, we prove the following proposition.

Proposition 1.5 (Lack of null-controllability at small time)**.** *Let $0 < T < 1$. The system (1.5) is not null-controllable at time T in the space $L^2(0,1) \times L^2(0,1)$.*

As a consequence of Theorem 1.3, we also achieve the null-controllability for the system (1.6) with a Dirichlet control on the density part. More precisely, we have the following result.

Theorem 1.6 (Dirichlet control on density)**.** *Let $T > 1$ and $b > 0$ such that $b^4+8b^2+5 < 4\pi^2$. Then, for any given initial state $(\rho_0, u_0) \in \dot{L}^2(0,1) \times L^2(0,1)$, there exists a boundary control $h \in L^2(0,T)$ acting through the density component such that the system (1.6) is null-controllable at time T, that is*

$$\rho(T,x) = u(T,x) = 0, \quad \forall x \in (0,1). \tag{1.12}$$

Indeed, by Theorem 1.3, there exists a control $p \in L^2(0,T)$ which drives the solution (ρ, u) of the system (1.5) to $(0,0)$ with initial state $(\rho_0, u_0) \in \dot{L}^2(0,1) \times L^2(0,1)$. Then, by showing $\rho(\cdot, 1) \in L^2(0,T)$, one can consider $h(t) := \rho(t,1) + p(t)$ for $t \in (0,T)$, which acts as a null-control for the system (1.6). Similar technique has been applied for instance in [10, 22].

To prove the main results of this paper, we notably use an Ingham-type inequality and the moments technique. In fact, we establish the following Ingham-type inequality which is of independent interest.

Proposition 1.7 (A combined Ingham-type inequality)**.** *Let $\{\lambda_k\}_{k\in\mathbb{N}^*}$ and $\{\gamma_k\}_{k\in\mathbb{Z}}$ be two sequences in $\mathbb{C}$ with the following properties: there is $N \in \mathbb{N}^*$ such that*

(i) *for all $k, j \in \mathbb{Z}$, $\gamma_k \neq \gamma_j$ unless $j = k$;*
(ii) *$\gamma_k = \beta + 2k\pi i + \nu_k$ for all $|k| \geq N$;*

where $\beta \in \mathbb{C}$ and $\{\nu_k\}_{|k|\geq N} \in \ell_2$.

Also, there exist constants $A_0 \geq 0$, $B_0 \geq \delta$ with $\delta > 0$ and some $\epsilon > 0$ for which $\{\lambda_k\}_{k\in\mathbb{N}^*}$ satisfies

(i) *for all $k, j \in \mathbb{N}^*$, $\lambda_k \neq \lambda_j$ unless $j = k$;*
(ii) *$\frac{-\mathrm{Re}(\lambda_k)}{|\mathrm{Im}(\lambda_k)|} \geq \widehat{c}$ for some $\widehat{c} > 0$ and $k \geq N$;*
(iii) *there exists some $r > 1$ such that $|\lambda_k - \lambda_j| \geq \delta\, |k^r - j^r|$ for all $k \neq j$ with $k, j \geq N$ and*
(iv) *$\epsilon(A_0 + B_0 k^r) \leq |\lambda_k| \leq A_0 + B_0 k^r$ for all $k \geq N$.*

We also assume that the families are disjoint, i.e.,

$$\{\gamma_k,\ k \in \mathbb{Z}\} \cap \{\lambda_k,\ k \in \mathbb{N}^*\} = \emptyset.$$

Then, for any time $T > 1$, there exists a positive constant C depending only on T such that

$$\int_0^T \left| \sum_{k\in\mathbb{N}^*} a_k e^{\lambda_k t} + \sum_{k\in\mathbb{Z}} b_k e^{\gamma_k t} \right|^2 dt \geq C \left(\sum_{k\in\mathbb{N}^*} |a_k|^2 e^{2\mathrm{Re}(\lambda_k)T} + \sum_{k\in\mathbb{Z}} |b_k|^2 \right), \tag{1.13}$$

for all sequences $\{a_k\}_{k\in\mathbb{N}^*}$ and $\{b_k\}_{k\in\mathbb{Z}}$ in ℓ_2.

Remark 1.8. *The first Ingham inequality was proved in 1936 by Ingham [35]. He considered a hyperbolic family of the form $(i\gamma_k)_{k\in\mathbb{N}^*}$, where $(\gamma_k)_{k\in\mathbb{N}^*}$ is a sequence of real numbers satisfying the gap condition $\inf_{k\in\mathbb{N}} |\gamma_{k+1} - \gamma_k| > 0$. Since then, there are many variations of this inequality available in the literature including the parabolic Ingham inequality (commonly known as the Müntz-Szász theorem). We refer to the works [6, 25, 29, 36, 39, 44, 45, 48, 53] for proofs of these variations of Ingham-type inequality.*

Zhang and Zuazua [54, 55, 56] proved a joint parabolic-hyperbolic Ingham-type inequality with a parabolic branch of the form $-k^2\pi^2 + 2 + O(k^{-1})$ and a hyperbolic branch of the form $(\frac{1}{2} + k)\pi i + O(|k|^{-1})$

(Lemma 4.1 in [54] or [56] and Lemma 4.5 in [55]). This result has been generalized by Komornik and Tenenbaum [40]. In this article, we prove a joint parabolic-hyperbolic Ingham-type inequality under more general assumptions on the parabolic and hyperbolic branches compare to the assumptions in [40, Theorem 1.1]. Our proof is based on a decoupling idea as mentioned in [58, Section 2.4] by Zuazua and [18, Theorem 4.2] by Chowdhury, Mitra, Ramaswamy and Renardy. In fact, our proof works with more general assumptions on the sequences $(\lambda_k)_{k\in\mathbb{N}^*}$ and $(\gamma_k)_{k\in\mathbb{Z}}$ for which each of the individual parabolic and hyperbolic Ingham inequalities hold.

1.4. **Literature on the controllability results related to the compressible Navier-Stokes equations.** In the past few years, the controllability of the compressible and incompressible fluids has turned into a very significant topic to the control community. Fernández-Cara et al. [30] proved the local exact distributed controllability of the incompressible Navier-Stokes system when a control is supported in a small open set; see also the references therein. A local null-controllability result of 3D Navier-Stokes system with distributed control for incompressible fluids having two vanishing components has been addressed in [23] by Coron and Lissy. Badra, Ervedoza and Guerrero [7] proved the local exact controllability to the trajectories for non-homogeneous (variable density) incompressible 2D Navier-Stokes equations using boundary controls for both density and velocity.

In the case of compressible Navier-Stokes equations, we first mention the work by Amosova [2] where she considered a compressible viscous fluid in 1D w.r.t. the Lagrangian coordinates with zero boundary condition on the velocity and an interior control acting on the velocity equation. She proved a local exact controllability result when the initial density is already on the targeted trajectory. Ervedoza, Glass, Guerrero and Puel [27] proved a local exact controllability result for the 1D compressible Navier-Stokes system in a bounded domain $(0, L)$ for regular initial data in $H^3(0, L) \times H^3(0, L)$ with two boundary controls, when time is large enough. This result has been improved by Ervedoza and Savel [28] by choosing the initial data from $H^1(0, L) \times H^1(0, L)$; see also a generalized result [26] by Ervedoza, Glass and Guerrero for dimensions 2 and 3.

We also refer that Chowdhury, Ramaswamy and Raymond [20] established a null-controllability and stabilizability result of a linearized (around a constant steady-state $(Q_0, 0)$, $Q_0 > 0$) 1D compressible Navier-Stokes equations. The authors proved that their system is null-controllable in $H^1_0 \times L^2$ by a distributed control acting everywhere in the velocity equation. Their result is proved to be sharp in the following sense: the null-controllability cannot be achieved by a localized interior control (or by a boundary control) acting on the velocity part.

Martin, Rosier and Rouchon in [47] considered the wave equation with structural damping in 1D. Using the spectral analysis and method of moments, they obtained that their equation is null-controllable with a moving distributed control for regular initial conditions in $H^{s+2} \times H^s$ for $s > 15/2$ at sufficiently large time. See also [12] by Chaves-Silva, Rosier and Zuazua for the higher dimensional case.

The 1D compressible Navier–Stokes equations linearized around a constant steady state with periodic boundary conditions is closely related to the structurally damped wave equation studied in [47]. Chowdhury and Mitra [17] studied the interior null-controllability of the linearized (around constant steady state (Q_0, V_0), $Q_0 > 0, V_0 > 0$) 1D compressible Navier–Stokes system with periodic boundary conditions. Following the approach of [47], the authors in [17] established that their system is null-controllable by a localized interior control when the time is large enough, and for regular initial data in $\dot{H}^{s+1}_{\text{per}} \times H^s_{\text{per}}$ with $s > 13/2$. They also achieved that, for any $T > \frac{2\pi}{V_0}$, the system is approximately controllable at time T in $\dot{L}^2 \times L^2$ using a localized interior control (of the form $f(t, x) = h(t)g(x)$) and, is null-controllable at time T using periodic boundary control with regular initial data $\dot{H}^{s+1}_{\text{per}} \times \dot{H}^s_{\text{per}}$ for $s > 9/2$.

In [18], Chowdhury, Mitra, Ramaswamy and Renardy considered the one-dimensional compressible Navier–Stokes equations linearized around a constant steady state $(Q_0, V_0), Q_0 > 0, V_0 > 0$, with homogeneous periodic boundary conditions in the interval $(0, 2\pi)$. They proved that the linearized system with homogeneous periodic boundary conditions is null-controllable in $\dot{H}^1_{\text{per}} \times L^2$ by a localized interior control when the time $T > \frac{2\pi}{V_0}$. Moreover, in their work the distributed null-controllability result in $\dot{H}^1_{\text{per}} \times L^2$ is sharp in the sense that the controllability fails in $\dot{H}^s_{\text{per}} \times L^2$ for any $0 \le s < 1$. As usual, the large time for controllability is needed due to the presence of transport part and indeed, the null-controllability fails for small time; see [46] by Maity and [1] by Ahamed, Maity and Mitra.

Chowdhury [13] considered the same linearized Navier–Stokes system around (Q_0, V_0) with $Q_0 > 0, V_0 > 0$ in $(0, L)$ with homogeneous Dirichlet boundary conditions and an interior control acting only on the velocity equation on a open subset $(0,l) \subset (0,L)$. He proved the approximate controllability of the linearized system in $L^2(0,L)\times L^2(0,L)$ with a localized control in $L^2(0,T;L^2(0,l))$ when $T > \frac{L-l}{V_0}$.

In the context of the controllability of coupled transport-parabolic system (which is the main feature of linearized compressible Navier-Stokes equations), we must mention the work [43] by Lebeau and Zuazua where the distributed null-controllability of Thermoelasticity system has been studied. More recently, Beauchard, Koenig and Le Balc'h [8] considered the linear parabolic-transport system with constant coefficients and coupling of order zero and one with locally distributed controls posed on the one-dimensional torus $\mathbb{T}$. Following the approach of [43], they proved the null-controllability at sufficiently large time when there are as many controls as equations. On the other hand, when the control acts only on the transport (resp. parabolic) component, they obtained an algebraic necessary and sufficient condition on the coupling term for the null-controllability, and their controllability studies based on a detailed spectral analysis. According to the more general result established in [8], we can say that for a 2×2 coupled parabolic-transport system (with periodic boundary conditions), the null-controllability with one localized interior control holds true in $L^2(\mathbb{T})\times \dot{L}^2(\mathbb{T})$ (resp. in $\dot{H}^2(\mathbb{T})\times H^2(\mathbb{T})$) when the control acts only on the transport (resp. parabolic) component. More recently, the distributed null-controllability of underactuated linear parabolic-transport systems with constant coefficients in one-dimensional torus has been established in [38] by Koenig and Lissy for regular enough initial data and large time.

Finally, one may find few stabilization results for linearized compressible Navier-Stokes system available in [3], [14, 16, 20], [49, 50].

1.5. **Our approach and achievement of the present work.** As mentioned earlier, in compressible Navier-Stokes system, the interesting feature is the first order coupling between transport equation and momentum equation of parabolic type. It was shown in [17, 18] that the linearized compressible Navier-Stokes system with Periodic boundary conditions, there is a sequence of generalized eigenfunctions of the associated adjoint operator that forms a Riesz Basis for the state Hilbert space. The success in obtaining this result lies in the simplicity of the corresponding characteristic equations as well as the explicit structure of all eigenfunctions in terms of Fourier basis.

But for the operator $(A^*, D(A^*))$ defined in (1.9), the characteristic equation is a third order ODE and the eigenvalue equation is a non-standard transcendental equation, which is quite challenging to handle. In fact, the method (invariant subspace idea) used in [17, 18] is not practically applicable to our case. However, we manage to characterize the set of eigenvalues and eigenfunctions for the operator A^*. More precisely, the spectrum of A^* consists of: a parabolic part containing the eigenvalues λ_k^p such that $\mathrm{Re}(\lambda_k^p)$ behaves like $-k^2\pi^2$ for large enough $k\in\mathbb{N}^*$ while $\mathrm{Im}(\lambda_k^p)$ is bounded; a hyperbolic part made up of the eigenvalues λ_k^h such that $\mathrm{Im}(\lambda_k^h)$ behaves like $2k\pi$ for large enough $k\in\mathbb{Z}$ while $\mathrm{Re}(\lambda_k^h)$ is bounded; and a finite set of lower frequencies. The Riesz basis property of the set of (generalized) eigenfunctions has been then established by using an abstract result of B.-Z. Guo [32].

To study the boundary null-controllability, we mention that the usual extension method is not really convenient for the Navier-Stokes system. This is because, when we put one interior control in the system, then upon extending the domain and restricting the solution on the boundary will give rise to two boundary controls for the system. In this regard, we refer some earlier null-controllability results [27,28,50] with one interior control in the velocity equation or two boundary controls both for density and velocity.

The main novelty of the present work is that we directly handle the boundary null-controllability with only one control acting on the density or velocity part where the boundary conditions are of mixed type (in this regard, we mention the work [11] by Cerpa, Montoya and Zhang, where some mixed boundary conditions has been appeared in the context of KdV-Burgers equation). More precisely, when a control acts in velocity, we use the Ingham-type inequality given by Proposition 1.7 to prove an observability inequality for the adjoint to the system (1.4) in $(\dot{H}_\sharp^{\frac{1}{2}}(0,1))'\times L^2(0,1)$, leading to the null-controllability of (1.4) at time $T>1$ with initial data in $\dot{H}_\sharp^{\frac{1}{2}}(0,1)\times L^2(0,1)$. On the other hand, when a boundary control acts on the density part, we proceed in the following way: first, using the Ingham-type inequality (1.13) we obtain the null-controllability of the system (1.5) at time $T>1$ in the space $\dot{L}^2(0,1)\times H_0^1(0,1)$; secondly, we apply a parabolic-hyperbolic joint moments technique as developed in [34] by Hansen to conclude the null-controllability of the same system (1.5) in the space $\dot{H}_\sharp^s(0,1)\times L^2(0,1)$ for $s>\frac{1}{2}$ at $T>1$. Then, due to the linearity of the solution map of the system (1.5), these two results provide the null-controllability of that system in the space $\dot{L}^2(0,1)\times L^2(0,1)$ when $T>1$. And, consequently, we deduce the null-controllability of the system (1.6) at time $T>1$ in $\dot{L}^2(0,1)\times L^2(0,1)$, which consists of full Dirichlet boundary conditions.

1.6. **Paper organization.** The paper is organized as follows.

- In Section 2, we discuss the well-posedness results of the main systems and some associated results have been proved in the Appendix.
- We split the spectral analysis for the associated adjoint operator into two sections for the ease of reading. Section 3 contains a short description of the spectral properties whereas the detailed analysis is prescribed in Section 8.
- In Section 4, we obtain the lower bounds for the observation terms which are crucial to determine the null-controllability for the system (1.4) or (1.5).
- Section 6 is devoted to prove the null-controllability of the system (1.4), that is Theorem 1.2. An Ingham-type inequality (Proposition 1.7), proved in Section 5, is the main ingredient for the required null-controllability proof.
- Then, in Section 7, we prove the null-controllability of the system (1.5), that is Theorem 1.3 by using both the method of moments and the Ingham-type inequality obtained in Section 5. As a consequence, we conclude the result in Theorem 1.6. Further, a lack of null controllability result (Proposition 1.5) for this system (1.5) is also included in this section.
- Finally, we conclude our paper by providing some open question and remarks in Section 9.

2. Well-posedness of the system

Let us first recall the operator A^* defined by (1.9). Then, we write the adjoint system associated to the control problems (1.4) and (1.5): let (σ, v) be the adjoint state and the system reads as

$$\begin{cases} -\sigma_t - \sigma_x - b v_x = f, & \text{in } (0,T)\times(0,1), \\ -v_t - v_{xx} - v_x - b\sigma_x = g, & \text{in } (0,T)\times(0,1), \\ \sigma(t,0) = \sigma(t,1), & \text{for } t\in(0,T), \\ v(t,0) = v(t,1) = 0, & \text{for } t\in(0,T), \\ \sigma(T,x) = \sigma_T(x), \quad v(T,x) = v_T(x), & \text{in } (0,1). \end{cases} \tag{2.1}$$

Shortly, one may express it by

$$-V'(t) = A^*V(t) + F(t), \quad \forall t\in(0,T), \quad V(T) = V_T, \tag{2.2}$$

where the state is $V := (\sigma, v)$, given final data is $V_T := (\sigma_T, v_T)$ and source term is $F := (f, g)$.

To show the well-posedness of the solutions to (1.4) and (1.5), let us first write the following lemma.

Lemma 2.1. *The operator A (resp. A^*) is maximal dissipative in $L^2(0,1)\times L^2(0,1)$, that is, $(A, D(A))$ (resp. $(A^*, D(A^*))$) generates a strongly continuous semigroup of contractions in $L^2(0,1)\times L^2(0,1)$.*

The proof of Lemma 2.1 can be done in a standard fashion. For the sake of completeness, we give the proof in Appendix A.1. As a consequence of this result, we now guarantee the existence of a strong solution of the linearized compressible Navier-Stokes equation (1.4) (resp. (1.5)) when there is no control input acting on the system.

Lemma 2.2. *For any given $(\rho_0, u_0)\in\mathcal{D}(A)$, the system (1.4) with $q=0$ (or the system (1.5) with $p=0$) admits a unique strong solution $(\rho,u)\in\mathcal{C}^1([0,T];L^2(0,1)\times L^2(0,1))\cap\mathcal{C}^0([0,T];\mathcal{D}(A))$.*

Once we have the existence of semigroup generated by the operator A^*, we can write the following result:

Proposition 2.3. *For any given $F := (f,g)\in L^2(0,T;L^2(0,1)\times L^2(0,1))$ and $V_T = (\sigma_T, v_T)\in L^2(0,1)\times L^2(0,1)$, there exists a unique weak solution $V := (\sigma, v)$ to the system (2.2) in the space $\mathcal{C}([0,T];L^2(0,1))\times[\mathcal{C}([0,T];L^2(0,1))\cap L^2(0,T;H^1_0(0,1))]$ with the estimate*

$$\|(\sigma,v)\|_{\mathcal{C}^0([0,T];L^2(0,1)\times L^2(0,1))} + \|v\|_{L^2(0,T;H^1_0(0,1))} \le C\left(\|F\|_{L^2(0,T;L^2(0,1)\times L^2(0,1))} + \|V_T\|_{L^2(0,1)\times L^2(0,1)}\right).$$

Moreover, we have the hidden regularity property $\sigma(\cdot,1)\in L^2(0,T)$.

In particular, if $F\in L^2(0,T;H^1(0,1)\times L^2(0,1))$ and $V_T = (0,0)$, the solution (σ,v) to (2.2) belongs to $\mathcal{C}^0([0,T];H^1_\sharp(0,1))\times[\mathcal{C}^0([0,T];H^1_0(0,1))\cap L^2(0,T;H^2(0,1))]$.

The proof of this result can be adapted from the work [31, Chapter IV, Sec. 4.3]; we omit the details here. For the hidden regularity property, we give a detailed proof in Appendix B.

Now, we can define the notion of solutions to the control systems (1.4) and (1.5) in the sense of transposition (see for instance [21]) where a non-trivial boundary source term is appearing.

Definition 2.4. *We write the following definitions based on the act of the control:*

- *For given initial state* $U_0 := (\rho_0, u_0) \in L^2(0,1) \times L^2(0,1)$ *and boundary data* $q \in L^2(0,T)$*, a function* $U := (\rho, u) \in L^2(0,T; (H^1_\sharp(0,1))') \times L^2(0,T; L^2(0,1))$ *is a solution to the system* (1.4) *if for any given* $F := (f,g) \in L^2(0,T; H^1(0,1)) \times L^2(0,T; L^2(0,1))$*, the following identity holds true:*

$$\int_0^T \langle \rho(t,\cdot), f(t,\cdot)\rangle_{(H^1)',H^1}\, dt + \int_0^T \int_0^1 u(t,x) g(t,x) dx dt \\ = \langle U_0(\cdot), V(0,\cdot)\rangle_{L^2\times L^2} + \int_0^T \big[b\sigma(t,1) + v_x(t,1)\big] q(t) dt,$$

where $V := (\sigma, v)$ *is the unique weak solution to the adjoint system* (2.2) *with* $V_T = (0,0)$.

- *For given initial state* $U_0 := (\rho_0, u_0) \in L^2(0,1) \times L^2(0,1)$ *and boundary data* $p \in L^2(0,T)$*, a function* $U := (\rho, u) \in L^2(0,T; L^2(0,1)) \times L^2(0,T; L^2(0,1))$ *is a solution to the system* (1.5) *if for any given* $F := (f,g) \in L^2(0,T; L^2(0,1)) \times L^2(0,T; L^2(0,1))$*, the following identity holds true:*

$$\int_0^T \int_0^1 \rho(t,x) f(t,x) dx dt + \int_0^T \int_0^1 u(t,x) g(t,x) dx dt = \langle U_0(\cdot), V(0,\cdot)\rangle_{L^2\times L^2} + \int_0^T \sigma(t,1) p(t) dt,$$

where $V := (\sigma, v)$ *is the unique weak solution to the adjoint system* (2.2) *with* $V_T = (0,0)$.

Let us state the following theorems that concern the existence and uniqueness of solutions to the control problems (1.4) and (1.5).

Theorem 2.5. *For every* $q \in L^2(0,T)$ *and every* $U_0 := (\rho_0, u_0) \in L^2(0,1) \times L^2(0,1)$*, the system* (1.4) *has a unique solution* $U := (\rho, u)$ *belonging to the space* $\mathcal{C}^0([0,T]; (H^1_\sharp(0,1))') \times [\mathcal{C}^0([0,T]; H^{-1}(0,1)) \cap L^2(0,T; L^2(0,1))]$ *in the sense of transposition.*

Moreover, this solution (ρ, u) *satisfies the following estimate*

$$\|\rho\|_{\mathcal{C}^0([0,T];(H^1_\sharp(0,1))')} + \|u\|_{\mathcal{C}^0([0,T];H^{-1}(0,1))\cap L^2(0,T;L^2(0,1))} \leq C \left(\|(\rho_0, u_0)\|_{L^2(0,1)\times L^2(0,1)} + \|q\|_{L^2(0,T)} \right)$$

for some constant $C > 0$.

The proof for Theorem 2.5 will be followed from [19, Section 3]. In fact, if $(\rho_0, u_0) \in L^2(0,1) \times L^2(0,1)$ and $q \in L^2(0,T)$, the solution (ρ, u) of (1.4) belong to $L^2(0,T; (H^1_\sharp(0,1))') \times L^2(0,T; L^2(0,L))$. Using the continuity estimate for the transport equation and properties of the heat equation, we can deduce that $\rho \in \mathcal{C}^0([0,T]; (H^1_\sharp(0,1))')$ and $u \in \mathcal{C}^0([0,T]; H^{-1}(0,1))$.

Theorem 2.6. *For every* $p \in L^2(0,T)$ *and* $U_0 := (\rho_0, u_0) \in L^2(0,1) \times L^2(0,1)$*, the system* (1.5) *has a unique solution* $U := (\rho, u)$ *belonging to the space* $L^2(0,T; L^2(0,1)) \times L^2(0,T; L^2(0,1))$ *in the sense of transposition and the operator defined by*

$$(U_0, p) \mapsto U(U_0, p),$$

is linear and continuous from $(L^2(0,1) \times L^2(0,1)) \times L^2(0,T)$ *into* $L^2(0,T; L^2(0,1)) \times L^2(0,T; L^2(0,1))$.

Moreover, the solution satisfies the following regularity result,

$$(\rho, u) \in \mathcal{C}^0([0,T]; L^2(0,1)) \times [\mathcal{C}^0([0,T]; L^2(0,1)) \cap L^2(0,T; H^1_0(0,1))] \tag{2.3}$$

with the estimate

$$\|\rho\|_{\mathcal{C}^0([0,T];L^2(0,1))} + \|u\|_{\mathcal{C}^0([0,T];L^2(0,1))\cap L^2(0,T;H^1_0(0,1))} \\ \leq C \left(\|(\rho_0, u_0)\|_{L^2(0,1)\times L^2(0,1)} + \|p\|_{L^2(0,T)} \right), \tag{2.4}$$

for some constant $C > 0$.

Further, we have the hidden regularity property $\rho(\cdot, 1) \in L^2(0,T)$.

We give a sketch of the proof for Theorem 2.6 in Appendix A.2-B.

3. A short description of the spectral properties of the adjoint operator

In this section, we briefly describe the spectral properties of the adjoint operator A^* associated to our control system (1.4) or (1.5). This part is crucial in our analysis but it is the most technical part, and thus a detailed study will be presented in Section 8.

3.1. **The eigenvalue problem.** Let us denote $\Phi := (\xi, \eta)$ and consider the following eigenvalue problem:

$$A^*\Phi = \lambda\Phi, \quad \text{for } \lambda \in \mathbb{C},$$

which is explicitly given by

$$\begin{aligned} \xi'(x) + b\eta'(x) &= \lambda\xi(x), \quad x \in (0,1), \\ \eta''(x) + \eta'(x) + b\xi'(x) &= \lambda\eta(x), \quad x \in (0,1), \\ \xi(0) &= \xi(1), \\ \eta(0) &= \eta(1) = 0. \end{aligned} \tag{3.1}$$

We prove the following proposition.

Proposition 3.1. *The following results are true.*

- (i) *We have* $\ker A^* = \operatorname{span}\{(1,0)\}$.
- (ii) *All non-zero eigenvalues of* A^* *have negative real parts.*
- (iii) *The resolvent operator associated with* A^* *is compact and hence the spectrum of* A^* *is discrete.*
- (iv) *Let* $b > 0$ *be such that* $b^4 + 8b^2 + 5 < 4\pi^2$*. Then, the eigenvalues of* A^* *are geometrically simple.*

A quick observation tells that: when $\lambda = 0$, then $c(1,0)$ with $c \neq 0$ are the only eigenfunctions of the operator A^*, which is nothing but the part (i) of the above proposition. Proofs of the other parts are given in Section 8.

3.2. **The set of eigenvalues.** Let us write the properties of the eigenvalues of the operator A^*. More precisely, we have the following lemma.

Lemma 3.2. *Let* $(A^*, D(A^*))$ *be the operator given by* (1.9)*. Then, there exist* $k_0, n_0 \in \mathbb{N}^*$ *such that* A^* *has three sets of eigenvalues: the parabolic part* $\{\lambda_k^p\}_{k\geq k_0}$*, the hyperbolic part* $\{\lambda_k^h\}_{|k|\geq k_0}$ *and a finite family* $\{0\} \cup \{\widehat{\lambda}_n\}_{n=1}^{n_0}$ *of lower frequencies. Moreover, the parabolic and hyperbolic branches satisfy the following asymptotic properties:*

$$\lambda_k^p = -k^2\pi^2 + O(1), \quad \text{for all } k \geq k_0 \text{ large}, \tag{3.2a}$$

$$\lambda_k^h = -b^2 - 2ik\pi + O(|k|^{-1}), \quad \text{for all } |k| \geq k_0 \text{ large}. \tag{3.2b}$$

The proof of the above lemma is one of the crucial part of our work and it is heavy; the details have been provided in Sections 8.1 and 8.3.

For simplicity, we set $\lambda_0 = 0$ and the associated eigenfunction by $\Phi_{\lambda_0} = (1,0)$. We further denote the set of eigenvalues associated to the parabolic and hyperbolic parts respectively by

$$\Lambda_p := \{\lambda_k^p,\ k \geq k_0\}, \quad \Lambda_h := \{\lambda_k^h,\ |k| \geq k_0\}, \tag{3.3}$$

and for the lower frequencies by

$$\Lambda_0 := \{\widehat{\lambda}_n,\ 1 \leq n \leq n_0\}. \tag{3.4}$$

Finally, the set of all eigenvalues are denoted by $\sigma(A^*)$, where

$$\sigma(A^*) := \{\lambda_0\} \cup \Lambda_0 \cup \Lambda_p \cup \Lambda_h. \tag{3.5}$$

3.3. **The set of eigenfunctions.** We start by writing the following proposition.

Proposition 3.3. *Let* k_0 *be as given by Lemma 3.2. Then, the operator* A^* *has the following sets of (generalized) eigenfunctions: the parabolic part* $\{\Phi_{\lambda_k^p}\}_{k\geq k_0}$*, the hyperbolic part* $\{\Phi_{\lambda_k^h}\}_{|k|\geq k_0}$*, the singleton set* $\{\Phi_{\lambda_0}\}$ *and a finite set* $\{\Phi_\lambda^i;\ \lambda \in \Lambda_0,\ i = 0, \ldots, m_\lambda - 1\}$*, where* $m_\lambda \geq 1$ *is the length of Jordan chain associated to each of the eigenvalues* $\lambda \in \Lambda_0$.

Furthermore, we have the following:

1. *The parabolic part of the eigenfunctions*

$$\Phi_{\lambda_k^p} := (\xi_{\lambda_k^p}, \eta_{\lambda_k^p}) \tag{3.6}$$

have asymptotic expressions for large $k \geq k_0$, given by

$$\xi_{\lambda_k^p}(x) = \frac{ib}{k\pi} e^{-\frac{1}{2}(1+x)} \cos(k\pi(1-x)) + e^{x\left(-k^2\pi^2+O(1)\right)} \times O\left(\frac{1}{k}\right) + O\left(\frac{1}{k^2}\right), \tag{3.7}$$

$$\eta_{\lambda_k^p}(x) = e^{-\frac{1}{2}(1+x)} \sin(k\pi(1-x)) + O\left(\frac{1}{k}\right), \tag{3.8}$$

for all $x \in (0,1)$ and the hyperbolic part of the eigenfunctions

$$\Phi_{\lambda_k^h} := (\xi_{\lambda_k^h}, \eta_{\lambda_k^h}) \tag{3.9}$$

have asymptotic expressions for large $|k| \geq k_0$, given by

$$\xi_{\lambda_k^h}(x) = \frac{2i}{be^{\frac{1}{\sqrt{|k|}}}} \operatorname{sgn}(k) e^{-\frac{1}{2}-i\operatorname{sgn}(k)\sqrt{|k\pi|}} e^{-2ik\pi x} + O\left(|k|^{-1}\right), \tag{3.10}$$

$$\begin{aligned}\eta_{\lambda_k^h}(x) = &\frac{1}{k\pi e^{\frac{1}{\sqrt{|k|}}}} \operatorname{sgn}(k) e^{-\frac{1}{2}-i\operatorname{sgn}(k)\sqrt{|k\pi|}} e^{-2ik\pi x} \\ &+ \frac{1}{k\pi e^{\frac{1}{\sqrt{|k|}}}} \operatorname{sgn}(k) e^{-(1-x)\left(\sqrt{|k\pi|}-\frac{1}{2}-i\operatorname{sgn}(k)\sqrt{|k\pi|}\right)} + O\left(|k|^{-1}\right),\end{aligned} \tag{3.11}$$

for all $x \in (0,1)$, where the sgn *function is defined as*

$$\operatorname{sgn}(k) = \begin{cases} 1 & \text{when } k \geq 0, \\ -1 & \text{when } k < 0, \end{cases} \tag{3.12}$$

2. *The eigenfamily, denoted by*

$$\mathcal{E}(A^*) := \left\{\Phi_{\lambda_k^p},\, k \geq k_0\right\} \cup \left\{\Phi_{\lambda_k^h},\, |k| \geq k_0\right\} \cup \left\{\Phi_{\lambda_0}\right\} \cup \left\{\Phi_\lambda^i;\, \lambda \in \Lambda_0,\, i = 0, \ldots, m_\lambda - 1\right\}, \tag{3.13}$$

forms a Riesz basis in $L^2(0,1) \times L^2(0,1)$.

The last property (Riesz basis) can also be proved in the space $(H_\sharp^{s_1}(0,1))' \times H^{-s_2}(0,1)$ (for $s_1, s_2 \geq 0$) by normalizing the eigenfunctions suitably, as written below.

Corollary 3.4. *Let $s_1, s_2 \geq 0$ be given. The family of (generalized) eigenfunctions*

$$\mathcal{E}(A^*) := \left\{k^{s_2}\Phi_{\lambda_k^p},\, k \geq k_0\right\} \cup \left\{k^{s_1}\Phi_{\lambda_k^h},\, |k| \geq k_0\right\} \cup \left\{\Phi_{\lambda_0}\right\} \cup \left\{\Phi_\lambda^i;\, \lambda \in \Lambda_0,\, i = 0, \ldots, m_\lambda - 1\right\},$$

forms a Riesz basis in $(H_\sharp^{s_1}(0,1))' \times H^{-s_2}(0,1)$.

We have taken the same finitely many eigenfunctions as before, which can be ensured by choosing a suitable multiple of the generalized eigenfunctions. We will use this Riesz basis property (with appropriate s_1 and s_2) to prove the required observability inequalities, see the proof of our main results in Sections 6–7.

The existence of parabolic and hyperbolic parts of the family of eigenfunctions are proved in Sections 8.2–8.4. Then, using a result from [32], we shall prove the existence of lower frequencies of eigenvalues $\{\widehat{\lambda}_n\}_{n=1}^{n_0}$ and the associated (generalized) eigenfunctions. Moreover, we will show that the set of eigenfunctions $\mathcal{E}(A^*)$ forms a Riesz basis for $L^2(0,1) \times L^2(0,1)$.

Lemma 3.5 (Bounds of the eigenfunctions)**.** *Recall the eigenfunctions $\Phi_{\lambda_k^p} = (\xi_{\lambda_k^p}, \eta_{\lambda_k^p})$, $\forall k \geq k_0$ and $\Phi_{\lambda_k^h} = (\xi_{\lambda_k^h}, \eta_{\lambda_k^h})$, $\forall |k| \geq k_0$ given by (3.7)–(3.8) and (3.10)–(3.11) respectively. Then there exist constants $C_1, C_2 > 0$ independent in k, such that we have the following.*

1. *For any $s \geq 0$ and $k \geq k_0$, we have*

$$\begin{cases} C_1 k^{-s-1} \leq \|\xi_{\lambda_k^p}\|_{(H_\sharp^s(0,1))'} \leq C_2 k^{-s-1}, \\ C_1 k^{-s} \leq \|\eta_{\lambda_k^p}\|_{H^{-s}(0,1)} \leq C_2 k^{-s}. \end{cases} \tag{3.14}$$

2. *On the other hand, for any $|k| \geq k_0$ and $s \geq 0$, we have*

$$\begin{cases} C_1\, |k|^{-s} \leq \|\xi_{\lambda_k^h}\|_{(H_\sharp^s(0,1))'} \leq C_2 |k|^{-s}, \\ C_1\, |k|^{-s-1} \leq \|\eta_{\lambda_k^h}\|_{H^{-s}(0,1)} \leq C_2 |k|^{-s-1}. \end{cases} \tag{3.15}$$

Again, the proofs can be found in Section 8.5.

Riesz basis property of the (generalized) eigenfunctions. Let us first recall the following result.

Theorem 3.6 (B.-Z. GUO [32]). *Let $\mathcal{A}$ be a densely defined discrete operator (i.e., the resolvent of $\mathcal{A}$ is compact) in a Hilbert space H. Let $\{\phi_n\}_1^\infty$ be a Riesz basis of H. If there are an integer $N \geq 0$ and a sequence of generalized eigenvectors $\{\psi_n\}_{N+1}^\infty$ of $\mathcal{A}$ such that*

$$\sum_{N+1}^{\infty} \|\phi_n - \psi_n\|^2 < +\infty,$$

then the following results hold.

- (i) *There are a constant $M > N$ and generalized eigenvectors $\{\psi_{n0}\}_1^M$ of $\mathcal{A}$ such that $\{\psi_{n0}\}_1^M \cup \{\psi_n\}_{M+1}^\infty$ forms a Riesz Basis for H.*
- (ii) *Let $\{\psi_{n0}\}_1^M \cup \{\psi_n\}_{M+1}^\infty$ correspond to the eigenvalues $\{\lambda_n\}_1^\infty$ of $\mathcal{A}$. Then the spectrum $\sigma(\mathcal{A}) = \{\lambda_n\}_1^\infty$, where λ_n is counted according to its algebraic multiplicity.*
- (iii) *If there is an $M_0 > 0$ such that $\lambda_n \neq \lambda_m$ for all $m, n > M_0$, then there is an $N_0 > M_0$ such that all λ_n are algebraically simple if $n > N_0$.*

The first assumption of Theorem 3.6 is true in our case since we know that the resolvent operator of A^* is compact, thanks to the Proposition 3.1–part (iii). So, the next duty is to find a known family $\{\Psi_k,\ k \in \mathbb{N}^*;\ \widetilde{\Psi}_k,\ k \in \mathbb{Z}\}$ that defines a Riesz basis for $L^2(0,1) \times L^2(0,1)$ and, is quadratically close to the countable family $\left\{\Phi_{\lambda_k^p},\ k \geq k_0\right\} \cup \left\{\Phi_{\lambda_k^h},\ |k| \geq k_0\right\}$. Precisely, our goal is to show the following:

$$\sum_{k\geq k_0} \left\|\Phi_{\lambda_k^p} - \Psi_k\right\|^2_{L^2\times L^2} + \sum_{|k|\geq k_0} \left\|\Phi_{\lambda_k^h} - \widetilde{\Psi}_k\right\|^2_{L^2\times L^2} < +\infty. \tag{3.16}$$

To this end, let us consider the following functions:

$$\Psi_k(x) := \begin{pmatrix} \phi_k \\ \psi_k \end{pmatrix} = \begin{pmatrix} 0 \\ 2ie^{-\frac{1}{2}(1+x)}\sin(k\pi(1-x)) \end{pmatrix}, \quad \forall k \in \mathbb{N}^*, \tag{3.17a}$$

$$\widetilde{\Psi}_k(x) := \begin{pmatrix} \widetilde{\phi}_k \\ \widetilde{\psi}_k \end{pmatrix} = \begin{pmatrix} \frac{2i}{be^{\frac{1}{\sqrt{|k|}}}}\operatorname{sgn}(k)e^{-\frac{1}{2}-i\operatorname{sgn}(k)\sqrt{|k\pi|}}e^{-2ik\pi x} \\ 0 \end{pmatrix}, \quad \forall k \in \mathbb{Z}, \tag{3.17b}$$

for $x \in (0,1)$. It can be shown that the family $\{\Psi_k,\ k \in \mathbb{N}^*;\ \widetilde{\Psi}_k,\ k \in \mathbb{Z}\}$ of above functions forms a Riesz basis for $L^2(0,1) \times L^2(0,1)$ and we have the following result.

Lemma 3.7. *The family $\{\Psi_k,\ k \in \mathbb{N}^*;\ \widetilde{\Psi}_k,\ k \in \mathbb{Z}\}$ given by (3.17a)–(3.17b) is quadratically close to the family of eigenfunctions $\left\{\Phi_{\lambda_k^p},\ k \geq k_0\right\} \cup \left\{\Phi_{\lambda_k^h},\ |k| \geq k_0\right\}$.*

Proof. Looking at the expressions of the eigenfunctions $\Phi_{\lambda_k^p}$, $\Phi_{\lambda_k^h}$ for large modulus of k, given by (3.6)–(3.7)–(3.8) and (3.9)–(3.10)–(3.11) (resp.) and the known functions Ψ_k, $\widetilde{\Psi}_k$ given by (3.17a)–(3.17b), it is straightforward to compute that

$$\left\|\Phi_{\lambda_k^p} - \Psi_k\right\|^2_{L^2\times L^2} \leq \frac{C}{k^2}, \quad \forall k \geq k_0 \text{ large enough},$$

and

$$\left\|\Phi_{\lambda_k^h} - \widetilde{\Psi}_k\right\|^2_{L^2\times L^2} \leq \frac{C}{k^2}, \quad \forall |k| \geq k_0 \text{ large enough},$$

which implies the required property (3.16). □

Sketch of the proof for Proposition 3.3. The proof of part 1 is lengthy and it has been postponed to Sections 8.2–8.4.

Now, thanks to Lemma 3.7, we can apply the point (i) of Theorem 3.6 to ensure the existence of eigenmodes for lower frequencies. Accordingly, there exist an $n_0 \in \mathbb{N}^*$ and a finite set eigenvalues

$$\Lambda_0 := \{\widehat{\lambda}_n\}_1^{n_0}$$

of the operator A^*. But there may exist some generalized eigenfunctions corresponding to the eigenvalues of the finite set Λ_0. Thus, for each $\lambda \in \Lambda_0$, we associate a Jordan chain of length $m_\lambda \geq 1$, denoted by $\Phi_\lambda^0, \ldots, \Phi_\lambda^{m_\lambda - 1}$ which verify

$$(A^* - \lambda I)\Phi_\lambda^i = \Phi_\lambda^{i-1}, \quad \forall i \in \{1, \ldots, m_\lambda - 1\}, \quad \lambda \in \Lambda_0,$$

where in particular $\Phi^0_\lambda := \Phi_\lambda$, the eigenfunction corresponding to λ. Moreover, by virtue of Theorem 3.6, we can guarantee that the family, given by

$$\mathcal{E}(A^*) := \left\{\Phi_{\lambda_k^p},\, k \geq k_0\right\} \cup \left\{\Phi_{\lambda_k^h},\, |k| \geq k_0\right\} \cup \left\{\Phi_{\lambda_0}\right\} \cup \left\{\Phi^i_\lambda;\, \lambda \in \Lambda_0,\, i = 0, \ldots, m_\lambda - 1\right\},$$

forms a Riesz basis in $L^2(0,1) \times L^2(0,1)$.

The proof ends. □

Remark 3.8. *In the same way, one can prove that the set of eigenvalues and (generalized) eigenfunctions of A (denoted by $\sigma(A)$ and $\mathcal{E}(A)$ respectively) have similar properties as of the eigenpairs of A^*.*

In this case, we can find some $\widetilde{k}_0 \in \mathbb{N}^*$ (large enough) such that A has the eigenvalues of parabolic and hyperbolic nature for $k \geq \widetilde{k}_0$ and $|k| \geq \widetilde{k}_0$ respectively. For later use, we denote the eigenfunctions of A, respectively by $\widetilde{\Phi}^p_k$, $k \geq \widetilde{k}_0$ and $\widetilde{\Phi}^h_k$, $|k| \geq \widetilde{k}_0$ corresponding to the parabolic and hyperbolic branches of eigenvalues.

Moreover, using the result of Theorem 3.6, we can show that the set $\mathcal{E}(A)$ forms a Riesz basis for the space $L^2(0,1) \times L^2(0,1)$.

4. Estimations of the observation terms

In this section, we are going to find some lower bounds of the observation terms associated to our control systems. In this regard, we use the notations $\mathcal{B}^*_\rho$ and $\mathcal{B}^*_u$ which represent the observation operators for the density and velocity case respectively, and their formal expressions are given below.

- The observation operator corresponding to (1.5) (control in density) is defined by

$$\mathcal{B}^*_\rho = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbb{1}_{\{x=1\}} : D(A^*) \to \mathbb{R}, \tag{4.1}$$

  such that

$$\mathcal{B}^*_\rho \Phi = \xi(1), \quad \forall\, \Phi = (\xi, \eta) \in D(A^*). \tag{4.2}$$

- The observation operator corresponding to (1.4) (control in velocity) is defined by

$$\mathcal{B}^*_u = b\mathbb{1}_{\{x=1\}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathbb{1}_{\{x=1\}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \frac{\partial}{\partial x} : D(A^*) \to \mathbb{R}, \tag{4.3}$$

  such that

$$\mathcal{B}^*_u \Phi = b\xi(1) + \eta'(1), \quad \forall\, \Phi = (\xi, \eta) \in D(A^*). \tag{4.4}$$

4.1. **Characteristics of the observation terms.** Let us pick any

$$\Phi := (\xi, \eta) \in \left\{\Phi_\lambda;\ \lambda \in \Lambda_p \cup \Lambda_h \cup \Lambda_0\right\},$$

and recall the eigenvalue problem (3.1). Substituting the first equation of (3.1) in the second one, we get

$$\eta''(x) - (b^2 - 1)\eta'(x) + b\lambda\xi(x) - \lambda\eta(x) = 0, \quad \forall x \in (0,1). \tag{4.5}$$

Differentiating, we have

$$\eta'''(x) - (b^2 - 1)\eta''(x) + b\lambda\xi'(x) - \lambda\eta'(x) = 0, \quad \forall x \in (0,1).$$

By substituting $b\xi' = \lambda\eta - \eta'' - \eta'$ in above, we get a third order ode satisfied only by η as follows

$$\begin{cases} \eta'''(x) - (\lambda + b^2 - 1)\eta''(x) - 2\lambda\eta'(x) + \lambda^2\eta(x) = 0, \quad \forall x \in (0,1), \\ \eta(0) = 0,\ \eta(1) = 0, \\ \eta''(0) - (b^2 - 1)\eta'(0) = \eta''(1) - (b^2 - 1)\eta'(1). \end{cases} \tag{4.6}$$

Let m_1, m_2 and m_3 be roots of the cubic auxiliary equation (associated to (4.6))

$$m^3 - (\lambda + b^2 - 1)m^2 - 2\lambda m + \lambda^2 = 0. \tag{4.7}$$

Then, we have the following result which states some properties of the roots m_1, m_2 and m_3.

Lemma 4.1. *The following statements hold:*

- *Roots of the cubic equation (4.7) has multiplicity less than 3.*
- *If $b > 0$ is such that $b^4 + 8b^2 + 5 < 4\pi^2$, the relation $e^{m_1} = e^{m_2} = e^{m_3}$ cannot hold.*

Proof. From the relation between roots and the coefficients, we have

$$\begin{cases} m_1 + m_2 + m_3 = \lambda + b^2 - 1, \\ m_1 m_2 + m_2 m_3 + m_3 m_1 = -2\lambda, \\ m_1 m_2 m_3 = -\lambda^2. \end{cases} \tag{4.8}$$

We prove all the statements separately.

- Let $m_1 = m_2 = m_3 = m$. Then, we have from the first equation of (4.8)

$$m = \frac{1}{3}(\lambda + b^2 - 1).$$

Next, from the second and third equations of (4.8), we have $3m^2 = -2\lambda$ and $m^3 = -\lambda^2$ which further yields

$$(\lambda + b^2 - 1)^2 = -6\lambda, \quad (\lambda + b^2 - 1)^3 = -27\lambda^2, \tag{4.9}$$

so that $\lambda + b^2 - 1 = \frac{9}{2}\lambda$. By means of the first equality in (4.9), we then have $\lambda = -\frac{8}{27}$ which eventually gives

$$b^2 = 1 + \frac{7}{2}\lambda = 1 - \frac{28}{27} = -\frac{1}{27} < 0,$$

and this is not possible. Hence m_1, m_2 and m_3 cannot be equal together.

- Let us now assume

$$e^{m_1} = e^{m_2} = e^{m_3},$$

that is,

$$m_2 = m_1 + 2il\pi, \quad m_3 = m_1 + 2in\pi,$$

for some $l, n \in \mathbb{Z}$. From the first equation of (4.8), we have that

$$3m_1 + 2il\pi + 2in\pi = \lambda + b^2 - 1, \quad \text{i.e.,} \quad m_1 = \frac{1}{3}(\lambda + b^2 - 1 - 2il\pi - 2in\pi), \tag{4.10}$$

and so,

$$m_2 = \frac{1}{3}(\lambda + b^2 - 1 + 4il\pi - 2in\pi), \quad m_3 = \frac{1}{3}(\lambda + b^2 - 1 - 2il\pi + 4in\pi). \tag{4.11}$$

Substituting the above m_1, m_2, m_3 in the second equation of (4.8), we deduce (upon simplifications)

$$\lambda^2 + 2(b^2 + 2)\lambda + 4(l^2 - ln + n^2)\pi^2 + (b^2 - 1)^2 = 0.$$

Solving the above equation, we get some particular values of λ, namely

$$\begin{aligned} \lambda &= \frac{-2(b^2+2) \pm \sqrt{4(b^2+2)^2 - 16\pi^2(l^2 - ln + n^2) - 4(b^2-1)^2}}{2} \\ &= -b^2 - 2 \pm \sqrt{3(2b^2+1) - 4\pi^2(l^2 - ln + n^2)}. \end{aligned}$$

Since $l, n \in \mathbb{Z}$, one has $l^2 - ln + n^2 \geq 0$[1] and $l^2 - ln + n^2 = 0$ if and only if $l = n = 0$[2]. Thus for $(l, n) \neq (0, 0)$ the values of λ are

$$\lambda = -b^2 - 2 \pm i\sqrt{4\pi^2(l^2 - ln + n^2) - 3(2b^2+1)}. \tag{4.12}$$

Note that $4\pi^2(l^2 - ln + n^2) - 3(2b^2+1)$ is always non-negative under the assumption $b^4 + 8b^2 + 5 < 4\pi^2$ and for all $(l, n) \neq (0, 0)$.

On the other hand, putting the values of m_1, m_2, m_3 (given by (4.10)–(4.11)) in the third equation of (4.8), we get

$$(\lambda + b^2 - 1 - 2il\pi - 2in\pi)(\lambda + b^2 - 1 + 4il\pi - 2in\pi)(\lambda + b^2 - 1 - 2il\pi + 4in\pi) = -27\lambda^2,$$

which further yields

$$\begin{aligned} &\lambda^3 + 3(b^2+8)\lambda^2 + (3(b^2-1)^2 + 12l^2\pi^2 - 12ln\pi^2 + 12n^2\pi^2)\lambda + (b^2-1)^3 \\ &+ 12\pi^2(b^2-1)(l^2 - ln + n^2) - 16il^3\pi^3 + 24il^2n\pi^3 + 24iln^2\pi^3 - 16in^3\pi^3 = 0. \end{aligned}$$

[1] For $ln = 0$, $l^2 - ln + n^2 = l^2 + n^2 \geq 0$, for $ln < 0$, $l^2 - ln + n^2 > 0$ and for $ln > 0$, $l^2 - ln + n^2 = (l-n)^2 + ln > 0$.

[2] If $l^2 - ln + n^2 = 0$ and $n \neq 0$ then $(\frac{l}{n})^2 - (\frac{l}{n}) + 1 = 0$ has no real solutions. Therefore $n = 0$ and hence $l = 0$.

The real part of above equality satisfies

$$\begin{aligned}\mathrm{Re}(\lambda^3)+3(b^2+8)\mathrm{Re}(\lambda^2)+[3(b^2-1)^2+12\pi^2(l^2-ln+n^2)]\mathrm{Re}(\lambda)\\ +(b^2-1)^3+12\pi^2(b^2-1)(l^2-ln+n^2)=0.\end{aligned} \tag{4.13}$$

Now, from (4.12), one may find that

$$\begin{aligned}&\mathrm{Re}(\lambda)=-(b^2+2),\\ &\mathrm{Re}(\lambda^2)=b^4+10b^2+7-4\pi^2(l^2-ln+n^2),\\ &\mathrm{Re}(\lambda^3)=-b^6-24b^4-57b^2-26+12\pi^2(b^2+2)(l^2-ln+n^2).\end{aligned}$$

Replacing the above values in (4.13), we obtain

$$\begin{aligned}&-b^6-24b^4-57b^2-26+12\pi^2(b^2+2)(l^2-ln+n^2)\\ &+3(b^2+8)[b^4+10b^2+7-4\pi^2(l^2-ln+n^2)]\\ &-[3(b^2-1)^2+12\pi^2(l^2-ln+n^2)](b^2+2)+(b^2-1)^3+12\pi^2(b^2-1)(l^2-ln+n^2)=0\end{aligned}$$

Simplifying, we eventually have

$$27b^4+216b^2+135-108\pi^2(l^2-ln+n^2)=0,$$

so that

$$l^2-ln+n^2=\frac{27b^4+216b^2+135}{108\pi^2}=\frac{b^4+8b^2+5}{4\pi^2}<1,$$

by our assumption $b^4+8b^2+5<4\pi^2$, which is a contradiction as $l^2-ln+n^2\geq 1$ for any $(l,n)\neq(0,0)$.

Therefore, the only possibility could be $l=n=0$, but in that case, the expressions (4.10) and (4.11) provides us $m_1=m_2=m_3$, which is again a contradiction to the first part of the lemma. Hence, the results of this lemma are true. □

We are now ready to prove that all the observation terms are non-zero for both density and velocity control cases. For $\lambda=0$, the eigenfunction is $(1,0)$, and thus from the expressions of observation terms (4.2) and (4.4), we immediately get

$$\mathcal{B}_\rho^*(1,0)=1,\quad \mathcal{B}_u^*(1,0)=b,$$

which are non-zero.

We thus focus only on the case when $\lambda\neq 0$. In such situation, for any eigenfunction Φ of A^*, the observation terms can be rewritten as

$$\mathcal{B}_\rho^*\Phi=-\frac{1}{b\lambda}\left(\eta''(1)-(b^2-1)\eta'(1)\right), \tag{4.14}$$

$$\mathcal{B}_u^*\Phi=-\frac{1}{\lambda}\left(\eta''(1)-(\lambda+b^2-1)\eta'(1)\right), \tag{4.15}$$

where we have used the equation (4.5).

We now prove the proposition written below.

Proposition 4.2. *We have the following results for any non-zero eigenvalue λ of A^*.*

1. *Let $b>0$ be such that $b^4+8b^2+5<4\pi^2$. Then, the solution η of (4.6) satisfies $\eta''(1)\neq(b^2-1)\eta'(1)$.*
2. *There exists a countable set $\mathcal{N}\subset(0,\infty)$ such that for all $b\in(0,\infty)\setminus\mathcal{N}$ with $b^4+8b^2+5<4\pi^2$, the solution η of (4.6) satisfies $\eta''(1)\neq(\lambda+b^2-1)\eta'(1)$.*

Proof. (1) To prove the first part, we suppose on contrary that $\eta''(1)=(b^2-1)\eta'(1)$. This will also give us $\eta''(0)=(b^2-1)\eta'(0)$ since $\xi(0)=\xi(1)$ and consequently, $\eta''(1)-(b^2-1)\eta'(1)=\eta''(0)-(b^2-1)\eta'(0)$. We will use the Fourier transform technique together with some complex analytic arguments to prove that $\eta=0$ on $(0,1)$. This kind of technique is applied in many works, see for instance [52] for KdV the equation.

Let us define an extension map $\vartheta:\mathbb{R}\to\mathbb{R}$ by

$$\vartheta(x)=\begin{cases}\eta(x), & x\in(0,1),\\ 0, & x\in\mathbb{R}\setminus(0,1).\end{cases} \tag{4.16}$$

Then the transformed equation for (4.6) is

$$\begin{aligned}&\vartheta'''(x) - (\lambda + b^2 - 1)\vartheta''(x) - 2\lambda\vartheta'(x) + \lambda^2\vartheta(x)\\&= -\eta''(1)\delta_{x=1} + \eta''(0)\delta_{x=0} - \eta'(1)\left[\delta'_{x=1} - (\lambda + b^2 - 1)\delta_{x=1}\right] + \eta'(0)\left[\delta'_{x=0} - (\lambda + b^2 - 1)\delta_{x=0}\right]\end{aligned} \tag{4.17}$$

for all $x \in \mathbb{R}$.

Let us use the conditions $\eta''(1) = (b^2 - 1)\eta'(1)$ and $\eta''(0) = (b^2 - 1)\eta'(0)$ in (4.17), which gives

$$\begin{aligned}&\vartheta'''(x) - (\lambda + b^2 - 1)\vartheta''(x) - 2\lambda\vartheta'(x) + \lambda^2\vartheta(x)\\&= -\eta'(1)\left[\delta'_{x=1} - \lambda\delta_{x=1}\right] + \eta'(0)\left[\delta'_{x=0} - \lambda\delta_{x=0}\right], \quad \forall x \in \mathbb{R}.\end{aligned} \tag{4.18}$$

Observe that, the existence of an η satisfying (4.6) is equivalent to the existence of α, β, λ with $(\alpha, \beta) \neq (0, 0)$, such that

$$\begin{aligned}&\vartheta'''(x) - (\lambda + b^2 - 1)\vartheta''(x) - 2\lambda\vartheta'(x) + \lambda^2\vartheta(x)\\&= -\alpha\left[\delta'_{x=1} - \lambda\delta_{x=1}\right] + \beta\left[\delta'_{x=0} - \lambda\delta_{x=0}\right], \quad \forall x \in \mathbb{R}.\end{aligned} \tag{4.19}$$

Without loss of generality, we can assume $\alpha \neq 0$. Indeed, $\alpha = \eta'(1) = 0$ implies $\eta''(1) = 0$ from our assumption and thus from the equation (4.6), one has $\eta = 0$.

Taking Fourier transform on both sides of (4.19), we get

$$\begin{aligned}&\left((iz)^3 - (\lambda + b^2 - 1)(iz)^2 - 2\lambda(iz) + \lambda^2\right)\hat{\vartheta}(z)\\&= -\alpha(ize^{-iz} - \lambda e^{-iz}) + \beta(iz - \lambda), \quad \text{for } z \in \mathbb{C},\end{aligned}$$

which yields

$$\hat{\vartheta}(z) = \frac{(-\alpha e^{-iz} + \beta)(iz - \lambda)}{(iz)^3 - (\lambda + b^2 - 1)(iz)^2 - 2\lambda(iz) + \lambda^2}, \quad \text{for } z \in \mathbb{C}.$$

Since $\hat{\vartheta}$ is the Fourier transform of a function $\eta \in H^1_0(0,1)$, by the Paley-Wiener theorem, the function $\hat{\vartheta}$ is entire. Thus, the roots of $(iz)^3 - (\lambda + b^2 - 1)(iz)^2 - 2\lambda(iz) + \lambda^2$ are also the roots of $(-\alpha e^{-iz} - \beta)(\lambda - iz)$ with the same multiplicity. So, the main work is to find the roots of

$$(-\alpha e^{-iz} + \beta)(iz - \lambda) = 0, \quad \text{for } z \in \mathbb{C}. \tag{4.20}$$

In fact, rewriting $\hat{\vartheta}$ as a function $iz \in \mathbb{C}$, we have

$$\hat{\vartheta}(iz) = \frac{(-\alpha e^{z} + \beta)(-z - \lambda)}{-z^3 - (\lambda + b^2 - 1)z^2 + 2\lambda z + \lambda^2}, \quad \text{for } z \in \mathbb{C}. \tag{4.21}$$

In (4.21), the roots of $(-\alpha e^z + \beta)(-z - \lambda)$ are $z = -\lambda$ and the zeros of $e^z = \frac{\beta}{\alpha}$ (as we have $\alpha \neq 0$). We also note that $-\lambda$ is not a root of the polynomial equation

$$-z^3 - (\lambda + b^2 - 1)z^2 + 2\lambda z + \lambda^2 = 0, \tag{4.22}$$

since $\lambda b \neq 0$.

Let r_1, r_2, r_3 be the roots of the equation (4.22). Then one must have

$$e^{r_1} = e^{r_2} = e^{r_3} = \frac{\beta}{\alpha},$$

which is not possible, due to Lemma 4.1.

Therefore, the only possibility is $\alpha = \beta = 0$, which gives (comparing (4.18) and (4.19)) that $\eta'(0) = \eta'(1) = 0$. But, we have the boundary condition $\eta(0) = \eta(1) = 0$ and by assumption $\eta''(1) - (b^2 - 1)\eta'(1) = \eta''(0) - (b^2 - 1)\eta'(0)$, i.e., $\eta''(1) = \eta''(0) = 0$. Consequently, $\eta = 0$ in $(0, 1)$ and thus $\xi = 0$ in $(0, 1)$.

So our assumption was false, and that the assertion of first part holds true.

(2) To prove the second statement, we assume on contrary that

$$\eta''(1) = (\lambda + b^2 - 1)\eta'(1). \tag{4.23}$$

Now, our claim is to show that $\eta = 0$ in $(0, 1)$. We note here that the Fourier transform technique used earlier will not work here due to the difficulty of the boundary condition $\eta''(1) = (\lambda + b^2 - 1)\eta'(1)$. However, we use a different complex analytic method, addressed for instance in [42], to conclude the proof.

Consider the following adjoint system of (4.6) as

$$\begin{cases} -\theta'''(x) - (\lambda + b^2 - 1)\theta''(x) + 2\lambda\theta'(x) + \lambda^2\theta(x) = 0, \\ \theta(0) = 0, \quad \theta'(0) = 0, \quad \theta'(1) \neq 0. \end{cases} \tag{4.24}$$

Multiplying the equation (4.6) by θ and then integrating by parts, we obtain

$$\eta''(1)\theta(1) - \eta'(1)\theta'(1) - (\lambda + b^2 - 1)\eta'(1)\theta(1) = 0.$$

Then, due to our assumption (4.23), we get

$$\eta'(1)\theta'(1) = 0. \tag{4.25}$$

Let us make the following claim.

Claim. There exists a countable set $\mathcal{N}$ such that for any $b \in (0,\infty) \setminus \mathcal{N}$ with $b^4 + 8b^2 + 5 < 4\pi^2$, the equation (4.24) has a non-trivial solution.

Proof of the Claim. Let m_1^*, m_2^*, m_3^* be roots of the following auxiliary equation

$$-m^3 - (\lambda + b^2 - 1)m^2 + 2\lambda m + \lambda^2 = 0. \tag{4.26}$$

Since b satisfies $b^4 + 8b^2 + 5 < 4\pi^2$, the roots of (4.26) does not satisfy $e^{m_1^*} = e^{m_2^*} = e^{m_3^*}$, thanks to Lemma 4.1. Note also that the map $b \mapsto m(b)$ is injective. In fact, $m(b_1) = m(b_2)$ implies $(b_1^2 - b_2^2)m(b_1) = 0$ and hence $b_1 = b_2$ (since $m(b_1) \neq 0$ for any $\lambda \neq 0$). We then write the solution θ of (4.24) as

$$\theta(x) = C_1 e^{m_1^* x} + C_2 e^{m_2^* x} + C_3 e^{m_3^* x}, \quad x \in (0,1). \tag{4.27}$$

Consider the following system of equations

$$\begin{aligned} C_1 + C_2 + C_3 &= 0 \\ C_1 m_1^* + C_2 m_2^* + C_3 m_3^* &= 0 \\ C_1 m_1^* e^{m_1^*} + C_2 m_2^* e^{m_2^*} + C_3 m_3^* e^{m_3^*} &= \theta'(1), \end{aligned}$$

which has a solution if and only if the matrix

$$\mathcal{R}_b := \begin{pmatrix} 1 & 1 & 1 \\ m_1^* & m_2^* & m_3^* \\ m_1^* e^{m_1^*} & m_2^* e^{m_2^*} & m_3^* e^{m_3^*} \end{pmatrix} \tag{4.28}$$

is invertible. The determinant of $\mathcal{R}_b$ is given by

$$\det(\mathcal{R}_b) = m_2^* m_3^* (e^{m_2^*} - e^{m_3^*}) + m_3^* m_1^* (e^{m_3^*} - e^{m_1^*}) + m_1^* m_2^* (e^{m_1^*} - e^{m_2^*}). \tag{4.29}$$

We now characterize all $b \in (0,\infty)$ such that $\det(\mathcal{R}_b) \neq 0$. Let us define three entire functions $F_i : \mathbb{C} \to \mathbb{C}$ $(i = 1,2,3)$ by

$$F_1(z) : = z\left[(m_2^* - m_3^*)e^z - m_2^* e^{m_2^*} + m_3^* e^{m_3^*}\right] + m_2^* m_3^* \left(e^{m_2^*} - e^{m_3^*}\right) \tag{4.30}$$

$$F_2(z) : = z\left[(m_3^* - m_1^*)e^z + m_1^* e^{m_1^*} - m_3^* e^{m_3^*}\right] + m_3^* m_1^* \left(e^{m_3^*} - e^{m_1^*}\right) \tag{4.31}$$

$$F_3(z) : = z\left[(m_1^* - m_2^*)e^z - m_1^* e^{m_1^*} + m_2^* e^{m_2^*}\right] + m_1^* m_2^* \left(e^{m_1^*} - e^{m_2^*}\right). \tag{4.32}$$

We first consider the function F_1. Note that if $F_1(0) = 0$, then $e^{m_2^*} = e^{m_3^*}$, which implies $F_1(z) = (m_2^* - m_3^*)z(e^z - e^{m_3^*})$ and hence $F_1(m_1^*) \neq 0$, else $e^{m_1^*} = e^{m_2^*} = e^{m_3^*}$ which is not possible due to Lemma 4.1. Therefore, the function F_1 does not vanish identically. This implies that the zero set of F_1, defined as

$$Z_{F_1} := \{z \in \mathbb{C} \ : \ F_1(z) = 0\} \tag{4.33}$$

is at most countable. In a similar manner, we can say that the zero sets of F_2 and F_3, defined as

$$Z_{F_2} : = \{z \in \mathbb{C} \ : \ F_2(z) = 0\}, \tag{4.34}$$

$$Z_{F_3} : = \{z \in \mathbb{C} \ : \ F_3(z) = 0\} \tag{4.35}$$

are at most countable. Since the map $b \mapsto m(b)$ is injective, the set

$$\mathcal{N}_j := \{b \in (0,\infty) \ : \ F_j(m_j(b)) = 0\} \tag{4.36}$$

for $j = 1, 2, 3$, is also at most countable. Let us then define the set

$$\mathcal{N} := \bigcup_{j=1}^{3} \mathcal{N}_j. \tag{4.37}$$

From the construction of the set $\mathcal{N}$, it is clear that for all $b \in (0,\infty) \setminus \mathcal{N}$ with $b^4 + 8b^2 + 5 < 4\pi^2$, $\det(\mathcal{R}_b)$ is non-zero. This proves our claim.

From the previous fact, we can see that for $b \in (0,\infty) \setminus \mathcal{N}$ with $b^4 + 8b^2 + 5 < 4\pi^2$, solution of the adjoint equation (4.24) verifies $\theta'(1) \neq 0$, which implies from (4.25) that $\eta'(1) = 0$. Hence $\eta \equiv 0$ on $(0, 1)$.

This completes the proof of the Lemma. □

4.2. **Lower bounds of the observation terms.** The next lemmas show that the observation terms satisfy some lower bounds which are not exponentially small. In fact, these lower bounds are crucial to conclude the null-controllability of the concerned systems (1.4) and (1.5).

Lemma 4.3 (Observation estimates: control on density)**.** *There exist constants $C_1, C_2 > 0$, independent in k, such that we have the following observation estimates for the parabolic and hyperbolic parts of the set of eigenfunctions of A^*, namely*

$$\frac{C_1}{k\pi} \leq |\mathcal{B}^*_\rho \Phi_{\lambda^p_k}| \leq \frac{C_2}{k\pi}, \quad \textit{for } k \geq k_0, \tag{4.38a}$$

$$C_1 \leq |\mathcal{B}^*_\rho \Phi_{\lambda^h_k}| \leq C_2, \quad \textit{for } |k| \geq k_0, \tag{4.38b}$$

where the number k_0 is introduced by Lemma 3.2.

Proof. Using the definition of $\mathcal{B}^*_\rho$ introduced by (4.2), we have

$$\mathcal{B}^*_\rho \Phi_{\lambda^p_k} = \xi_{\lambda^p_k}(1), \quad \forall k \geq k_0,$$
$$\mathcal{B}^*_\rho \Phi_{\lambda^h_k} = \xi_{\lambda^h_k}(1), \quad \forall |k| \geq k_0.$$

(i) Let us recall the expressions of $\xi_{\lambda^p_k}$ from (3.7), so that we have

$$\xi_{\lambda^p_k}(1) = \frac{ib}{k\pi} e^{-1} + e^{-k^2\pi^2 + O(1)} \times O\left(\frac{1}{k}\right) + O\left(\frac{1}{k^2}\right)$$

From the above expression, it is easy to observe that

$$k\pi \left|\xi_{\lambda^p_k}(1)\right| \to be^{-1} \quad \text{as } k \to +\infty,$$

and thus the result (4.38a) holds for large enough k.

(ii) On the other hand, from the expression of $\xi_{\lambda^h_k}$ given by (3.10), we have

$$\xi_{\lambda^h_k}(1) = \frac{2i}{b} \operatorname{sgn}(k) e^{-\frac{1}{2} - i \operatorname{sgn}(k)\sqrt{|k\pi|}} + O\left(|k|^{-1}\right),$$

and so,

$$\left|\xi_{\lambda^h_k}(1)\right| \to \frac{2}{b} e^{-\frac{1}{2}} \quad \text{as } k \to +\infty.$$

As a consequence, the estimate (4.38b) follows.

The proof is completed. □

Lemma 4.4 (Observation estimates: control in velocity)**.** *There exist some constants $C_1, C_2 > 0$, independent in k, such that we have the following observation estimates:*

$$C_1 k\pi \leq |\mathcal{B}^*_u \Phi_{\lambda^p_k}| \leq C_2 k\pi, \quad \textit{for large } k, \tag{4.39a}$$

$$\frac{C_1}{\sqrt{|k\pi|}} \leq |\mathcal{B}^*_u \Phi_{\lambda^h_k}| \leq \frac{C_2}{\sqrt{|k\pi|}}, \quad \textit{for large } k, \tag{4.39b}$$

Proof. Using the definition of $\mathcal{B}^*_u$ given by (4.3)–(4.4), we have

$$\mathcal{B}^*_u \Phi_{\lambda^p_k} = b\xi_{\lambda^p_k}(1) + \eta'_{\lambda^p_k}(1), \quad \forall k \geq k_0,$$
$$\mathcal{B}^*_u \Phi_{\lambda^h_k} = b\xi_{\lambda^h_k}(1) + \eta'_{\lambda^h_k}(1), \quad \forall |k| \geq k_0.$$

(i) Recall the expressions of $\xi_{\lambda_k^p}$ and $\eta_{\lambda_k^p}$, given by (3.7) and (3.8) respectively, so that we have

$$b\xi_{\lambda_k^p}(1) + \eta'_{\lambda_k^p}(1) = \frac{ib^2}{k\pi} e^{-1} + be^{-k^2\pi^2 + O(1)} \times O\left(\frac{1}{k}\right) + k\pi e^{-1} + O\left(\frac{1}{k}\right).$$

Observe that,

$$\frac{1}{k\pi}\left|b\xi_{\lambda_k^p}(1) + \eta'_{\lambda_k^p}(1)\right| \to e^{-1} \quad \text{as } k \to +\infty,$$

and hence the estimate (4.39a) holds.

(ii) For the set of eigenfunctions (3.10)–(3.11) associated to λ_k^h, the observation terms are

$$b\xi_{\lambda_k^h}(1) + \eta'_{\lambda_k^h}(1) = \operatorname{sgn}(k)\frac{\sqrt{|k\pi|} - \frac{1}{2} - i\operatorname{sgn}(k)\sqrt{|k\pi|}}{k\pi e^{\frac{1}{\sqrt{|k|}}}} + O\left(|k|^{-1}\right)$$

Here, one can show that

$$\sqrt{|k\pi|}\left|b\xi_{\lambda_k^h}(1) + \eta'_{\lambda_k^h}(1)\right| \to \sqrt{2} \quad \text{as } k \to +\infty,$$

which concludes the required observation estimate (4.39b).

The proof ends. □

5. A combined parabolic-hyperbolic Ingham-type inequality

This section is devoted to prove the Ingham-type inequality stated in Proposition 1.7 which will be intensively used to prove the controllability results of this paper. We will closely follow the decoupling idea given by [18, Theorem 4.2] and [58, Section 2.4].

Proof of Proposition 1.7. Recall the sequences $\{\lambda_k\}_{k\in\mathbb{N}^*}$ and $\{\gamma_k\}_{k\in\mathbb{Z}}$ and the hypothesis of Proposition 1.7. We denote $\tilde{\lambda}_k = \lambda_k - \beta$, $\forall k \in \mathbb{N}^*$ and $\tilde{\gamma}_k = \gamma_k - \beta$, $\forall k \in \mathbb{Z}$. Let $N \in \mathbb{N}^*$ be as given in the hypothesis. Then, we have the following known parabolic and hyperbolic Ingham inequalities

$$\int_0^T \left|\sum_{k\geq N} a_k e^{\tilde{\lambda}_k(T-t)}\right|^2 dt \geq C \sum_{k\geq N} |a_k|^2 e^{2\mathrm{Re}(\tilde{\lambda}_k)T} \quad \text{for any } T > 0, \tag{5.1}$$

$$C_1 \sum_{|k|\geq N} |b_k|^2 \leq \int_0^T \left|\sum_{|k|\geq N} b_k e^{\tilde{\gamma}_k(T-t)}\right|^2 dt \leq C_2 \sum_{|k|\geq N} |b_k|^2 \quad \text{for any } T > 1, \tag{5.2}$$

see for instance, [25, 29, 33, 35, 39, 44, 45, 48].

Let us denote

$$U^p(t) = \sum_{k\geq N} a_k e^{\tilde{\lambda}_k(T-t)}, \quad U^h(t) = \sum_{|k|\geq N} b_k e^{\tilde{\gamma}_k(T-t)}, \quad t \geq 0, \tag{5.3}$$

and

$$U(t) = U^p(t) + U^h(t), \quad t \geq 0. \tag{5.4}$$

Motivating from [58], we define for $t > 1$

$$\widetilde{U}^p(t) = U^p(t) - U^p(t-1) = \sum_{k\geq N} a_k \left(1 - e^{\tilde{\lambda}_k}\right) e^{\tilde{\lambda}_k(T-t)}, \tag{5.5a}$$

$$\widetilde{U}^h(t) = U^h(t) - U^h(t-1) = \sum_{|k|\geq N} b_k \left(1 - e^{\tilde{\gamma}_k}\right) e^{\tilde{\gamma}_k(T-t)}, \tag{5.5b}$$

and

$$\widetilde{U}(t) = \widetilde{U}^p(t) + \widetilde{U}^h(t) = U(t) - U(t-1). \tag{5.6}$$

Then, we have

$$\begin{aligned}\int_1^T \left|\widetilde{U}(t)\right|^2 dt &\leq \int_1^T |U(t)|^2 dt + \int_1^T |U(t-1)|^2 dt \\ &\leq C\int_0^T |U(t)|^2 dt.\end{aligned}$$

We now compute the L^2-norms of the functions $\widetilde{U}^p$ and $\widetilde{U}^h$ separately. Applying the hyperbolic Ingham inequality given by (5.2), we get

$$\int_1^T \left|\widetilde{U}^h(t)\right|^2 dt \leq C \sum_{|k|\geq N} |b_k|^2 \left|1 - e^{\tilde{\gamma}_k}\right|^2.$$

Since $1 - e^{\tilde{\gamma}_k} = 1 - e^{\nu_k}$ and $\{\nu_k\}_{|k|\geq N} \in \ell_2$, we can choose N large enough such that $\left|1 - e^{\tilde{\gamma}_k}\right|^2 < \varepsilon$ for all $|k| \geq N$. Thus, it follows that,

$$\int_1^T \left|\widetilde{U}^h(t)\right|^2 dt \leq C\varepsilon \sum_{|k|\geq N} |b_k|^2. \tag{5.7}$$

Now, recall (5.6) so that one has $\widetilde{U}^p(t) = \widetilde{U}(t) - \widetilde{U}^h(t)$. Using the triangle inequality, we get

$$\begin{aligned} \int_1^T \left|\widetilde{U}^p(t)\right|^2 dt &\leq C \int_1^T \left|\widetilde{U}(t)\right|^2 dt + C \int_1^T \left|\widetilde{U}^h(t)\right|^2 dt \\ &\leq C \int_0^T |U(t)|^2 dt + C\varepsilon \sum_{|k|\geq N} |b_k|^2. \end{aligned} \tag{5.8}$$

Let be $0 < \tau < T$. Applying the parabolic Ingham inequality (5.1) to the quantity $\widetilde{U}^p(t)$ (given by (5.5a)), we obtain

$$\begin{aligned} \int_{T-\tau}^T \left|\widetilde{U}^p(t)\right|^2 dt = \int_0^\tau \left|\widetilde{U}^p(T-t)\right|^2 dt &\geq C \sum_{k\geq N} |a_k|^2 |1 - e^{\tilde{\lambda}_k}|^2 e^{2\mathrm{Re}(\tilde{\lambda}_k)\tau} \\ &\geq C \sum_{k\geq N} |a_k|^2 e^{2\mathrm{Re}(\tilde{\lambda}_k)\tau}, \end{aligned}$$

thanks to the properties of $\tilde{\lambda}_k$. Note that the above constant C depends on τ. Let us now choose $\tau > 0$ small enough such that $T - \tau > 1$. Thus, we get

$$\int_1^T \left|\widetilde{U}^p(t)\right|^2 dt \geq \int_{T-\tau}^T \left|\widetilde{U}^p(t)\right|^2 dt \geq C \sum_{k\geq N} |a_k|^2 e^{2\mathrm{Re}(\tilde{\lambda}_k)\tau}. \tag{5.9}$$

Recall the function $U^p(t)$ given by (5.3), we deduce that

$$\begin{aligned} \int_0^{T-\tau} |U^p(t)|^2 dt &\leq \sum_{k\geq N} |a_k|^2 \int_0^{T-\tau} e^{2\mathrm{Re}(\tilde{\lambda}_k)(T-t)} dt \\ &\leq \sum_{k\geq N} |a_k|^2 \left| \frac{e^{\mathrm{Re}(\tilde{\lambda}_k)\tau} - e^{2\mathrm{Re}(\tilde{\lambda}_k)T}}{2\mathrm{Re}(\tilde{\lambda}_k)} \right| \\ &\leq C \sum_{k\geq N} |a_k|^2 e^{2\mathrm{Re}(\tilde{\lambda}_k)\tau}, \end{aligned} \tag{5.10}$$

thanks to fact that $\left|\mathrm{Re}(\tilde{\lambda}_k)\right|^2 \geq C$ for $k \geq N$ large enough (combining the hypothesis (ii) and (iv) in Proposition 1.7 satisfied by $\{\lambda_k\}_{k\in\mathbb{N}^*}$).

Now, using the facts (5.9) and (5.8) in (5.10), we have

$$\int_0^{T-\tau} |U^p(t)|^2 dt \leq C \left(\int_0^T |U(t)|^2 dt + \varepsilon \sum_{|k|\geq N} |b_k|^2 \right). \tag{5.11}$$

Since $T - \tau > 1$, applying the hyperbolic Ingham inequality (5.2) to $U^h(t)$ and then following a triangle inequality, we have

$$\begin{aligned} \sum_{|k|\geq N} |b_k|^2 \leq C \int_0^{T-\tau} \left|U^h(t)\right|^2 dt &\leq C \left(\int_0^{T-\tau} |U(t)|^2 dt + \int_0^{T-\tau} |U^p(t)|^2 dt \right) \\ &\leq C \left(\int_0^T |U(t)|^2 dt + \varepsilon \sum_{|k|\geq N} |b_k|^2 \right), \end{aligned}$$

thanks to the estimate (5.11).

Now, fix $\varepsilon > 0$ small enough such that $1 - C\varepsilon > 0$. As a consequence, there is some constant $C > 0$ depending only on T such that, we have

$$\sum_{|k|\geq N} |b_k|^2\, dt \leq C \int_0^T |U(t)|^2\, dt. \tag{5.12}$$

On the other hand, using the parabolic Ingham inequality to $U^p(t)$, followed by a triangle inequality, hyperbolic Ingham inequality to $U_h(t)$ and the result (5.12), we obtain

$$\begin{aligned}\sum_{k\geq N} |a_k|^2 e^{2\mathrm{Re}(\tilde{\lambda}_k)T} \leq C \int_0^T |U^p(t)|^2\, dt &\leq C \left(\int_0^T |U(t)|^2\, dt + \int_0^T \left|U^h(t)\right|^2 dt \right) \\ &\leq C \left(\int_0^T |U(t)|^2\, dt + \sum_{|k|\geq N} |b_k|^2\, dt \right) \\ &\leq C \int_0^T |U(t)|^2\, dt.\end{aligned}$$

Thus, eventually we have

$$\sum_{k\geq N} |a_k|^2 e^{2\mathrm{Re}(\tilde{\lambda}_k)T} + \sum_{|k|\geq N} |b_k|^2 \leq C \int_0^T |U(t)|^2\, dt. \tag{5.13}$$

Recall that $\tilde{\lambda}_k = \lambda_k - \beta$, $\tilde{\gamma}_k = \gamma_k - \beta$, and that

$$\begin{aligned}\int_0^T |U(t)|^2\, dt &= \int_0^T \left| \sum_{k\geq N} a_k e^{(\lambda_k-\beta)(T-t)} + \sum_{|k|\geq N} b_k e^{(\gamma_k-\beta)(T-t)} \right|^2 dt \\ &\leq C \int_0^T \left| \sum_{k\geq N} a_k e^{\lambda_k(T-t)} + \sum_{|k|\geq N} b_k e^{\gamma_k(T-t)} \right|^2 dt.\end{aligned} \tag{5.14}$$

Moreover, it is easy to see that

$$e^{2\mathrm{Re}(\tilde{\lambda}_k)T} = e^{2\mathrm{Re}(\lambda_k)T - 2\mathrm{Re}(\beta)T} \geq C e^{2\mathrm{Re}(\lambda_k)T}$$

for some $C > 0$ and thus combining (5.13) and (5.14), we obtain

$$\sum_{k\geq N} |a_k|^2 e^{2\mathrm{Re}(\lambda_k)T} + \sum_{|k|\geq N} |b_k|^2 \leq C \int_0^T \left| \sum_{k\geq N} a_k e^{\lambda_k(T-t)} + \sum_{|k|\geq N} b_k e^{\gamma_k(T-t)} \right|^2 dt.$$

Finally, adding the finitely many terms in the above summation using a similar idea as in [48, Theorem 4.3, Chapter 4] (since $\{\gamma_k\}_{k\in\mathbb{Z}}$ and $\{\lambda_k\}_{k\in\mathbb{N}^*}$ are disjoint), we can conclude that

$$\sum_{k\in\mathbb{N}^*} |a_k|^2 e^{2\mathrm{Re}(\lambda_k)T} + \sum_{k\in\mathbb{Z}} |b_k|^2 \leq C \int_0^T \left| \sum_{k\in\mathbb{N}^*} a_k e^{\lambda_k(T-t)} + \sum_{k\in\mathbb{Z}} b_k e^{\gamma_k(T-t)} \right|^2 dt. \tag{5.15}$$

This completes the proof. ☐

6. Null-controllability for the velocity case

In this section, we prove the null-controllability of the system (1.4) (that is, Theorem 1.2) by establishing a proper observability inequality. The parabolic-hyperbolic joint Ingham-type inequality as obtained in Section 5, is the main ingredient to conclude this result.

Let (ρ, u) be the solution to the system (1.4) with a boundary control q acting on the velocity part. The following lemma gives an equivalent criterion for the null-controllability of the concerned model (1.4).

Lemma 6.1. *The system (1.4) is null-controllable at time $T > 0$ in $\dot{H}^{\frac{1}{2}}_{\sharp}(0,1) \times L^2(0,1)$ if and only if there exists a control $q \in L^2(0,T)$ such that*

$$\left\langle \begin{pmatrix} \sigma(0) \\ v(0) \end{pmatrix}, \begin{pmatrix} \rho_0 \\ u_0 \end{pmatrix} \right\rangle_{(\dot{H}^{\frac{1}{2}}_{\sharp})' \times L^2, \dot{H}^{\frac{1}{2}}_{\sharp} \times L^2} = \int_0^T \left(\overline{b\sigma(t,1)} + \overline{v_x(t,1)} \right) q(t) dt, \tag{6.1}$$

where (σ, v) *is the solution to the adjoint system* (2.1) *with* $(f, g) = (0, 0)$ *and any given final data* $(\sigma_T, v_T) \in D(A^*)$.

With this result, we can now write the observability inequality that is required to prove null-controllability of the system (1.4). Recall the observation operator $\mathcal{B}_u^*$ defined by (4.3)–(4.4).

Theorem 6.2. *The system* (1.4) *is null-controllable at time* $T > 0$ *in the space* $\dot{H}^{\frac{1}{2}}_\sharp(0,1) \times L^2(0,1)$ *if and only if the following observability inequality*

$$\int_0^T \left|\mathcal{B}_u^*(\sigma(t), v(t))\right|^2 dt \geq C \left\|(\sigma(0), v(0))\right\|^2_{(\dot{H}^{\frac{1}{2}}_\sharp(0,1))' \times L^2(0,1)}, \tag{6.2}$$

where (σ, v) *is the solution to the adjoint system* (2.1) *with* $(f, g) = (0, 0)$ *and any given final data* $(\sigma_T, v_T) \in D(A^*)$.

Proof. We only give a proof of the null-controllability by assuming the observability inequality (6.2); for the other part we refer to the article [48]. To prove null-controllability of the system (1.4), it is enough to prove the existence of a minimizer of certain quadratic functional, see for instance [48, 57]. For this, we define the following set

$$\mathcal{H} := \left\{ (\sigma_T, v_T) \in (\dot{H}^{\frac{1}{2}}_\sharp(0,1))' \times L^2(0,1) \ : \ \exists\, q \in L^2(0,T) \text{ such that } \int_0^T \left|\mathcal{B}_u^*(\sigma(t), v(t))\right|^2 dt < \infty \right\}$$

and define a quadratic functional $J_u : \mathcal{H} \to \mathbb{R}$ by

$$J_u(\sigma_T, v_T) := \frac{1}{2}\int_0^T \left|\mathcal{B}_u^*(\sigma(t), v(t))\right|^2 dt + \left\langle \begin{pmatrix} \sigma(0) \\ v(0) \end{pmatrix}, \begin{pmatrix} \rho_0 \\ u_0 \end{pmatrix} \right\rangle_{(\dot{H}^{\frac{1}{2}}_\sharp)' \times L^2, \dot{H}^{\frac{1}{2}}_\sharp \times L^2}, \quad (\sigma_T, v_T) \in \mathcal{H}. \tag{6.3}$$

Here (σ, v) denotes the solution of the adjoint system (2.2) with this terminal data $(\sigma_T, v_T) \in \mathcal{H}$ and $(f, g) = (0, 0)$. We note here that the map J_u may not be coercive in H with respect to the usual $(\dot{H}^{\frac{1}{2}}_\sharp)' \times L^2$-norm. Thus, we define a new norm on $\mathcal{H}$ by

$$\left\|(\sigma_T, v_T)\right\|_{\mathcal{H}} := \left(\int_0^T \left|\mathcal{B}^*(\sigma(t), v(t))\right|^2 dt \right)^{\frac{1}{2}}.$$

Indeed, if $\left\|(\sigma_T, v_T)\right\|_{\mathcal{H}} = 0$ then $\mathcal{B}_u^*(\sigma(t), v(t)) = 0$ for all $t \in (0, T)$. The observability inequality (6.2) is then yields $(\sigma(0), v(0)) = (0, 0)$ and as a consequence of the backward uniqueness property of the adjoint system (2.2) with $(f, g) = (0, 0)$ (see Section 9), it follows that $(\sigma, v) \equiv (0, 0)$.

With this new norm on $\mathcal{H}$, the operator J_u is continuous and coercive in $\mathcal{H}$. Thus, it has a unique minimizer $(\hat{\sigma}_T, \hat{v}_T) \in \mathcal{H}$. Let $(\hat{\sigma}, \hat{v})$ be the solution of (2.2) with respect to this terminal data $(\hat{\sigma}_T, \hat{v}_T)$ and $(f, g) = (0, 0)$. Then the function $q = \mathcal{B}_u^*(\hat{\sigma}, \hat{v}) \in L^2(0, T)$ will be a null-control of the system (1.4). □

We are now ready to prove our first main result, i.e., Theorem 1.2 of our work.

Proof of Theorem 1.2. We prove each part separately.

Null-controllability in $\dot{H}^{\frac{1}{2}}_\sharp(0,1) \times L^2(0,1)$. Recall that the set of (generalized) eigenfunctions

$$\left\{\Phi_{\lambda_k^p},\, k \geq k_0\right\} \cup \left\{k^{\frac{1}{2}}\Phi_{\lambda_k^h},\, |k| \geq k_0\right\} \cup \left\{\Phi_\lambda^i;\, \lambda \in \Lambda_0,\, i = 0, ..., m_\lambda - 1\right\}$$

forms a Riesz basis in $(\dot{H}^{\frac{1}{2}}_\sharp(0,1))' \times L^2(0,1)$, due to Proposition 3.3 and Corollary 3.4, and thus one can consider any given final data $(\sigma_T, v_T) \in (\dot{H}^{\frac{1}{2}}_\sharp(0,1))' \times L^2(0,1)$ as follows:

$$(\sigma_T, v_T) = \sum_{k \geq k_0} a_k \Phi_{\lambda_k^p} + \sum_{|k| \geq k_0} b_k k^{\frac{1}{2}} \Phi_{\lambda_k^h} + \sum_{\lambda \in \Lambda_0} \sum_{j=0}^{m_\lambda - 1} c_{\lambda, j} \Phi_\lambda^j, \tag{6.4}$$

where $\sum_{k \geq k_0} |a_k|^2 + \sum_{|k| \geq k_0} |b_k|^2 < +\infty$, and $c_{\lambda,j}$ for $\lambda \in \Lambda_0$ and $j \in \{0, \cdots, m_\lambda - 1\}$ are constants.

Therefore, the solution to the adjoint system (2.1) with this terminal data (σ_T, v_T) and $(f,g)=(0,0)$ can be written as

$$(\sigma(t), v(t)) = \sum_{k\geq k_0} a_k e^{\lambda_k^p (T-t)}\Phi_{\lambda_k^p} + \sum_{|k|\geq k_0} b_k k^{\frac12} e^{\lambda_k^h (T-t)}\Phi_{\lambda_k^h} + \sum_{\lambda\in\Lambda_0}\sum_{j=0}^{m_\lambda -1} c_{\lambda,j}(T-t)^j e^{\lambda(T-t)}\Phi_\lambda^j, \tag{6.5}$$

for $t\in[0,T]$. Now, we find that

$$\begin{aligned}\mathcal{B}_u^*(\sigma(t), v(t)) &= b\sigma(t,1) + v_x(t,1)\\ &= \sum_{k\geq k_0} a_k\, e^{\lambda_k^p (T-t)}\mathcal{B}_u^*\Phi_{\lambda_k^p} + \sum_{|k|\geq k_0} b_k k^{\frac12}\, e^{\lambda_k^h (T-t)}\mathcal{B}_u^*\Phi_{\lambda_k^h} + \sum_{\lambda\in\Lambda_0}\sum_{j=0}^{m_\lambda -1} c_{\lambda,j}(T-t)^j e^{\lambda(T-t)}\mathcal{B}_u^*\Phi_\lambda^j,\end{aligned}$$

for $t\in(0,T)$. At this point, we may assume that

$$\mathcal{B}_u^*\Phi_\lambda^j \neq 0, \quad \forall \lambda\in\Lambda_0,\ j=1,\cdots, m_\lambda - 1,$$

which can be ensured as one can add any multiple of the eigenfunction to each (finitely many) generalized eigenfunction and adjust accordingly.

We start with $T>1$. Then, in one hand, using the Ingham-type inequality (1.13) for $|k|\geq k_0$ we have

$$\begin{aligned}&\int_0^T \left| \sum_{k\geq k_0} a_k\, e^{\lambda_k^p (T-t)}\mathcal{B}_u^*\Phi_{\lambda_k^p} + \sum_{|k|\geq k_0} b_k k^{\frac12}\, e^{\lambda_k^h (T-t)}\mathcal{B}_u^*\Phi_{\lambda_k^h}\right|^2 dt\\ &\geq C_1\Big(\sum_{k\geq k_0} \left| a_k \mathcal{B}_u^*\Phi_{\lambda_k^p}\right|^2 e^{2\mathrm{Re}(\lambda_k^p)T} + \sum_{|k|\geq k_0}\left| b_k k^{\frac12}\mathcal{B}_u^*\Phi_{\lambda_k^h}\right|^2\Big)\\ &\geq C_1\left(\sum_{k\geq k_0}|a_k|^2 k^2 e^{2\mathrm{Re}(\lambda_k^p)T} + \sum_{|k|\geq k_0}|b_k|^2\right),\end{aligned} \tag{6.6}$$

for some $C_1>0$, where we have also used the observation estimates given by Lemma 4.4.

On the other hand, thanks to the Riesz basis property (Corollary 3.4), we have

$$\left\| \sum_{k\geq k_0} a_k e^{\lambda_k^p T}\Phi_{\lambda_k^p} + \sum_{|k|\geq k_0} b_k k^{\frac12} e^{\lambda_k^h T}\Phi_{\lambda_k^h}\right\|^2_{(\dot H^{\frac12}_\sharp)'\times L^2} \leq C_2\Big(\sum_{k\geq k_0}|a_k|^2 e^{2\mathrm{Re}(\lambda_k^p)T} + \sum_{|k|\geq k_0}|b_k|^2 e^{2\mathrm{Re}(\lambda_k^h)T}\Big),$$

for some $C_2>0$. Thus, we deduce that

$$\begin{aligned}&\int_0^T \left| \sum_{k\geq k_0} a_k\, e^{\lambda_k^p (T-t)}\mathcal{B}_u^*\Phi_{\lambda_k^p} + \sum_{|k|\geq k_0} b_k k^{\frac12}\, e^{\lambda_k^h (T-t)}\mathcal{B}_u^*\Phi_{\lambda_k^h}\right|^2 dt\\ &\qquad\geq C\left\| \sum_{k\geq k_0} a_k e^{\lambda_k^p T}\Phi_{\lambda_k^p} + \sum_{|k|\geq k_0} b_k k^{\frac12} e^{\lambda_k^h T}\Phi_{\lambda_k^h}\right\|^2_{(\dot H^{\frac12}_\sharp)'\times L^2}\end{aligned} \tag{6.7}$$

for some $C>0$. But the solution (σ, v) also contains some finitely many terms as written in (6.5). Thus, to conclude the required observability inequality (6.2), we need to consider those finite number of terms in the inequality (6.7). Indeed, this can be done by using the strategy developed in [39] and [18, Section 4.2] since all the observation terms $\mathcal{B}_u^*\Phi\neq 0$ for any (generalized) eigenfunction Φ of A^* as long as we consider $b\notin\mathcal{N}$ with $b^4+8b^2+5<4\pi^2$ (see Proposition 4.2– Part 2). However, we give a detailed proof here for the sake of completeness.

Let $(\sigma_T, v_T)\in(\dot H^{\frac12}_\sharp(0,1))'\times L^2(0,1)$ be given. We write $(\sigma_T, v_T) = (\sigma_{T,1}, v_{T,1}) + (\sigma_{T,2}, v_{T,2})$ with

$$(\sigma_{T,1}, v_{T,1}) = \sum_{\lambda\in\Lambda_0}\sum_{j=0}^{m_\lambda-1} c_{\lambda,j}\Phi_\lambda^j, \text{ and } (\sigma_{T,2}, v_{T,2}) = \sum_{k\geq k_0} a_k\Phi_{\lambda_k^p} + \sum_{|k|\geq k_0} b_k k^{\frac12}\Phi_{\lambda_k^h}.$$

The corresponding solutions of the adjoint system (2.2) with these $(\sigma_{T,1}, v_{T,1})$, $(\sigma_{T,2}, v_{T,2})$ and $(f,g) = (0,0)$ are respectively

$$(\sigma_1(t), v_1(t)) = \sum_{\lambda\in\Lambda_0} \sum_{j=0}^{m_\lambda-1} c_{\lambda,j} e^{\lambda(T-t)} (T-t)^j \Phi_\lambda^j,$$
$$(\sigma_2(t), v_2(t)) = \sum_{k\geq k_0} a_k e^{\lambda_k^p(T-t)} \Phi_{\lambda_k^p} + \sum_{|k|\geq k_0} b_k k^{\frac12} e^{\lambda_k^h(T-t)} \Phi_{\lambda_k^h}.$$

From the previous computations (the case of high frequencies), we have the following inequality

$$\int_0^T \left|\mathcal{B}_u^*(\sigma_2(t), v_2(t))\right|^2 dt \geq C \left\|(\sigma_2(0), v_2(0))\right\|^2_{(\dot H_\sharp^{\frac12})'\times L^2}. \tag{6.8}$$

To prove the observability inequality (6.2), we have to include the observation term $\mathcal{B}_u^*(\sigma_1(t), v_1(t)) = \sum_{\lambda\in\Lambda_0}\sum_{j=0}^{m_\lambda-1} c_{\lambda,j} e^{\lambda(T-t)}(T-t)^j \mathcal{B}_u^*\Phi_\lambda^j$ in the above inequality. We give a detailed proof below by adding only one term, say for instance $e^{\lambda_{j_0}(T-t)}\left(c_{j_0}\mathcal{B}_u^*\Phi_{j_0} + (T-t)\tilde c_{j_0}\mathcal{B}_u^*\tilde\Phi_{j_0}\right)$ corresponding to the eigenvalue $\lambda = \lambda_{j_0} \in \Lambda_0$, where Φ_{j_0} and $\tilde\Phi_{j_0}$ denote the (generalized) eigenfunctions corresponding to λ_{j_0}. All the remaining finitely many terms can be added one by one using the same argument. We denote

$$\mathcal{F}(t) := \mathcal{B}_u^*(\sigma_2(t), v_2(t)) + e^{\lambda_{j_0}(T-t)}\left(c_{j_0}\mathcal{B}_u^*\Phi_{j_0} + (T-t)\tilde c_{j_0}\mathcal{B}_u^*\tilde\Phi_{j_0}\right), \quad \text{for } t\in(0,T), \tag{6.9}$$

and define

$$\mathcal{G}(t) := \mathcal{F}(t) - \frac{1}{2\delta}\int_{-\delta}^{\delta} e^{\lambda_{j_0}s}\mathcal{F}(t+s)ds, \quad t\in(\delta, T-\delta),$$

where we will choose $\delta > 0$ later accordingly. Then, one can obtain the following estimate (see for instance [39, Section 4.4]):

$$\int_\delta^{T-\delta} |\mathcal{G}(t)|^2\,dt \leq C\int_0^T |\mathcal{F}(t)|^2\,dt \tag{6.10}$$

for some constant $C > 0$.

On the other hand, we have

$$\mathcal{G}(t) = \sum_{k\geq k_0} a_k e^{\lambda_k^p(T-t)}\mathcal{B}_u^*\Phi_{\lambda_k^p}\left(1 - \frac{\sinh((\lambda_k^p - \lambda_{j_0})\delta)}{(\lambda_k^p-\lambda_{j_0})\delta}\right)$$
$$+ \sum_{|k|\geq k_0} b_k k^{\frac12} e^{\lambda_k^h(T-t)}\mathcal{B}_u^*\Phi_{\lambda_k^h}\left(1 - \frac{\sinh((\lambda_k^h - \lambda_{j_0})\delta)}{(\lambda_k^h-\lambda_{j_0})\delta}\right)$$

for $t\in(\delta, T-\delta)$. Since $T > 1$, choosing $\delta > 0$ small enough so that $T - 2\delta > 1$, we obtain by using the Ingham-type inequality (1.13)

$$\int_\delta^{T-\delta} |\mathcal{G}(t)|^2\,dt \geq C\left(\sum_{k\geq k_0}\left|a_k\mathcal{B}_u^*\Phi_{\lambda_k^p}\right|^2 e^{2\mathrm{Re}(\lambda_k^p)T} + \sum_{|k|\geq k_0}\left|b_k k^{\frac12}\mathcal{B}_u^*\Phi_{\lambda_k^h}\right|^2\right).$$

This can be ensured from the fact that $\inf_{k\geq k_0}|\lambda_k^p - \lambda_{j_0}|, \inf_{k\geq k_0}\left|\lambda_k^h - \lambda_{j_0}\right| > 0$, which then gives (by taking $\delta > 0$ suitably) that

$$\inf_{k\geq k_0}\left|1 - \frac{\sinh((\lambda_k^p-\lambda_{j_0})\delta)}{(\lambda_k^p-\lambda_{j_0})\delta}\right|, \ \inf_{k\geq k_0}\left|1 - \frac{\sinh((\lambda_k^h-\lambda_{j_0})\delta)}{(\lambda_k^h-\lambda_{j_0})\delta}\right| > 0.$$

Using this inequality, we readily have (see eq. (6.6)-(6.7))

$$\int_\delta^{T-\delta}|\mathcal{G}(t)|^2\,dt \geq C\left\|(\sigma_2(0), v_2(0))\right\|^2_{(\dot H_\sharp^{\frac12})'\times L^2}.$$

Combining this with the estimate (6.10), we deduce that

$$\int_0^T |\mathcal{F}(t)|^2\,dt \geq C\left\|(\sigma_2(0), v_2(0))\right\|^2_{(\dot H_\sharp^{\frac12})'\times L^2}. \tag{6.11}$$

Since $T > 1$, we can choose $\varepsilon > 0$ small enough so that $T - \varepsilon > 1$. Then, we obtain from the above inequality

$$\int_0^T |\mathcal{F}(t)|^2 \, dt \geq \int_\varepsilon^T |\mathcal{F}(t)|^2 \, dt \geq C \left\| (\sigma_2(\varepsilon), v_2(\varepsilon)) \right\|^2_{(\dot{H}^{\frac{1}{2}}_\sharp)' \times L^2}. \tag{6.12}$$

We now prove a weak admissibility inequality

$$\int_0^{\frac{\varepsilon}{2}} |\mathcal{B}^*_u(\sigma_2(t), v_2(t))|^2 \, dt \leq C \left\| (\sigma_2(\varepsilon), v_2(\varepsilon)) \right\|^2_{(\dot{H}^{\frac{1}{2}}_\sharp)' \times L^2}. \tag{6.13}$$

In fact, applying Hölder's inequality and the hyperbolic Ingham inequality (5.2) (right side), we deduce that

$$\begin{aligned}
\int_0^{\frac{\varepsilon}{2}} |\mathcal{B}^*_u(\sigma_2(t), v_2(t))|^2 \, dt &\leq 2 \int_0^{\frac{\varepsilon}{2}} \left| \sum_{k \geq k_0} a_k e^{\lambda^p_k (T-t)} \mathcal{B}^*_u \Phi_{\lambda^p_k} \right|^2 dt + 2 \int_0^{\frac{\varepsilon}{2}} \left| \sum_{|k| \geq k_0} b_k k^{\frac{1}{2}} e^{\lambda^h_k (T-t)} \mathcal{B}^*_u \Phi_{\lambda^h_k} \right|^2 dt \\
&\leq C \sum_{k \geq k_0} |a_k|^2 e^{2\mathrm{Re}(\lambda^p_k)(T-\varepsilon)} \sum_{k \geq k_0} \left| \mathcal{B}^*_u \Phi_{\lambda^p_k} \right|^2 e^{-2\mathrm{Re}(\lambda^p_k)(T-\varepsilon)} \int_0^{\frac{\varepsilon}{2}} e^{2\mathrm{Re}(\lambda^p_k)(T-t)} dt \\
&\quad + C \sum_{|k| \geq k_0} \left| b_k k^{\frac{1}{2}} \mathcal{B}^*_u \Phi_{\lambda^h_k} \right|^2 \\
&\leq C \sum_{k \geq k_0} |a_k|^2 e^{2\mathrm{Re}(\lambda^p_k)(T-\varepsilon)} + C \sum_{|k| \geq k_0} |b_k|^2,
\end{aligned}$$

thanks to the observation estimate (4.39b). On the other hand, using the Riesz basis property of the eigenfunctions (see Corollary 3.4), we obtain

$$\left\| (\sigma_2(\varepsilon), v_2(\varepsilon)) \right\|^2_{(\dot{H}^{\frac{1}{2}}_\sharp)' \times L^2} \geq C \sum_{k \geq k_0} |a_k|^2 e^{2\mathrm{Re}(\lambda^p_k)(T-\varepsilon)} + C \sum_{|k| \geq k_0} |b_k|^2.$$

Combining the above estimates, the weak admissibility inequality (6.13) follows. With this, we get from (6.12) that

$$\int_0^T |\mathcal{F}(t)|^2 \, dt \geq C \int_0^{\frac{\varepsilon}{2}} |\mathcal{B}^*_u(\sigma_2(t), v_2(t))|^2 \, dt. \tag{6.14}$$

We now introduce the finite dimensional space generated by the (generalized) eigenfunctions

$$\mathcal{X} := \mathrm{span} \left\{ \Phi_{j_0}, \tilde{\Phi}_{j_0} \right\}$$

and define the norms on $\mathcal{X}$ as

$$\left\| (\hat{\sigma}_{T,1}, \hat{v}_{T,1}) \right\|_1^2 := \int_0^{\frac{\varepsilon}{2}} \left| e^{\lambda_{j_0}(T-t)} \left(c_{j_0} \mathcal{B}^*_u \Phi_{j_0} + (T-t) \tilde{c}_{j_0} \mathcal{B}^*_u \tilde{\Phi}_{j_0} \right) \right|^2 dt, \tag{6.15}$$

$$\left\| (\hat{\sigma}_{T,1}, \hat{v}_{T,1}) \right\|_2 := \left\| (\hat{\sigma}_1(0), \hat{v}_1(0)) \right\|_{(\dot{H}^{\frac{1}{2}}_\sharp)' \times L^2}, \tag{6.16}$$

where $(\hat{\sigma}_1(t), \hat{v}_1(t)) = e^{\lambda_{j_0}(T-t)} \left(c_{j_0} \Phi_{j_0} + \tilde{c}_{j_0} \tilde{\Phi}_{j_0} \right)$ for $t \in (0, T)$ is the solution of the adjoint system (2.2) with the terminal data $(\hat{\sigma}_{T,1}, \hat{v}_{T,1}) \in \mathcal{X}$ and $(f, g) = (0, 0)$. In fact, the norms (6.15) and (6.16) are well-defined since we have $\mathcal{B}^* \Phi_{j_0}, \mathcal{B}^* \tilde{\Phi}_{j_0} \neq 0$ and $(\hat{\sigma}_1(0), \hat{v}_1(0)) = (0, 0)$ implies $\Phi_{j_0} = \tilde{\Phi}_{j_0} = 0$. Moreover, as any two norms in a finite dimensional space are equivalent, we deduce that

$$\int_0^{\frac{\varepsilon}{2}} \left| e^{\lambda_{j_0}(T-t)} \left(c_{j_0} \mathcal{B}^*_u \Phi_{j_0} + (T-t) \tilde{c}_{j_0} \mathcal{B}^*_u \tilde{\Phi}_{j_0} \right) \right|^2 dt \geq C \left\| (\hat{\sigma}_1(0), \hat{v}_1(0)) \right\|^2_{(\dot{H}^{\frac{1}{2}}_\sharp)' \times L^2}.$$

As a consequence, we obtain (recall the function $\mathcal{F}$ defined by (6.9))

$$\left\| (\hat{\sigma}_1(0), \hat{v}_1(0)) \right\|^2_{(\dot{H}^{\frac{1}{2}}_\sharp)' \times L^2} \leq C \int_0^{\frac{\varepsilon}{2}} |\mathcal{F}(t)|^2 \, dt + C \int_0^{\frac{\varepsilon}{2}} |\mathcal{B}^*_u(\sigma_2(t), v_2(t))|^2 \, dt \leq C \int_0^T |\mathcal{F}(t)|^2 \, dt,$$

thanks to the lower bound (6.14). This inequality together with (6.11), we deduce that

$$\int_0^T |\mathcal{F}(t)|^2 \, dt \geq C \left\| (\sigma(0) + \hat{\sigma}_1(0), v(0) + \hat{v}_1(0)) \right\|^2_{(\dot{H}^{\frac{1}{2}}_\sharp)' \times L^2}. \tag{6.17}$$

In a similar way, we can add the remaining finitely many terms in the above inequality. As a result, we eventually get for $T > 1$,

$$\int_0^T |\mathcal{B}_u^*(\sigma(t), v(t))|^2 dt \geq C\|(\sigma(0), v(0))\|^2_{(\dot{H}_\sharp^{\frac{1}{2}})' \times L^2}, \tag{6.18}$$

for given data $(\sigma_T, v_T) \in D(A^*)$.

This is a necessary and sufficient for the null-controllability of system (1.4) with given initial data $(\rho_0, u_0) \in \dot{H}_\sharp^{\frac{1}{2}}(0,1) \times L^2(0,1)$, when $T > 1$, which proves the first part of Theorem 1.2.

Lack of null-controllability for less regular initial states. Consider $(\sigma_{T,k}, v_{T,k}) = \Phi_{\lambda_k^h}$ for $|k| \geq k_0$. Then, the solution to the adjoint system (2.2) with this $(\sigma_{T,k}, v_{T,k})$ and $(f,g) = (0,0)$ reads as

$$(\sigma_k(t,x), v_k(t,x)) = e^{\lambda_k^h (T-t)} \Phi_{\lambda_k^h}(x), \quad \forall |k| \geq k_0, \quad (t,x) \in (0,T) \times (0,1).$$

Now, in one hand we have

$$\left\| \Phi_{\lambda_k^h} \right\|_{(\dot{H}_\sharp^s)' \times L^2} \geq \frac{C}{|k|^s}, \quad \forall |k| \geq k_0,$$

by Lemma 3.5–eq. (3.15), and thus

$$\|(\sigma_k(0), v_k(0))\|^2_{(\dot{H}_\sharp^s)' \times L^2} \geq \frac{C}{|k|^{2s}}, \quad \forall |k| \geq k_0.$$

for all $k \in \mathbb{Z}^*$, since $\mathrm{Re}(\lambda_k^h)$ is bounded. On the other hand, we have the following upper bounds of the observation terms, namely

$$\int_0^T |\mathcal{B}_u^*(\sigma_k(t), v_k(t))|^2 \, dt \leq \frac{C}{|k|}, \quad \forall |k| \geq k_0,$$

in view of Lemma 4.4–eq. (4.39b). Thus, if the observability inequality (6.18) holds, we would have

$$\frac{C}{|k|^{2s}} \leq \frac{C}{|k|} \implies |k|^{1-2s} \leq C,$$

which is not possible since $0 \leq s < \frac{1}{2}$. Therefore, the system (1.4) is not null-controllable at any time T whenever $0 < s < \frac{1}{2}$.

This concludes the proof of Theorem 1.2. □

7. Null-controllability for the density case

This section is devoted to prove the null-controllability of the system (1.5), more precisely Theorem 1.3. The proof is made of two steps:

- First, we use the Ingham-type inequality (1.13) (introduced as before) to show the null-controllability of (1.5) in the space $\dot{L}^2(0,1) \times H_0^1(0,1)$.
- Secondly, by developing the moments method for parabolic-hyperbolic coupled system (due to Hansen [34]), we prove that the same system (1.5) is null-controllable in the space $\dot{H}_\sharp^s(0,1) \times L^2(0,1)$ for any $s > \frac{1}{2}$.

As a consequence, we conclude the null-controllability of our system (1.6) in the space $\dot{L}^2(0,1) \times L^2(0,1)$.

The following lemma gives an equivalent criterion for the null-controllability of system (1.5).

Lemma 7.1. *Let $s_1, s_2 \geq 0$ be given. The system (1.5) is null-controllable at time $T > 0$ in $\dot{H}_\sharp^{s_1}(0,1) \times H_0^{s_2}(0,1)$ if and only if there exists a control $p \in L^2(0,T)$ such that*

$$\left\langle \begin{pmatrix} \sigma(0) \\ v(0) \end{pmatrix}, \begin{pmatrix} \rho_0 \\ u_0 \end{pmatrix} \right\rangle_{(\dot{H}_\sharp^{s_1})' \times H^{-s_2}, \dot{H}_\sharp^{s_1} \times H_0^{s_2}} = -\int_0^T \overline{\sigma(t,1)} p(t) dt, \tag{7.1}$$

where (σ, v) is the solution to the adjoint system (2.1) with $(f,g) = (0,0)$ and any given final data $(\sigma_T, v_T) \in D(A^*)$.

7.1. **Null-controllability in $\dot{L}^2 \times H_0^1$: using Ingham-type inequality.** We first write the following result, the proof of which is similar to the velocity case (Theorem 6.2) and so we omit the details here.

Theorem 7.2. *The system (1.5) is null-controllable at time $T > 0$ in the space $\dot{L}^2(0,1) \times H_0^1(0,1)$ if and only if the following observability inequality*

$$\int_0^T \left|\mathcal{B}_\rho^*(\sigma(t), v(t))\right|^2 dt \geq C \,\|(\sigma(0), v(0))\|^2_{\dot{L}^2 \times H^{-1}} \tag{7.2}$$

hold for every $(\sigma_T, v_T) \in D(A^*)$.

Let $(\sigma_T, v_T) \in \dot{L}^2(0,1) \times H^{-1}(0,1)$ be given. Since the set of (generalized) eigenfunctions

$$\left\{k \ \Phi_{\lambda_k^p},\, k \geq k_0\right\} \cup \left\{\Phi_{\lambda_k^h},\, |k| \geq k_0\right\} \cup \left\{\Phi_\lambda^i;\, \lambda \in \Lambda_0,\, i = 0, ..., m_\lambda - 1\right\}$$

forms a Riesz basis of $\dot{L}^2(0,1) \times H^{-1}(0,1)$, thanks to Corollary 3.4, we can write (σ_T, v_T) as

$$(\sigma_T, v_T) = \sum_{k \geq k_0} a_k k \ \Phi_{\lambda_k^p} + \sum_{|k| \geq k_0} b_k \Phi_{\lambda_k^h} + \sum_{\lambda \in \Lambda_0} \sum_{j=0}^{m_\lambda - 1} c_{\lambda,j} \Phi_\lambda^j.$$

Therefore, the solution to the adjoint system (2.1) with this terminal data (σ_T, v_T) and $(f, g) = (0, 0)$ can be written as

$$(\sigma(t), v(t)) = \sum_{k \geq k_0} a_k k \ e^{\lambda_k^p (T-t)} \Phi_{\lambda_k^p} + \sum_{|k| \geq k_0} b_k e^{\lambda_k^h (T-t)} \Phi_{\lambda_k^h} + \sum_{\lambda \in \Lambda_0} \sum_{j=0}^{m_\lambda - 1} c_{\lambda,j} (T-t)^j e^{\lambda(T-t)} \Phi_\lambda^j,$$

for $t \in [0, T]$. Note that

$$\mathcal{B}_\rho^*(\sigma(t), v(t)) = \sum_{k \geq k_0} a_k k \ e^{\lambda_k^p (T-t)} \mathcal{B}_\rho^* \Phi_{\lambda_k^p} + \sum_{|k| \geq k_0} b_k e^{\lambda_k^h (T-t)} \mathcal{B}_\rho^* \Phi_{\lambda_k^h} + \sum_{\lambda \in \Lambda_0} \sum_{j=0}^{m_\lambda - 1} c_{\lambda,j} (T-t)^j e^{\lambda(T-t)} \mathcal{B}_\rho^* \Phi_\lambda^j,$$

for all $t \in (0, T)$. Since $T > 1$, we use the Ingham-type inequality (1.13) to obtain

$$\begin{aligned}
&\int_0^T \left| \sum_{k \geq k_0} a_k k \, e^{\lambda_k^p (T-t)} \mathcal{B}_\rho^* \Phi_{\lambda_k^p} + \sum_{|k| \geq k_0} b_k \, e^{\lambda_k^h (T-t)} \mathcal{B}_\rho^* \Phi_{\lambda_k^h} \right|^2 dt \\
&\geq C_1 \Bigg(\sum_{k \geq k_0} \left| a_k k \mathcal{B}_\rho^* \Phi_{\lambda_k^p} \right|^2 e^{2\mathrm{Re}(\lambda_k^p) T} + \sum_{|k| \geq k_0} \left| b_k \mathcal{B}_\rho^* \Phi_{\lambda_k^h} \right|^2 \Bigg) \\
&\geq C_1 \left(\sum_{k \geq k_0} |a_k|^2 e^{2\mathrm{Re}(\lambda_k^p) T} + \sum_{|k| \geq k_0} |b_k|^2 \right),
\end{aligned} \tag{7.3}$$

for some $C_1 > 0$, where we also have used the observation estimates from Lemma 4.3.

On the other hand, we have

$$\begin{aligned}
&\left\| \sum_{k \geq k_0} a_k k \ e^{\lambda_k^p T} \Phi_{\lambda_k^p} + \sum_{|k| \geq k_0} b_k e^{\lambda_k^h T} \Phi_{\lambda_k^h} \right\|^2_{\dot{L}^2 \times H^{-1}} \\
&\leq C_2 \Bigg(\sum_{k \geq k_0} |a_k|^2 e^{2\mathrm{Re}(\lambda_k^p) T} + \sum_{|k| \geq k_0} |b_k|^2 e^{2\mathrm{Re}(\lambda_k^h) T} \Bigg),
\end{aligned}$$

for some $C_2 > 0$, thanks to the Riesz basis property (Corollary 3.4).

Thus we deduce that

$$\begin{aligned}
&\int_0^T \left| \sum_{k \geq k_0} a_k k \, e^{\lambda_k^p (T-t)} \mathcal{B}_\rho^* \Phi_{\lambda_k^p} + \sum_{|k| \geq k_0} b_k \, e^{\lambda_k^h (T-t)} \mathcal{B}_\rho^* \Phi_{\lambda_k^h} \right|^2 dt \\
&\geq C \left\| \sum_{k \geq k_0} a_k k \ e^{\lambda_k^p T} \Phi_{\lambda_k^p} + \sum_{|k| \geq k_0} b_k e^{\lambda_k^h T} \Phi_{\lambda_k^h} \right\|^2_{\dot{L}^2 \times H^{-1}},
\end{aligned} \tag{7.4}$$

for some $C > 0$.

On the other hand, since $b^4 + 8b^2 + 5 < 4\pi^2$, all the observation terms $\mathcal{B}_\rho^* \Phi \neq 0$ for any (generalized) eigenfunction Φ of A^* and hence it is enough to consider only the large frequencies of eigenvalues. In fact, the lower frequencies can be added one by one by proceeding in a similar way as in the proof of Theorem 1.2 to deduce the required observability inequality

$$\int_0^T |\mathcal{B}_\rho^*(\sigma(t), v(t))|^2 dt \geq C \left\|(\sigma(0), v(0))\right\|^2_{\dot{L}^2\times H^{-1}},$$

for given data $(\sigma_T, v_T) \in D(A^*)$ provided $T > 1$.

This proves the null-controllability of the system (1.5) at time $T > 1$ for given initial data $(\rho_0, u_0) \in \dot{L}^2(0,1) \times H_0^1(0,1)$.

7.2. **Null-controllability in $\dot{H}^s_\sharp \times L^2$, $s > \frac{1}{2}$ by moments method.** To prove the null-controllability of system (1.5) at $T > 1$ in the space $\dot{H}^s_\sharp(0,1) \times L^2(0,1)$ for $s > \frac{1}{2}$, we shall formulate and solve a set of moments problem using the strategy developed in [34]. For the sake of completeness, we recall the main results from [34] and use these results with respect to our setting.

7.2.1. *Parabolic-hyperbolic joint moments problem: results by S. W. Hansen.* Let us first recall some important results by S. W. Hansen [34] which will be used to prove the required null-controllability result of the system (1.5) in the space $\dot{H}^s_\sharp(0,1) \times L^2(0,1)$ for $s > \frac{1}{2}$.

The author in [34] made the following assumptions in his work.

Hypothesis 7.3. *Let $\{\lambda_k\}_{k\in\mathbb{N}^*}$ and $\{\gamma_k\}_{k\in\mathbb{Z}}$ be two sequences in $\mathbb{C}$ with the following properties:*

(H1) *for all $k, j \in \mathbb{Z}$, $\gamma_k \neq \gamma_j$ unless $j = k$,*
(H2) *$\gamma_k = \beta + ck\pi i + \nu_k$ for all $k \in \mathbb{Z}$,*

where $\beta \in \mathbb{C}$, $c > 0$ and $\{\nu_k\}_{k\in\mathbb{Z}} \in \ell_2$.

Also, there exist positive constants $A_0, B_0, \delta, \varepsilon$ and $0 \leq \theta < \pi/2$ for which $\{\lambda_k\}_{k\in\mathbb{N}^*}$ satisfies

(P1) *$|\arg(-\lambda_k)| \leq \theta$ for all $k \in \mathbb{N}^*$,*
(P2) *$|\lambda_k - \lambda_j| \geq \delta \left|k^2 - j^2\right|$ for all $k \neq j$, $k, j \in \mathbb{N}^*$,*
(P3) *$\varepsilon(A_0 + B_0k^2) \leq |\lambda_k| \leq A_0 + B_0k^2$ for all $k \in \mathbb{N}^*$.*

We also assume that the families are disjoint, i.e.,

$$\{\gamma_k,\ k \in \mathbb{Z}\} \cap \{\lambda_k,\ k \in \mathbb{N}^*\} = \emptyset.$$

Then, he introduced the following spaces: for any $0 \leq a < d$,

$$W_{[a,d]} = \text{closed span } \{e^{\gamma_k t}\}_{k\in\mathbb{Z}} \text{ in } L^2(a,d),$$
$$E_{[a,d]} = \text{closed span } \{e^{-\lambda_k t}\}_{k\in\mathbb{N}^*} \text{ in } L^2(a,d).$$

With these, the author in [34] has proved the following results.

Theorem 7.4. *Assume that the Hypothesis 7.3 holds true. Then, for each $T > 2/c$, where c is defined as in Hypothesis 7.3, the spaces $W_{[0,T]}$ and $E_{[0,T]}$ are uniformly separated. This does not hold for $T \leq 2/c$.*

The proof mainly relies upon the following lemma. Hereinafter, we denote $t_c = 2/c$.

Lemma 7.5. *For any $a \in \mathbb{R}$, $W_{[a,a+t_c]} = L^2(a, a + t_c)$. Furthermore, for $T \geq t_c$, $\{e^{\gamma_k t}\}_{k\in\mathbb{Z}}$ forms a Riesz basis for each of the spaces $W_{[a,a+T]}$.*

We refer [34] for the proofs of Theorem 7.4 and Lemma 7.5.

Let us write the following set of moments problem,

$$p_k = \int_0^T e^{\lambda_k t} f(t)dt, \qquad k \in \mathbb{N}^*, \tag{7.5}$$

$$h_k = \int_0^T e^{\gamma_k t} f(t)dt, \qquad k \in \mathbb{Z}. \tag{7.6}$$

The space of all sequences $\{p_k\}_{k\in\mathbb{N}^*} \cup \{h_k\}_{k\in\mathbb{Z}}$ for which there exists a $f \in L^2(0,T)$ that solves the set of equations (7.5)–(7.6) is called the moment space.

Now, we recall the following results from the same paper which relate Theorem 7.4 to the moments problem (7.5)–(7.6).

Proposition 7.6. *Let $\{h_k\}_{k\in\mathbb{Z}} \in \ell_2$. Then, for any $T \geq t_c$, there exists $f \in W_{[0,T]}$, which solves the moment problem (7.6). Moreover, any $\widetilde{f} \in L^2(0,T)$ given by $\widetilde{f} = f + \widehat{f}$ with $\widehat{f} \in W^\perp_{[0,T]}$ also solves (7.6).*

The proof follows as a consequence of Lemma 7.5.

Proposition 7.7. *Assume that for any $r > 0$, the sequence $\{p_k\}_{k\in\mathbb{N}^*}$ satisfies*

$$|p_k|e^{rk} \to 0 \quad \text{as } k \to +\infty. \tag{7.7}$$

Then, for given any $\tau > 0$, there exists $g \in E_{[0,\tau]}$, which solves the moment problem (7.5). Moreover, any $\widetilde{g} \in L^2(0,\tau)$ given by $\widetilde{g} = g + \widehat{g}$ with $\widehat{g} \in E^{\perp}_{[0,\tau]}$ also solves (7.5).

The proof of the above proposition is standard. It relies on the existence of bi-orthogonal family in the space $E_{[0,\tau]}$ to the family of exponentials $\{e^{\lambda_k t}\}_{k\in\mathbb{N}^*}$; see [33] for a proof.

Let us now present the main theorem that tells the solvability of the mixed moment problems (7.5)–(7.6).

Theorem 7.8. *Let any $T > t_c$ be given. Then, under Hypothesis 7.3, given any sequence $\{p_k\}_{k\in\mathbb{N}^*}$ satisfying (7.7) and any $\{h_k\}_{k\in\mathbb{Z}} \in \ell_2$, there exists a function $f \in L^2(0,T)$ that simultaneously solves the set of moments problem (7.5)–(7.6). This does not hold for $T \leq t_c$.*

The proof of above theorem can be found in [34, Theorem 4.11]. For the sake of completeness, we give the proof below.

Proof. For $T \leq t_c$, the set of moments problem (7.5)–(7.6) does not necessarily have a solution. Thus, we start with $T > t_c$. By Theorem 7.4, the spaces $E := E_{[0,T]}$ and $W := W_{[0,T]}$ are uniformly separated. Thus the space $V := E + W$ is closed in $L^2(0,T)$ with its norm $\|\cdot\|_V := \|\cdot\|_{L^2(0,T)}$ and so $V := E \oplus W$. Moreover, the orthogonal complements $E^\perp$ and $W^\perp$ of E and W (resp.) in V are also uniformly separated using a result by T. Kato [37, Chap. 4, §4] and therefore, $V = E^\perp \oplus W^\perp$. From this, one can show that the restrictions $P_E|_{W^\perp}$ and $P_W|_{E^\perp}$ are isomorphisms, where P_E and P_W are the orthogonal projections respectively onto E and W in V. By Propositions 7.7 and 7.6, there exist functions $f_1 \in E$ and $f_2 \in W$ which solve the equations (7.5) and (7.6) respectively. Set,

$$f = (P_E|_{W^\perp})^{-1} f_1 + (P_W|_{E^\perp})^{-1} f_2,$$

which simultaneously solves the equations (7.5)–(7.6) and moreover $f \in L^2(0,T)$. □

7.2.2. *Formulation of the parabolic-hyperbolic moments problem.* Let us recall the set of eigenvalues $\sigma(A^*)$, given by (3.5).

The sequence $\{\lambda^h_k\}_{|k|\geq k_0}$ satisfies (H1) and (H2) of Hypothesis 7.3 with

$$\beta = -b^2, \; c = 2, \; \nu_k = O(|k|^{-1}).$$

Moreover, it is easy to observe that $\{\lambda^p_k\}_{k\geq k_0}$ satisfies the properties (P1), (P2), (P3) of Hypothesis 7.3.

Thus, the spectrum $\sigma(A^*)$ satisfies Hypothesis 7.3 except for the finite set $\{\lambda_0\} \cup \{\widehat{\lambda}_n\}_{n=1}^{n_0}$. But this will not lead any problem to construct and solve the associated moments equations. Let us go to the detail.

General setting. We first recall Theorem 7.4 and Theorem 7.5. As per those results, our goal is to find uniformly separated spaces $\mathcal{W}_{[0,T]}$ and $\mathcal{E}_{[0,T]}$ in $L^2(0,T)$ for $T > t_c = 1$ (where $t_c = 2/c$ as introduced in Section 7.2.1 and in our case $c = 2$).

We start with $T > 1$. Then, we pick a subset of complex numbers $\{\widehat{\lambda}_{n_l}\}_{l=1}^{l_0}$ in such a way that

$$\mathcal{W}_{[a,a+1]} := \text{closed span } \left(\{e^{\lambda^h_k t}\}_{|k|\geq k_0} \cup \{e^{\widehat{\lambda}_{n_l} t}\}_{l=1}^{l_0}\right) \; \text{ in } L^2(a,a+1), \text{ for any } a \in \mathbb{R}, \tag{7.8}$$

equals the space $L^2(a,a+1)$; and moreover the above set forms a *Riesz basis* for the space $\mathcal{W}_{[a,a+T]}$ for each $T \geq 1$.

In particular,

$$\mathcal{W}_{[0,T]} = \text{closed span } \left(\{e^{\lambda^h_k t}\}_{|k|\geq k_0} \cup \{e^{\widehat{\lambda}_{n_l} t}\}_{l=1}^{l_0}\right) \; \text{ in } L^2(0,T). \tag{7.9}$$

Next, we consider the space

$$\mathcal{E}_{[0,T]} = \text{closed span } \left(\{e^{-\lambda^p_k t}\}_{k\geq k_0} \cup \{e^{-\lambda t}\}_{\lambda\in\Lambda_0} \cup \{1\}\right) \; \text{ in } L^2(0,T). \tag{7.10}$$

Then, we have the following result which follows from Theorem 7.4.

Lemma 7.9. *The spaces $\mathcal{W}_{[0,T]}$ and $\mathcal{E}_{[0,T]}$ defined by (7.9) and (7.10) respectively, are uniformly separated in $L^2(0,T)$ for $T > 1$. This does not hold for $T \leq 1$.*

The set of moments problem. To begin with, let us recall that the eigenvalues for parabolic and hyperbolic parts, namely Λ_p and Λ_h given by (3.3) are simple. Also, recall that the set of eigenfunctions

$$\mathcal{E}(A^*) = \left\{\Phi_{\lambda_k^p},\, k \geq k_0\right\} \cup \left\{k^s \Phi_{\lambda_k^h},\, |k| \geq k_0\right\} \cup \left\{\Phi_\lambda^i;\, \lambda \in \Lambda_0,\, i = 0, ..., m_\lambda - 1\right\}$$

of A^* defines a Riesz basis in $(\dot{H}^s_\sharp(0,1))' \times L^2(0,1)$ for any $s > 0$, thanks to Corollary 3.4. Thus, it is enough to check the control problem (7.1) for the eigenfunctions of A^*. In what follows, the problem (1.5) is null-controllable at given time $T > 1$ if and only if there exists some $p \in L^2(0,T)$ such that we have the following:

$$\begin{cases} -\displaystyle\int_0^T \overline{e^{\lambda_k^p(T-t)}} p(t)\, dt & = m_{1,k}, \quad \forall k \geq k_0, \\ -\displaystyle\int_0^T (T-t)^j \overline{e^{\lambda(T-t)}} p(t)\, dt & = m_\lambda^j, \quad \forall\, \lambda \in \Lambda_0,\ j = 0, 1, \ldots, m_\lambda - 1, \end{cases} \tag{7.11}$$

and

$$-\int_0^T \overline{e^{\lambda_k^h(T-t)}} p(t)\, dt = m_{2,k}, \quad \forall |k| \geq k_0, \tag{7.12}$$

where

$$\begin{cases} m_{1,k} = \dfrac{\overline{e^{\lambda_k^p T}} \left\langle \begin{pmatrix} \xi_{\lambda_k^p} \\ \eta_{\lambda_k^p} \end{pmatrix}, \begin{pmatrix} \rho_0 \\ u_0 \end{pmatrix} \right\rangle_{(\dot{H}^s_\sharp)' \times L^2, \dot{H}^s_\sharp \times L^2}}{\overline{\xi_{\lambda_k^p}(1)}}, \quad \forall k \geq k_0, \\ m_\lambda^j = \dfrac{\overline{e^{\lambda T}} \left\langle \begin{pmatrix} \xi_\lambda^j \\ \eta_\lambda^j \end{pmatrix}, \begin{pmatrix} \rho_0 \\ u_0 \end{pmatrix} \right\rangle_{(\dot{H}^s_\sharp)' \times L^2, \dot{H}^s_\sharp \times L^2}}{\overline{\xi_\lambda^j(1)}}, \quad \forall\, \lambda \in \Lambda_0,\ j = 0, 1 \ldots, m_\lambda - 1, \end{cases} \tag{7.13}$$

and

$$m_{2,k} = \frac{\overline{e^{\lambda_k^h T}} \left\langle \begin{pmatrix} \xi_{\lambda_k^h} \\ \eta_{\lambda_k^h} \end{pmatrix}, \begin{pmatrix} \rho_0 \\ u_0 \end{pmatrix} \right\rangle_{(\dot{H}^s_\sharp)' \times L^2, \dot{H}^s_\sharp \times L^2}}{\overline{\xi_{\lambda_k^h}(1)}}, \quad \forall |k| \geq k_0. \tag{7.14}$$

The above set of equations (7.11)–(7.12) are the so-called moments problem which are well-defined since $\mathcal{B}_\rho^* \Phi = \xi(1) \neq 0$ for any (generalized) eigenfunction $\Phi \in \mathcal{E}(A^*)$ as proved in Proposition 4.2–Part 1 under the assumption $b^4 + 8b^2 + 5 < 4\pi^2$. Let us now study the solvability of those equations.

Proof of the null-controllability result in $\dot{H}^s_\sharp \times L^2$, $s > \frac{1}{2}$. Let any parameter $s > 1/2$, initial data $(\rho_0, u_0) \in \dot{H}^s_\sharp(0,1) \times L^2(0,1)$ and time $T > 1$ be given. We now consider the finitely many complex numbers $(\widehat{\lambda}_{n_l})_{l=1}^{l_0}$ introduced earlier (see eq. (7.8)) in the above moments problem (hyperbolic part)

$$-\int_0^T \overline{e^{\widehat{\lambda}_{n_l}(T-t)}} p(t) dt = m_{2,l}, \quad \forall\ l = 1, \ldots, l_0, \tag{7.15}$$

where $m_{2,l} \in \mathbb{C}$ for all $l = 1, \ldots, l_0$. Then, our goal is to apply the result of Theorem 7.8 to solve the set of moments problem (7.11)–(7.12)-(7.15). To do this, it suffices to show the following facts: for any $r > 0$

$$|m_{1,k}| e^{rk} \to 0 \quad \text{as} \quad k \to +\infty, \tag{7.16}$$

and

$$\sum_{|k| \geq k_0} |m_{2,k}|^2 < +\infty. \tag{7.17}$$

– Recall the expression of $m_{1,k}$ for $k \geq k_0$ from (7.13). We have

$$\begin{aligned} |m_{1,k}| &\leq C \|(\rho_0, u_0)\|_{\dot{H}^s_\sharp \times L^2}\, e^{\mathrm{Re}(\lambda_k^p) T} \frac{\|\xi_{\lambda_k^p}\|_{(\dot{H}^s_\sharp)'} + \|\eta_{\lambda_k^p}\|_{L^2}}{|\xi_{\lambda_k^p}(1)|} \\ &\leq C \|(\rho_0, u_0)\|_{\dot{H}^s_\sharp \times L^2}\, e^{-k^2 \pi^2 T} k\pi \left(k^{-s-1} + 1\right), \end{aligned} \tag{7.18}$$

thanks to the bounds of the eigenfunctions (3.14) and observation estimate (4.38a). Indeed, the bound (7.18) directly implies the Claim (7.16) due to the presence of $e^{-k^2\pi^2 T}$ in the right hand side of (7.18).

Thus, in view of Proposition 7.7, there exists a function $p_1 \in \mathcal{E} := \mathcal{E}_{[0,T]}$ that solves the set of equations (7.11) for the case of simple eigenvalues. To add the finitely many generalized eigenfunctions, one can adapt the strategy developed for instance in [29] or [9], where the authors have proved the existence of bi-orthogonal family for a general sequence of type $\{t^j e^{\lambda_n t}\}_{j=0,\cdots,J;n\geq 1}$ for any $J \in \mathbb{N}^*$, where $\{\lambda_n\}_{n\geq 1}$ verifies the properties like (P1) and (P2) at least for large index $n \in \mathbb{N}^*$. As a consequence, we can find a $p_1 \in \mathcal{E}_{0,T}$ solving the parabolic moment problem (7.11).

– On the other hand, we show that $\{m_{2,k}\}_{|k|\geq k_0} \in \ell_2$. In this regard, we recall the bounds of the eigenfunctions given by (3.15) and the observation estimate (4.38b), which yields

$$\begin{aligned}\sum_{|k|\geq k_0} |m_{2,k}|^2 &\leq C\|(\rho_0,u_0)\|^2_{\dot{H}^s_\sharp \times L^2} \sum_{|k|\geq k_0} \frac{\|\xi_{\lambda_k^h}\|^2_{(\dot{H}^s_\sharp)'} + \|\eta_{\lambda_k^h}\|^2_{L^2}}{|\xi_{\lambda_k^h}(1)|^2} \\ &\leq C\|(\rho_0,u_0)\|^2_{\dot{H}^s_\sharp \times L^2} \sum_{|k|\geq k_0} \left(|k|^{-2s} + |k|^{-2}\right) \\ &\leq C\|(\rho_0,u_0)\|^2_{\dot{H}^s_\sharp \times L^2}.\end{aligned}$$

The above series converges due to the sharp choice $s > 1/2$ and indeed, it is clear that for $s \leq 1/2$, the series $\displaystyle\sum_{|k|\geq k_0} \frac{1}{|k|^{2s}}$ diverges.

Therefore, in view of Proposition 7.6, there exists a function $p_2 \in \mathcal{W} := \mathcal{W}_{[0,T]}$ that solves the set of equations (7.12)–(7.15).

Now, as consequence of Lemma 7.9, the space

$$\mathcal{V} := \mathcal{E} + \mathcal{W} \tag{7.19}$$

is closed and thus a Hilbert space with $\|\cdot\|_{\mathcal{V}} := \|\cdot\|_{L^2(0,T)}$, so $\mathcal{V} = \mathcal{E} \oplus \mathcal{W}$. Likewise, we have $\mathcal{V} := \mathcal{E}^\perp \oplus \mathcal{W}^\perp$. Therefore, the restrictions $P_{\mathcal{E}}|_{\mathcal{W}^\perp}$ and $P_{\mathcal{W}}|_{\mathcal{E}^\perp}$ are isomorphisms, where $P_{\mathcal{E}}$ and $P_{\mathcal{W}}$ denote the orthogonal projections from $\mathcal{V}$ onto $\mathcal{E}$ and $\mathcal{W}$ respectively. Let us set

$$p := (P_{\mathcal{E}}|_{\mathcal{W}^\perp})^{-1} p_1 + (P_{\mathcal{W}}|_{\mathcal{E}^\perp})^{-1} p_2, \tag{7.20}$$

which certainly belongs to the space $L^2(0,T)$ and simultaneously solves the set of moments problem (7.11)–(7.12)–(7.15) for $T > 1$ and any $\rho_0 \in \dot{H}^s_\sharp(0,1)$ for $s > 1/2$, $u_0 \in L^2(0,1)$. This concludes the proof of the result of this section.

7.3. Null-controllability result with $\dot{L}^2 \times L^2$ initial data.

Proof of Theorem 1.3. We start with $b^4 + 8b^2 + 5 < 4\pi^2$ and pick any initial data $(\rho_0, u_0) \in \dot{L}^2(0,1) \times L^2(0,1)$ for the system (1.5). We express the initial data as

$$(\rho_0, u_0) = (\rho_0, 0) + (0, u_0),$$

and consider the following two systems

$$\begin{cases} \rho_{1,t} + \rho_{1,x} + bu_{1,x} = 0, & \text{in } (0,T)\times(0,1), \\ u_{1,t} - u_{1,xx} + u_{1,x} + b\rho_{1,x} = 0, & \text{in } (0,T)\times(0,1), \\ \rho_1(t,0) = \rho_1(t,1) + p_1(t), & \text{on } (0,T), \\ u_1(t,0) = 0, \quad u_1(t,1) = 0, & \text{on } (0,T), \\ \rho_1(0,x) = \rho_0(x), \quad u_1(0,x) = 0, & \text{in } (0,1), \end{cases} \tag{7.21}$$

and

$$\begin{cases} \rho_{2,t} + \rho_{2,x} + bu_{2,x} = 0, & \text{in } (0,T)\times(0,1), \\ u_{2,t} - u_{2,xx} + u_{2,x} + b\rho_{2,x} = 0, & \text{in } (0,T)\times(0,1), \\ \rho_2(t,0) = \rho_2(t,1) + p_2(t), & \text{on } (0,T), \\ u_2(t,0) = 0, \quad u_2(t,1) = 0, & \text{on } (0,T), \\ \rho_2(0,x) = 0, \quad u_2(0,x) = u_0(x), & \text{in } (0,1). \end{cases} \tag{7.22}$$

Here $p_1, p_2 \in L^2(0,T)$ are boundary controls which are to be determined.

Now, from the analysis pursued in Section 7.1, if we start with initial data $(\rho_0, 0)$ with $\rho_0 \in \dot{L}^2(0,1)$, then there exists a control $p_1 \in L^2(0,T)$ such that the solution (ρ_1, u_1) to the system (7.21) verifies

$$(\rho_1(T,\cdot), u_1(T,\cdot)) = (0,0), \quad \text{in } (0,1).$$

On the other hand, it is also known from Section 7.2 that, starting with initial data $(0, u_0)$ with $u_0 \in L^2(0,1)$, we can find a control $p_2 \in L^2(0,T)$ such that the solution (ρ_2, u_2) to the system (7.22) satisfies

$$(\rho_2(T,\cdot), u_2(T,\cdot)) = (0,0), \quad \text{in } (0,1).$$

Let us define $p(t) = p_1(t) + p_2(t)$ for $t \in (0,T)$. Then $p \in L^2(0,T)$, and the solution (ρ, u) to the main system (1.5), with this control p and the prescribed initial state $(\rho_0, u_0) \in \dot{L}^2(0,1) \times L^2(0,1)$, satisfies

$$(\rho(T,\cdot), u(T,\cdot)) = (0,0) \quad \text{in } (0,1).$$

□

Proof of Theorem 1.6. We have already shown the existence of a null-control $p \in L^2(0,T)$ for the system (1.5). Now, to prove the existence of a null-control $h \in L^2(0,T)$ for the control problem (1.6), all we need to show that $\rho(\cdot,1) \in L^2(0,T)$, where ρ is the solution component of the system (1.5) associated with the control function $p \in L^2(0,T)$. But the proof for $\rho(\cdot,1) \in L^2(0,T)$ follows from a hidden regularity result given in Appendix B (Lemma B.1).

Hence, we define $h(t) = \rho(t,1) + p(t)$ for all $t \in (0,T)$, which plays the role of a Dirichlet (null) control function for the main system (1.6). The proof is complete. □

7.4. **Lack of null-controllability at small time.** This section is devoted to prove the lack of null-controllability result of the system (1.5) for $0 < T < 1$, that is precisely Proposition 1.5. In this regard, we mention the work [8] where the authors proved the lack of null-controllability for a transport-parabolic system with localized interior control. Similar result has been treated in [15] in the context of boundary controllability for a transport-elliptic system (the so-called creeping flow model).

Proof of Proposition 1.5. Let $0 < T < 1$. Consider the transport equation

$$\begin{cases} \tilde{\sigma}_t(t,x) + \tilde{\sigma}_x(t,x) - b^2\tilde{\sigma}(t,x) = 0, & (t,x) \in (0,T) \times (0,1), \\ \tilde{\sigma}(t,0) = \tilde{\sigma}(t,1), & t \in (0,T), \\ \tilde{\sigma}(T,x) = \tilde{\sigma}_T(x), & x \in (0,1), \end{cases} \tag{7.23}$$

with $\tilde{\sigma}_T \in L^2(0,1)$. Since $T < 1$, there exists a nontrivial function $\tilde{\sigma}_T \in C^\infty(0,1)$ with $\text{supp}(\tilde{\sigma}_T) \subset (T,1)$ such that the associated solution $\tilde{\sigma}$ of (7.23) satisfies $\tilde{\sigma}(t,0) = \tilde{\sigma}(t,1) = 0$ for all $t \in (0,T)$ and $\tilde{\sigma} \neq 0$ in $(0,T) \times (0,1)$. Let $N > 0$ be a fixed integer. We define the polynomial

$$P^N(x) := \prod_{l=-N}^{N} (x - l), \quad x \in (0,1)$$

and the function

$$\tilde{\sigma}_T^N := P^N\left(-i\frac{d}{dx}\right)\tilde{\sigma}_T.$$

We write the terminal state $\tilde{\sigma}_T \in L^2(0,1)$ as

$$\tilde{\sigma}_T(x) := \sum_{n\in\mathbb{Z}} a_n e^{2in\pi x}, \quad x \in (0,1).$$

Then the above function $\tilde{\sigma}_T^N$ becomes

$$\tilde{\sigma}_T^N(x) = \sum_{n\in\mathbb{Z}} a_n \prod_{l=-N}^{N} \left(-i\frac{d}{dx} - l\right) e^{2in\pi x} = \sum_{n\in\mathbb{Z}} a_n \prod_{l=-N}^{N} (n-l)\, e^{2in\pi x} = \sum_{n\in\mathbb{Z}} a_n P^N(n) e^{2in\pi x},$$

for $x \in (0,1)$. Note that $P^N(n) = 0$ for all $|n| \leq N$ and therefore

$$\tilde{\sigma}_T^N(x) = \sum_{|n|\geq N+1} a_n P^N(n) e^{2in\pi x}.$$

With this $\tilde{\sigma}_T^N$, let us now consider the following system

$$\begin{cases} \tilde{\sigma}_t(t,x) + \tilde{\sigma}_x(t,x) - b^2\tilde{\sigma}(t,x) = 0, & (t,x) \in (0,T) \times (0,1), \\ \tilde{\sigma}(t,0) = \tilde{\sigma}(t,1), & t \in (0,T), \\ \tilde{\sigma}(T,x) = \tilde{\sigma}_T^N(x), & x \in (0,1). \end{cases} \tag{7.24}$$

Since $\mathrm{supp}(\tilde{\sigma}_T^N) \subset \mathrm{supp}(\tilde{\sigma}_T) \subset (T,1)$, the solution $\tilde{\sigma}$ to this system (7.24) satisfies $\tilde{\sigma}^N(t,0) = \tilde{\sigma}^N(t,1) = 0$ for all $t \in (0,T)$. On the other hand, we consider the following adjoint system

$$\begin{cases} \sigma_t(t,x) + \sigma_x(t,x) + bv_x(t,x) = 0, & (t,x) \in (0,T) \times (0,1), \\ v_t(t,x) - v_{xx}(t,x) + v_x(t,x) + b\sigma_x(t,x) = 0, & (t,x) \in (0,T) \times (0,1), \\ \sigma(t,0) = \sigma(t,1), & t \in (0,T), \\ v(t,0) = 0, \quad v(t,1) = 0, & t \in (0,T), \\ \sigma(T,x) = \tilde{\sigma}_T^N(x), \quad v(T,x) = v_T^N(x), & x \in (0,1), \end{cases} \tag{7.25}$$

where we choose v_T^N such that

$$(\tilde{\sigma}_T^N, v_T^N)^\dagger = \sum_{|n| \geq N+1} \tilde{a}_n^h \Phi_{\lambda_n^h}$$

with $\tilde{a}_n^h := \frac{a_n P^N(n)}{\xi_{\lambda_n^h}(1)}$ for all $|n| \geq N+1$ (note that $\xi_{\lambda_n^h}(1) \neq 0$, thanks to the eigen equation). We write the solutions to the systems (7.24) and (7.25) respectively as

$$\tilde{\sigma}^N(t,x) = \sum_{|n| \geq N+1} a_n P^N(n) e^{(-2in\pi - b^2)(T-t)} e^{2in\pi x}, \tag{7.26}$$

$$\sigma^N(t,x) = \sum_{|n| \geq N+1} \frac{a_n P^N(n)}{\xi_{\lambda_n^h}(1)} e^{\lambda_n^h (T-t)} \xi_{\lambda_n^h}, \tag{7.27}$$

$$v^N(t,x) = \sum_{|n| \geq N+1} \frac{a_n P^N(n)}{\xi_{\lambda_n^h}(1)} e^{\lambda_n^h (T-t)} \eta_{\lambda_n^h}, \tag{7.28}$$

for $(t,x) \in [0,T] \times [0,2\pi]$. We prove that the solution component σ^N of (7.25) approximates the solution $\tilde{\sigma}^N$ of (7.24) at the point $x = 1$. Indeed,

$$\begin{aligned} & \left\| \sigma^N(\cdot,1) - \tilde{\sigma}^N(\cdot,1) \right\|^2_{L^2(0,T)} \\ & \leq \sum_{|n| \geq N+1} |a_n|^2 \left| P^N(n) \right|^2 \left\| e^{\lambda_n^h (T-t)} - e^{\left(-2in\pi - b^2\right)(T-t)} e^{2in\pi} \right\|^2_{L^2(0,T)} \\ & \leq \sum_{|n| \geq N+1} |a_n|^2 \left| P^N(n) \right|^2 \left\| e^{O(\frac{1}{n})(T-t)} - 1 \right\|^2_{L^2(0,T)} \\ & \leq \sum_{|n| \geq N+1} \frac{1}{|n|^2} |a_n|^2 \left| P^N(n) \right|^2, \end{aligned}$$

and therefore

$$\left\| \sigma^N(\cdot,1) - \tilde{\sigma}^N(\cdot,1) \right\|^2_{L^2(0,T)} \leq \frac{C}{|N|^2} \sum_{|n| \geq N+1} |a_n|^2 \left| P^N(n) \right|^2.$$

Let us now suppose that the following observability inequality holds

$$\int_0^T \left| \sigma^N(t,1) \right|^2 dt \geq C \left\| (\sigma^N(0), v^N(0)) \right\|^2_{(L^2(0,1))^2}. \tag{7.29}$$

Then, we have

$$\begin{aligned}\left\|(\sigma^N(0), v^N(0))\right\|^2_{(L^2(0,1))^2} &\leq C\int_0^T \left|\sigma^N(t,1)\right|^2 dt \\ &\leq C\int_0^T \left(\left|\sigma^N(t,1)-\tilde{\sigma}^N(t,1)\right|^2 + \left|\tilde{\sigma}^N(t,1)\right|^2\right) dt \\ &\leq \frac{C}{N^2}\sum_{|n|\geq N+1} |a_n|^2 \left|P^N(n)\right|^2,\end{aligned}$$

as we have $\tilde{\sigma}^N(t,0) = 0 = \tilde{\sigma}^N(t,1)$ for all $t \in (0,T)$. Thus we get

$$\left\|\sigma^N(0)\right\|^2_{L^2(0,1)} \leq \left\|(\sigma^N(0), v^N(0))\right\|^2_{(L^2(0,1))^2} \leq \frac{C}{N^2}\sum_{|n|\geq N+1} |a_n|^2 \left|P^N(n)\right|^2 \leq \frac{C}{N^2}\left\|\sigma^N(0)\right\|^2_{L^2(0,1)},$$

since $\mathrm{Re}(\nu_n^h)$ is bounded and $\left\|\xi_{\lambda_n^h}\right\|_{L^2(0,1)} \geq C$, $\left|\xi_{\lambda_n^h}(0)\right| \geq C$. Therefore, $1 \leq \frac{C}{N^2}$ for all N and hence the above inequality is not true. This shows that the observability inequality (7.29) cannot hold; as a consequence the system is not null-controllable at time T. This completes the proof. □

8. Detailed spectral analysis of the adjoint operator

In this section, we study the detailed spectral analysis of the adjoint operator A^*. We hereby recall the eigenvalue problem (3.1) from Section 3 which has been rewritten below,

$$\begin{aligned}\xi'(x) + b\eta'(x) &= \lambda\xi(x), \quad x \in (0,1),\\ \eta''(x) + \eta'(x) + b\xi'(x) &= \lambda\eta(x), \quad x \in (0,1),\\ \xi(0) &= \xi(1),\\ \eta(0) = 0, \quad \eta(1) &= 0.\end{aligned} \tag{8.1}$$

We divide the analysis into several steps. Let us begin by the following results.

Proof of point (ii)-Proposition 3.1: All non-trivial eigenvalues have negative real parts. Let $\lambda \neq 0$. Multiplying the first equation of (8.1) by $\overline{\xi}$, the second one by $\overline{\eta}$ and then integrating, we obtain

$$\begin{aligned}\int_0^1 \overline{\xi(x)}\xi'(x)dx + b\int_0^1 \overline{\xi(x)}\eta'(x)dx &= \lambda\int_0^1 |\xi(x)|^2 dx\\ \int_0^1 \overline{\eta(x)}\eta''(x)dx + \int_0^1 \overline{\eta(x)}\eta'(x)dx + b\int_0^1 \overline{\eta(x)}\xi'(x)dx &= \lambda\int_0^1 |\eta(x)|^2 dx.\end{aligned}$$

Adding these two equations, we get

$$\begin{aligned}\int_0^1 \overline{\xi(x)}\xi'(x)dx + \int_0^1 \overline{\eta(x)}\eta'(x)dx + b\int_0^1 \overline{\xi(x)}\eta'(x)dx + b\int_0^1 \overline{\eta(x)}\xi'(x)dx&\\ + \int_0^1 \overline{\eta(x)}\eta''(x)dx = \lambda\int_0^1 |\xi(x)|^2 dx + \lambda\int_0^1 |\eta(x)|^2 dx&,\end{aligned} \tag{8.2}$$

where we have used the following fact

$$\int_0^1 \overline{\xi(x)}\xi'(x)dx = \frac{1}{2}\int_0^1 \frac{d}{dx}|\xi(x)|^2 dx + i\int_0^1 \mathrm{Im}(\overline{\xi(x)}\xi'(x))dx = i\int_0^1 \mathrm{Im}(\overline{\xi(x)}\xi'(x))dx, \tag{8.3}$$

thanks to the boundary condition $\xi(0) = \xi(1)$.

Similarly, we can obtain

$$\int_0^1 \overline{\eta(x)}\eta'(x)dx = i\int_0^1 \mathrm{Im}(\overline{\eta(x)}\eta'(x))dx. \tag{8.4}$$

Using the relations (8.3), (8.4) in (8.2) and performing an integration by parts, we deduce that

$$\begin{aligned}i\int_0^1 \left(\mathrm{Im}(\overline{\xi(x)}\xi'(x)) + \mathrm{Im}(\overline{\eta(x)}\eta'(x))\right)dx + b\int_0^1 \xi'(x)\overline{\eta(x)}dx - b\int_0^1 \overline{\xi'(x)}\eta(x)dx - \int_0^1 |\eta'(x)|^2 dx&\\ = \lambda\int_0^1 |\xi(x)|^2 dx + \lambda\int_0^1 |\eta(x)|^2 dx&,\end{aligned}$$

from which it is clear that

$$\operatorname{Re}(\lambda) = -\frac{\|\eta'\|_{L^2}^2}{\|\xi\|_{L^2}^2 + \|\eta\|_{L^2}^2} < 0, \tag{8.5}$$

since $\eta' = 0$ is not possible. If yes, then from the boundary condition $\eta(0) = \eta(1) = 0$, we have $\eta = 0$ and this yields that $\xi = c$, for some constant c, which is possible if and only if $\lambda = 0$. Therefore, when $\lambda \neq 0$, then one has the condition (8.5).

Remark 8.1. *It can be easily seen that the first component ξ satisfies $\int_0^1 \xi = 0$ provided $\lambda \neq 0$.*

Proof of point (iii)- Proposition 3.1: compactness of the resolvent to the adjoint operator. In this section, we are going to prove the part (iii) of Proposition 3.1.

For any $\lambda \notin \sigma(A^*)$, denote the resolvent operator associated to A^* by $R(\lambda, A^*) := (\lambda I - A^*)^{-1}$ (where $\sigma(A^*)$ is the spectrum of A^* defined by (3.5)).

Let $\{Y_n\}_n = \{(f_n, g_n)\}_n$ be a bounded sequence in $\mathbf{Z} := L^2(0,1) \times L^2(0,1)$. Our claim is to prove that for any $\lambda > 0$ the sequence $\left\{R(\lambda; A^*)Y_n\right\}_n$ contains a convergent subsequence. Let $X_n = (\sigma_n, v_n) = R(\lambda; A^*)Y_n \in D(A^*)$, that is

$$(\lambda I - A^*)X_n = Y_n. \tag{8.6}$$

More explicitly,

$$\begin{cases} \lambda\sigma_n - (\sigma_n)_x - b(v_n)_x = f_n & \text{in } (0,1), \\ \lambda v_n - b(\sigma_n)_x - (v_n)_x - (v_n)_{xx} = g_n & \text{in } (0,1), \\ \sigma_n(0) = \sigma_n(1), \quad v_n(0) = v_n(1) = 0. \end{cases} \tag{8.7}$$

Taking inner product with X_n in the equation (8.6), we get

$$\lambda \langle X_n, X_n \rangle_{\mathbf{Z}} - \langle A^* X_n, X_n \rangle_{\mathbf{Z}} = \langle X_n, Y_n \rangle_{\mathbf{Z}}.$$

Considering only the real parts, we see

$$\lambda \|X_n\|_{\mathbf{Z}}^2 - \operatorname{Re}(\langle A^* X_n, X_n \rangle_{\mathbf{Z}}) = \operatorname{Re}(\langle X_n, Y_n \rangle_{\mathbf{Z}}).$$

Now, recall that the operator A^* is dissipative, i.e., $\operatorname{Re}(\langle A^* X_n, X_n \rangle_{\mathbf{Z}}) \leq 0$; in what follows, we have

$$\lambda \|X_n\|_{\mathbf{Z}}^2 \leq \operatorname{Re}(\langle X_n, Y_n \rangle_{\mathbf{Z}}) \leq |\langle X_n, Y_n \rangle_{\mathbf{Z}}| \leq \frac{\lambda}{2} \|X_n\|_{\mathbf{Z}}^2 + \frac{1}{2\lambda} \|Y_n\|_{\mathbf{Z}}^2.$$

In other words,

$$\|X_n\|_{\mathbf{Z}}^2 \leq \frac{1}{\lambda^2} \|Y_n\|_{\mathbf{Z}}^2.$$

Thus, the sequence $\{X_n\}_n$ is bounded in $\mathbf{Z}$. We now prove that $\{X_n\}_n$ is in fact bounded in $H^1_\sharp(0,1) \times H^1_0(0,1)$. Multiplying the second equation of (8.7) by u_n, we get

$$\lambda \int_0^1 |v_n|^2\, dx - b \int_0^1 (\sigma_n)_x \bar{v}_n dx - \int_0^1 (v_n)_{xx} \bar{v}_n dx = \int_0^1 g_n \bar{v}_n dx.$$

Performing an integration by parts, we obtain

$$\lambda \int_0^1 |v_n|^2\, dx + b \int_0^1 \sigma_n (\bar{v}_n)_x dx + \int_0^1 |(v_n)_x|^2\, dx = \int_0^1 g_n \bar{v}_n dx,$$

from which, it follows that

$$\begin{aligned} \lambda \int_0^1 |v_n|^2\, dx + \int_0^1 |(v_n)_x|^2\, dx &= \operatorname{Re}\left(\int_0^1 g_n \bar{v}_n dx\right) - b\operatorname{Re}\left(\int_0^1 \sigma_n (\bar{v}_n)_x dx\right) \\ &\leq \left|\int_0^1 g_n \bar{v}_n dx\right| + b\left|\int_0^1 \sigma_n (\bar{v}_n)_x dx\right| \\ &\leq \frac{1}{2\lambda} \int_0^1 |g_n|^2\, dx + \frac{\lambda}{2} \int_0^1 |v_n|^2\, dx + \frac{b^2}{2} \int_0^1 |\sigma_n|^2\, dx + \frac{1}{2} \int_0^1 |(v_n)_x|^2\, dx. \end{aligned}$$

After simplification, we have

$$\frac{\lambda}{2} \int_0^1 |v_n|^2\, dx + \frac{1}{2} \int_0^1 |(v_n)_x|^2\, dx \leq \frac{1}{2\lambda} \int_0^1 |g_n|^2\, dx + \frac{b^2}{2} \int_0^1 |\sigma_n|^2\, dx,$$

that is, the sequence $\{v_n\}_n$ is bounded in $H_0^1(0,1)$. Then, the first equation of (8.7) gives

$$(\sigma_n)_x = \lambda\sigma_n - b(v_n)_x - f_n,$$

which shows that the sequence $\{(\sigma_n)_x\}_n$ is bounded in $L^2(0,1)$.

So, we have proved that $\{X_n\}_n$ is a bounded sequence in $H_\sharp^1(0,1) \times H_0^1(0,1)$ (which is compactly embedded in $\mathbf{Z}$) and therefore, $\{X_n\}_n$ is relatively compact in $\mathbf{Z}$.

This completes the proof.

Proof of point (iv)-Proposition 3.1: all eigenvalues are geometrically simple. Let $b > 0$ be such that $b^4 + 8b^2 + 5 < 4\pi^2$. On contrary, let us assume that for any eigenvalue λ, there are two distinct eigenfunctions $\Phi_1 := (\xi_1, \eta_1)$ and $\Phi_2 := (\xi_2, \eta_2)$ of A^*. We prove that Φ_1 and Φ_2 are linearly dependent.

Let be $\theta_1, \theta_2 \in \mathbb{C} \setminus \{0\}$ and consider the linear combination $\Phi := \theta_1\Phi_1 + \theta_2\Phi_2$. Then $\Phi := (\xi, \eta)$ also satisfies the eigenvalue problem (8.1). We now choose θ_1, θ_2 in such a way that $\xi(0) = 0$ (a particular choice is $\theta_1 = -\frac{\theta_2\xi_2(0)}{\xi_1(0)}$). Then, in the same spirit of Proposition 4.2–Part 1, we can conclude that $\Phi = 0$.

This ensures the assumption that each eigenvalue of A^* has geometric multiplicity 1.

8.1. **Determining the eigenvalues for large modulus.** We write the set of equations (8.1) satisfied by ξ and η into a single equation of η as obtained in (4.6), given by

$$\eta'''(x) - (\lambda + b^2 - 1)\eta''(x) - 2\lambda\eta'(x) + \lambda^2\eta(x) = 0, \quad \forall x \in (0,1), \tag{8.8a}$$

$$\eta(0) = \eta(1) = 0, \quad \eta''(0) - (b^2 - 1)\eta'(0) = \eta''(1) - (b^2 - 1)\eta'(1). \tag{8.8b}$$

Then, the auxiliary equation associated to (8.8a) is

$$m^3 - (\lambda + b^2 - 1)m^2 - 2\lambda m + \lambda^2 = 0. \tag{8.9}$$

Introduce $\mu = -\lambda \in \mathbb{C}$ and $a_1 = \mu - b^2 + 1$, $a_2 = 2\mu$, $a_3 = \mu^2$, so that the roots of cubic polynomial (8.9) are given by

$$\begin{cases} m_1 = -\dfrac{1}{3}\left(a_1 + C + \dfrac{D_0}{C}\right), \\ m_2 = -\dfrac{1}{3}\left(a_1 + \dfrac{(-1 + i\sqrt{3})}{2} C + \dfrac{(-1 - i\sqrt{3})}{2} \dfrac{D_0}{C}\right), \\ m_3 = -\dfrac{1}{3}\left(a_1 + \dfrac{(-1 - i\sqrt{3})}{2} C + \dfrac{(-1 + i\sqrt{3})}{2} \dfrac{D_0}{C}\right), \end{cases} \tag{8.10}$$

with

$$D_0 = a_1^2 - 3a_2, \quad D_1 = 2a_1^3 - 9a_1a_2 + 27a_3, \quad C = \left(\frac{D_1 + \sqrt{D_1^2 - 4D_0^3}}{2}\right)^{1/3}.$$

Exerting the values of a_1, a_2, a_3, we can find

$$D_0 = \mu^2 + (b^2 - 1)^2 - 2(2 + b^2)\mu,$$
$$D_1 = 2\mu^3 + (15 - 6b^2)\mu^2 + (6b^4 + 6b^2 - 12)\mu - 2b^6 + 6b^4 - 6b^2 + 2.$$

From the above expressions, we calculate

$$D_1^2 - 4D_0^3 = 108\mu^5 - (324b^2 + 27)\mu^4 + O(\mu^3).$$

Using the binomial expansion and approximating for large $|\mu|$, we obtain

$$\begin{aligned} \sqrt{D_1^2 - 4D_0^3} &= 6\sqrt{3}\mu^{5/2}\left[1 - \left(\frac{12b^2 + 1}{4\mu} + O(\mu^{-2})\right)\right]^{1/2} \\ &= 6\sqrt{3}\mu^{5/2}\left[1 - \frac{12b^2 + 1}{8\mu} + O(\mu^{-2})\right] \\ &= 6\sqrt{3}\mu^{5/2} - \frac{6\sqrt{3}}{8}(12b^2 + 1)\mu^{3/2} + O(\mu^{1/2}). \end{aligned}$$

In terms of the above quantities, we have

$$C = \left[\mu^3 + 3\sqrt{3}\mu^{5/2} + \frac{(15 - 6b^2)}{2}\mu^2 - \frac{3\sqrt{3}}{8}(12b^2 + 1)\mu^{3/2} + O(\mu)\right]^{1/3}$$

Now, using binomial expansion and simplifying, one can obtain for large modulus of μ, that

$$\begin{aligned}
C &= \mu\Bigg[1 + \left(\sqrt{3}\mu^{-1/2} + \frac{(15-6b^2)}{6}\mu^{-1} - \frac{\sqrt{3}}{8}(12b^2+1)\mu^{-3/2} + O(\mu^{-2})\right) \\
&\qquad - \frac{1}{9}\left(27\mu^{-1} + 3\sqrt{3}(15-6b^2)\mu^{-3/2} + O(\mu^{-2})\right) \\
&\qquad + \frac{5}{81}\left(81\sqrt{3}\mu^{-3/2} + O(\mu^{-2})\right) + O(\mu^{-2})\Bigg] \\
&= \mu\left[1 + \sqrt{3}\mu^{-1/2} - \frac{2b^2+1}{2}\mu^{-1} + \frac{\sqrt{3}}{8}(4b^2-1)\mu^{-3/2} + O(\mu^{-2})\right] \\
&= \mu + \sqrt{3}\mu^{1/2} - \frac{2b^2+1}{2} + \frac{\sqrt{3}}{8}(4b^2-1)\mu^{-1/2} + O(\mu^{-1}).
\end{aligned}$$

Similarly we have,

$$\begin{aligned}
\frac{D_0}{C} &= \left(\frac{D_1 - \sqrt{D_1^2 - 4D_0^3}}{2}\right)^{\frac{1}{3}} \\
&= \left[\mu^3 - 3\sqrt{3}\mu^{5/2} + \frac{(15-6b^2)}{2}\mu^2 + \frac{3\sqrt{3}}{8}(12b^2+1)\mu^{3/2} + O(\mu)\right]^{1/3} \\
&= \mu - \sqrt{3}\mu^{1/2} - \frac{2b^2+1}{2} - \frac{\sqrt{3}}{8}(4b^2-1)\mu^{-1/2} + O(\mu^{-1}).
\end{aligned}$$

So, the characteristic roots are (recall (8.10))

$$\begin{aligned}
m_1 &= -\frac{1}{3}\Bigg[\mu - b^2 + 1 + \left(\mu + \sqrt{3}\mu^{1/2} - \frac{2b^2+1}{2} + \frac{\sqrt{3}}{8}(4b^2-1)\mu^{-1/2} + O(\mu^{-1})\right) \\
&\qquad + \left(\mu - \sqrt{3}\mu^{1/2} - \frac{2b^2+1}{2} - \frac{\sqrt{3}}{8}(4b^2-1)\mu^{-1/2} + O(\mu^{-1})\right)\Bigg] \\
&= -\mu + b^2 + O(\mu^{-1}),
\end{aligned}$$

$$\begin{aligned}
m_2 &= -\frac{1}{3}\Bigg[\mu - b^2 + 1 + \frac{-1+i\sqrt{3}}{2}\left(\mu + \sqrt{3}\mu^{1/2} - \frac{2b^2+1}{2} + \frac{\sqrt{3}}{8}(4b^2-1)\mu^{-1/2} + O(\mu^{-1})\right) \\
&\qquad + \frac{-1-i\sqrt{3}}{2}\left(\mu - \sqrt{3}\mu^{1/2} - \frac{2b^2+1}{2} - \frac{\sqrt{3}}{8}(4b^2-1)\mu^{-1/2} + O(\mu^{-1})\right)\Bigg] \\
&= -\frac{1}{2} - i\mu^{1/2} + O(\mu^{-1/2}),
\end{aligned}$$

$$\begin{aligned}
m_3 &= -\frac{1}{3}\Bigg[\mu - b^2 + 1 + \frac{-1-i\sqrt{3}}{2}\left(\mu + \sqrt{3}\mu^{1/2} - \frac{2b^2+1}{2} + \frac{\sqrt{3}}{8}(4b^2-1)\mu^{-1/2} + O(\mu^{-1})\right) \\
&\qquad + \frac{-1+i\sqrt{3}}{2}\left(\mu - \sqrt{3}\mu^{1/2} - \frac{2b^2+1}{2} - \frac{\sqrt{3}}{8}(4b^2-1)\mu^{-1/2} + O(\mu^{-1})\right)\Bigg] \\
&= -\frac{1}{2} + i\mu^{1/2} + O(\mu^{-1/2}).
\end{aligned}$$

Together, we write

$$\begin{cases}
m_1 = -\mu + b^2 + O(\mu^{-1}), \\
m_2 = -\dfrac{1}{2} - i\mu^{1/2} + O(\mu^{-1/2}), \\
m_3 = -\dfrac{1}{2} + i\mu^{1/2} + O(\mu^{-1/2}),
\end{cases} \tag{8.11}$$

with $\mu = -\lambda$ as mentioned earlier. Since, for large modulus of μ, the roots m_1, m_2 and m_3 are distinct, we can write the general solution to the equation (8.8a) as

$$\eta(x) = C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x}, \quad x \in (0,1), \tag{8.12}$$

for some constants $C_1, C_2, C_3 \in \mathbb{C}$.

Using the boundary conditions (8.8b), we get a system of linear equations in C_1, C_2 and C_3, given by

$$\begin{cases} C_1 + C_2 + C_3 = 0, \\ C_1 e^{m_1} + C_2 e^{m_2} + C_3 e^{m_3} = 0, \\ C_1 m_1^2 (1 - e^{m_1}) + C_2 m_2^2 (1 - e^{m_2}) + C_3 m_3^2 (1 - e^{m_3}) = 0. \end{cases} \tag{8.13}$$

These system of equations (8.13) has a nontrivial solution if and only if

$$\det \begin{pmatrix} 1 & 1 & 1 \\ e^{m_1} & e^{m_2} & e^{m_3} \\ m_1^2 (1 - e^{m_1}) & m_2^2 (1 - e^{m_2}) & m_3^2 (1 - e^{m_3}) \end{pmatrix} = 0.$$

Expanding the determinant, we obtain

$$m_1^2 (1 - e^{m_1}) (e^{m_3} - e^{m_2}) + m_2^2 (1 - e^{m_2}) (e^{m_1} - e^{m_3}) + m_3^2 (1 - e^{m_3}) (e^{m_2} - e^{m_1}) = 0. \tag{8.14}$$

We shall now compute the determinant term by term for large $|\mu|$.

- Plugging the values of m_1, m_2 and m_3 as given in (8.11), we obtain

$$\begin{aligned} & m_1^2 (1 - e^{m_1}) (e^{m_3} - e^{m_2}) \\ &= \left(-\mu + b^2 + O(\mu^{-1/2})\right)^2 \left(1 - e^{-\mu + b^2 + O(\mu^{-1/2})}\right) \left(e^{-1/2 + i\mu^{1/2} + O(\mu^{-1/2})} - e^{-1/2 - i\mu^{1/2} + O(\mu^{-1/2})}\right) \\ &= \left(\mu^2 - 2b^2\mu + O(\mu^{1/2})\right) \left(1 - e^{-\mu + b^2 + O(\mu^{-1})}\right) \left(e^{-1/2 + O(\mu^{-1/2})} \left(\cos(\mu^{1/2}) + i \sin(\mu^{1/2})\right)\right. \\ &\qquad \left. - e^{-1/2 + O(\mu^{-1/2})} \left(\cos(\mu^{1/2}) - i \sin(\mu^{1/2})\right)\right) \\ &= \left(\mu^2 - 2b^2\mu + O(\mu^{1/2})\right) \left(1 - e^{-\mu + b^2 + O(\mu^{-1})}\right) \Big[O(\mu^{-1/2}) e^{-1/2 + O(\mu^{-1/2})} \cos(\mu^{1/2}) \\ &\qquad + i(2 + O(\mu^{-\frac{1}{2}})) e^{-1/2 + O(\mu^{-1/2})} \sin(\mu^{1/2})\Big], \end{aligned} \tag{8.15}$$

where we have used the facts that

$$e^{-1/2 + O(\mu^{-1/2})} - e^{-1/2 + O(\mu^{-1/2})} = e^{-1/2 + O(\mu^{-1/2})} \left(1 - e^{O(\mu^{-\frac{1}{2}})}\right) = e^{-1/2 + O(\mu^{-1/2})} \times O(\mu^{-\frac{1}{2}}),$$

and

$$e^{-1/2 + O(\mu^{-1/2})} + e^{-1/2 + O(\mu^{-1/2})} = e^{-1/2 + O(\mu^{-1/2})} \left(1 + e^{O(\mu^{-\frac{1}{2}})}\right) = e^{-1/2 + O(\mu^{-1/2})} \times (2 + O(\mu^{-\frac{1}{2}})).$$

- We also compute

$$\begin{aligned} & m_2^2 (1 - e^{m_2}) (e^{m_1} - e^{m_3}) \\ &= \left(-\frac{1}{2} - i\mu^{1/2} + O(\mu^{-\frac{1}{2}})\right)^2 \left(1 - e^{-1/2 - i\mu^{1/2} + O(\mu^{-1/2})}\right) \left(e^{-\mu + b^2 + O(\mu^{-1})} - e^{-\frac{1}{2} + i\mu^{\frac{1}{2}} + O(\mu^{-\frac{1}{2}})}\right) \\ &= \left(-\mu + i\mu^{\frac{1}{2}} + O(1)\right) \left(e^{-\mu + b^2 + O(\mu^{-1})} + e^{-1 + O(\mu^{-\frac{1}{2}})} - e^{-\mu + b^2 - \frac{1}{2} - i\mu^{\frac{1}{2}} + O(\mu^{-\frac{1}{2}})} - e^{-\frac{1}{2} + i\mu^{\frac{1}{2}} + O(\mu^{-\frac{1}{2}})}\right) \\ &= \left(-\mu + i\mu^{\frac{1}{2}} + O(1)\right) \Big[e^{-\mu + b^2 + O(\mu^{-1})} + e^{-1 + O(\mu^{-\frac{1}{2}})} - e^{-\mu + b^2 - \frac{1}{2} + O(\mu^{-\frac{1}{2}})} \left(\cos(\mu^{\frac{1}{2}}) - i \sin(\mu^{\frac{1}{2}})\right) \\ &\qquad - e^{-\frac{1}{2} + O(\mu^{-\frac{1}{2}})} \left(\cos(\mu^{\frac{1}{2}}) + i \sin(\mu^{\frac{1}{2}})\right)\Big]. \end{aligned}$$

- Finally, we have

$$
\begin{aligned}
&m_3^2\,(1-e^{m_3})\,(e^{m_2}-e^{m_1})\\
&=\Big(-\frac{1}{2}+i\mu^{1/2}+O(\mu^{-\frac12})\Big)^2\Big(1-e^{-\frac12+i\mu^{\frac12}+O(\mu^{-\frac12})}\Big)\times\Big(e^{-\frac12-i\mu^{\frac12}+O(\mu^{-\frac12})}-e^{-\mu+b^2+O(\mu^{-1})}\Big)\\
&=\Big(-\mu-i\mu^{\frac12}+O(1)\Big)\Big[-e^{-\mu+b^2+O(\mu^{-1})}-e^{-1+O(\mu^{-\frac12})}+e^{-\mu+b^2-\frac12+O(\mu^{-\frac12})}\Big(\cos(\mu^{\frac12})+i\sin(\mu^{\frac12})\Big)\\
&\qquad\qquad+e^{-\frac12+O(\mu^{-\frac12})}\Big(\cos(\mu^{\frac12})-i\sin(\mu^{\frac12})\Big)\Big].
\end{aligned}
$$

- We add now the last two terms, in what follows

$$
\begin{aligned}
(8.16)\qquad &m_2^2\,(1-e^{m_2})\,(e^{m_1}-e^{m_3})+m_3^2\,(1-e^{m_3})\,(e^{m_2}-e^{m_1})\\
&=-\,\mu O(\mu^{-\frac12})e^{-\mu+b^2+O(\mu^{-1})}-\mu O(\mu^{-\frac12})e^{-1+O(\mu^{-\frac12})}+\big(2i\mu^{\frac12}+O(1)\big)e^{-\mu+b^2+O(\mu^{-1})}\\
&\quad+\big(2i\mu^{\frac12}+O(1)\big)e^{-1+O(\mu^{-\frac12})}+i\sin\mu^{\frac12}\Big[\big(-2\mu+O(\mu^{\frac12})\big)e^{-\mu+b^2-\frac12+O(\mu^{-\frac12})}\\
&\quad+i\mu^{\frac12}O(\mu^{-\frac12})e^{-\mu+b^2-\frac12+O(\mu^{-\frac12})}+\big(2\mu+O(\mu^{\frac12})\big)e^{-\frac12+O(\mu^{-\frac12})}+i\mu^{\frac12}O(\mu^{-\frac12})e^{-\frac12+O(\mu^{-\frac12})}\Big]\\
&\quad+\cos\mu^{\frac12}\Big[\big(\mu+O(1)\big)O(\mu^{-\frac12})e^{-\mu+b^2-\frac12+O(\mu^{-\frac12})}+\big(\mu+O(1)\big)O(\mu^{-\frac12})e^{-\frac12+O(\mu^{-\frac12})}\\
&\qquad-i\mu^{\frac12}(2+O(\mu^{-\frac12}))e^{-\mu+b^2-\frac12+O(\mu^{-\frac12})}-i\mu^{\frac12}(2+O(\mu^{-\frac12}))e^{-\frac12+O(\mu^{-\frac12})}\Big].
\end{aligned}
$$

We get after adding (8.15) and (8.16),

$$
\begin{aligned}
&m_1^2\,(1-e^{m_1})\,(e^{m_3}-e^{m_2})+m_2^2\,(1-e^{m_2})\,(e^{m_1}-e^{m_3})+m_3^2\,(1-e^{m_3})\,(e^{m_2}-e^{m_1})\\
&=-\,\mu O(\mu^{-\frac12})e^{-\mu+b^2+O(\mu^{-1})}-\mu O(\mu^{-\frac12})e^{-1+O(\mu^{-\frac12})}\\
&\quad+\big(2i\mu^{\frac12}+O(1)\big)e^{-\mu+b^2+O(\mu^{-1})}+\big(2i\mu^{\frac12}+O(1)\big)e^{-1+O(\mu^{-\frac12})}\\
&\quad+i\sin\mu^{\frac12}\Big[\Big(-2\mu^2+O(\mu^{\frac32})\Big)e^{-\mu+b^2-\frac12+O(\mu^{-\frac12})}+\Big(2\mu^2+O(\mu^{\frac32})\Big)e^{-\frac12+O(\mu^{-\frac12})}\Big]\\
&\quad+\cos\mu^{\frac12}\Big[\Big(-\mu^2O(\mu^{-\frac12})+(2b^2+1)\mu O(\mu^{-\frac12})-2i\mu^{\frac12}+O(1)\Big)e^{-\mu+b^2-\frac12+O(\mu^{-\frac12})}\\
&\qquad+\Big(\mu^2O(\mu^{-\frac12})-\mu O(\mu^{-\frac12})-2i\mu^{\frac12}+O(1)\Big)e^{-\frac12+O(\mu^{-\frac12})}\Big].
\end{aligned}
$$

Now, replacing the above quantity in the equation (8.14), and then dividing it by μ^2 (since $\mu\neq 0$), we obtain the equation

$$
(8.17)\qquad F(\mu)=0,
$$

where

$$
\begin{aligned}
F(\mu)=&-2\sin\mu^{\frac12}\Big(e^{-\mu+b^2}-1\Big)+O(\mu^{-\frac12})\sin\mu^{\frac12}\,e^{-\mu+b^2+O(\mu^{-\frac12})}+O(\mu^{-\frac12})\sin\mu^{\frac12}\,e^{O(\mu^{-\frac12})}\\
&+\cos\mu^{\frac12}\Big[O(\mu^{-\frac12})e^{-\mu+b^2+O(\mu^{-\frac12})}+O(\mu^{-\frac12})e^{O(\mu^{-\frac12})}\Big]\\
&+O(\mu^{-\frac32})e^{-\mu+b^2+\frac12+O(\mu^{-\frac12})}+O(\mu^{-\frac32})e^{-\frac12+O(\mu^{-\frac12})}.
\end{aligned}
$$

Application of Rouche's theorem. Let G be a function of μ, defined as

$$
G(\mu)=-2\sin(\mu^{\frac12})\Big(e^{-\mu+b^2}-1\Big).
$$

Then

$$\begin{aligned} F(\mu)-G(\mu) =& \underbrace{\sin(\mu^{\frac{1}{2}})\left(O(\mu^{-\frac{1}{2}})e^{-\mu+b^2+O(\mu^{-\frac{1}{2}})}+O(\mu^{-\frac{1}{2}})e^{O(\mu^{-\frac{1}{2}})}\right)}_{I_1} \\ &+\underbrace{\cos(\mu^{\frac{1}{2}})\left(O(\mu^{-\frac{1}{2}})e^{-\mu+b^2+O(\mu^{-\frac{1}{2}})}+O(\mu^{-\frac{1}{2}})e^{O(\mu^{-\frac{1}{2}})}\right)}_{I_2} \\ &+\underbrace{O(\mu^{-\frac{3}{2}})e^{-\mu+b^2+\frac{1}{2}+O(\mu^{-\frac{1}{2}})}+O(\mu^{-\frac{3}{2}})e^{-\frac{1}{2}+O(\mu^{-\frac{1}{2}})}}_{I_3} \\ :=& I_1+I_2+I_3. \end{aligned}$$

Since the function G has two branches of zeros, we will calculate them separately and in each case, we use the Rouche's theorem to talk about the zeros of the function F.

Case 1. We first observe that $\mu = k^2\pi^2$ is a zero of G for each $k \in \mathbb{N}^*$ and consider the following region in the complex plane

$$\mathcal{R}_k = \left\{z = x+iy \in \mathbb{C} \ : \ k\pi - \frac{\pi}{2} \le x \le k\pi + \frac{\pi}{2}, \ \ -\frac{\pi}{2} \le y \le \frac{\pi}{2}\right\}, \quad \text{for } k \in \mathbb{N}^*. \tag{8.18}$$

Our goal is to prove that $|F(\mu) - G(\mu)| < |G(\mu)|$ on $\partial\mathcal{R}_k$. It is sufficient to prove that

$$\left|\frac{F(\mu)-G(\mu)}{G(\mu)}\right| \to 0 \ \text{ for } \mu \in \partial\mathcal{R}_k \text{ such that } \mathrm{Re}(\mu) \to +\infty. \tag{8.19}$$

To avoid difficulties in notations, we denote $w = \mu^{\frac{1}{2}}$ and without loss of generality, we simply write I_1, I_2 and I_3 as the functions w. Note that

$$\left|\frac{I_1(w)}{G(w)}\right| = \left|\frac{O(w^{-1})e^{-w^2+b^2+O(w^{-1})}+O(w^{-1})e^{O(w^{-1})}}{e^{-w^2+b^2}-1}\right| \le \frac{C}{|w|}\frac{\left|e^{-w^2+b^2}\right|+1}{\left|e^{-w^2+b^2}-1\right|},$$

and since $\frac{\left|e^{-w^2+b^2}\right|+1}{\left|e^{-w^2+b^2}-1\right|}$ is bounded when $\mathrm{Re}(w) \to +\infty$, therefore

$$\left|\frac{I_1(w)}{G(w)}\right| \to 0, \quad \text{as } \mathrm{Re}(w) \to +\infty.$$

We now compute

$$\left|\frac{I_2(w)}{G(w)}\right| = \left|\frac{\cos(w)}{\sin(w)}\right| \frac{\left|O(w^{-1})e^{-w^2+b^2+O(w^{-1})}+O(w^{-1})e^{O(w^{-1})}\right|}{\left|e^{-w^2+b^2}-1\right|} \le \frac{C}{|w|}\left|\frac{\cos(w)}{\sin(w)}\right| \frac{\left|e^{-w^2+b^2}\right|+1}{\left|e^{-w^2+b^2}-1\right|},$$

which yields

$$\left|\frac{I_2(w)}{G(w)}\right| \to 0, \quad \text{for } w \in \partial\mathcal{R}_k \text{ such that } \mathrm{Re}(w) \to +\infty,$$

because of the fact that $\left|\frac{\cos(w)}{\sin(w)}\right|$ is bounded on $\partial\mathcal{R}_k$. We can say similarly for the third term that

$$\left|\frac{I_3(w)}{G(w)}\right| \to 0, \quad \text{for } w \in \partial\mathcal{R}_k \text{ such that } \mathrm{Re}(w) \to +\infty,$$

as we have

$$\left|\frac{I_3(w)}{G(w)}\right| \le \frac{C}{|w|^3}\left|\frac{1}{\sin(w)}\right| \frac{\left|e^{-w^2+b^2+\frac{1}{2}}\right|+1}{\left|e^{-w^2+b^2}-1\right|}.$$

Case 2. When $\sin(\mu^{\frac{1}{2}}) \neq 0$, $G(\mu) = 0$ gives $e^{-\mu+b^2} - 1 = 0$, that is $\mu = b^2 + 2ik\pi$ for $k \in \mathbb{Z}$. In this case, we consider the following region in the complex plane

$$\mathcal{S}_k = \left\{z = x+iy \in \mathbb{C} \ : \ b^2 - \frac{\pi}{2} \le x \le b^2 + \frac{\pi}{2}, \ \ 2k\pi - \frac{\pi}{2} \le y \le 2k\pi + \frac{\pi}{2}\right\}. \tag{8.20}$$

We need to show that $|F(\mu) - G(\mu)| < |G(\mu)|$ on $\partial\mathcal{S}_k$. In particular, we prove that

$$\left|\frac{F(\mu)-G(\mu)}{G(\mu)}\right| \to 0 \ \text{ for } \mu \in \partial\mathcal{S}_k \text{ such that } \mathrm{Im}(\mu) \to +\infty.$$

We compute

$$\left|\frac{I_1(\mu)}{G(\mu)}\right| = \frac{1}{\left|\sin(\mu^{\frac{1}{2}})\right|}\left|\frac{O(\mu^{-\frac{1}{2}})e^{-\mu+b^2+O(\mu^{-\frac{1}{2}})} + O(\mu^{-1})e^{O(\mu^{-\frac{1}{2}})}}{e^{-\mu+b^2}-1}\right| \leq \frac{C}{|\mu|^{\frac{1}{2}}}\frac{1}{\left|\sin(\mu^{\frac{1}{2}})\right|}\frac{\left|e^{-\mu+b^2}\right|+1}{\left|e^{-\mu+b^2}-1\right|},$$

$$\left|\frac{I_2(\mu)}{G(\mu)}\right| = \left|\frac{\cos(\mu^{\frac{1}{2}})}{\sin(\mu^{\frac{1}{2}})}\right|\frac{\left|O(\mu^{-\frac{1}{2}})e^{-\mu+b^2+O(\mu^{-\frac{1}{2}})} + O(\mu^{-\frac{1}{2}})e^{O(\mu^{-\frac{1}{2}})}\right|}{\left|e^{-\mu+b^2}-1\right|} \leq \frac{C}{|\mu|^{\frac{1}{2}}}\left|\frac{\cos(\mu^{\frac{1}{2}})}{\sin(\mu^{\frac{1}{2}})}\right|\frac{\left|e^{-\mu+b^2}\right|+1}{\left|e^{-\mu+b^2}-1\right|},$$

and

$$\left|\frac{I_3(\mu)}{G(\mu)}\right| \leq \frac{C}{|\mu|^{\frac{3}{2}}}\frac{1}{\left|\sin(\mu^{\frac{1}{2}})\right|}\frac{\left|e^{-\mu+b^2+\frac{1}{2}}\right|+1}{\left|e^{-\mu+b^2}-1\right|}.$$

On $\partial\mathcal{S}_k$, $\left|\cos(\mu^{\frac{1}{2}})\right|$ and $\left|\sin(\mu^{\frac{1}{2}})\right|$ has both lower and upper bounds and $\frac{\left|e^{-\mu+b^2}\right|+1}{\left|e^{-\mu+b^2}-1\right|}$, $\frac{\left|e^{-\mu+b^2+\frac{1}{2}}\right|+1}{\left|e^{-\mu+b^2}-1\right|}$ are bounded. Therefore, for each $j=1,2,3$, we have

$$\left|\frac{I_j(\mu)}{G(\mu)}\right| \to 0, \quad \text{for } \mu\in\partial\mathcal{S}_k \text{ such that } \mathrm{Im}(\mu)\to+\infty.$$

Thus, combining the above two cases, we conclude that there exists some $k_0\in\mathbb{N}^*$ sufficiently large, such that

$$|F(\mu)-G(\mu)| < |G(\mu)|, \quad \forall\mu\in\partial\mathcal{R}_k\cup\partial\mathcal{S}_k \text{ and for large } k. \tag{8.21}$$

Since any two regions $\mathcal{R}_k$ and $\mathcal{R}_l$ are disjoint for $k\neq l$ and in each region $\mathcal{R}_k$, there is exactly one root of G (more precisely, the square-root of μ), the same is true for the function F, thanks to the Rouche's theorem. Similar phenomenon holds true in the region $\mathcal{S}_k$. To be more precise, we have the following.

On the region $\mathcal{R}_k$: parabolic part. For $k\geq k_0$, the function F has a unique root in $\mathcal{R}_k$ of the form

$$\mu_k^{\frac{1}{2}} = (k\pi+c_k)+id_k,$$

with $|c_k|,\ |d_k|\leq\frac{\pi}{2}$. Therefore, the first set of eigenvalues are given by

$$\lambda_k^p := -\mu_k := -k^2\pi^2 - 2c_kk\pi - 2id_kk\pi - (c_k^2-d_k^2) - 2ic_kd_k, \quad \forall k\geq k_0. \tag{8.22}$$

On the region $\mathcal{S}_k$: hyperbolic part. On the other hand, for $|k|\geq k_0$, the function F has a unique root in $\mathcal{S}_k$ of the form

$$\tilde{\mu}_k = b^2+\alpha_{1,k}+i(2k\pi+\alpha_{2,k}),$$

with $|\alpha_{1,k}|,\ |\alpha_{2,k}|\leq\frac{\pi}{2}$.

Therefore, the second set of eigenvalues are given by

$$\lambda_k^h := -\tilde{\mu}_k := -b^2-\alpha_{1,k}-i(2k\pi+\alpha_{2,k}), \quad \forall|k|\geq k_0. \tag{8.23}$$

This indeed proves the results (3.2a) and (3.2b) of our Lemma 3.2.

8.2. **Computing the eigenfunctions for large frequencies.** From the set of equations (8.13), one can obtain the following values of C_1, C_2, C_3

$$\begin{cases} C_1 = e^{m_2}-e^{m_3}, \\ C_2 = e^{m_3}-e^{m_1}, \\ C_3 = e^{m_1}-e^{m_2}. \end{cases} \tag{8.24}$$

Note that C_1, C_2 and C_3 cannot be simultaneously zero for large $|\mu|$. Once we have that, one can easily obtain the function $\eta(x)$, defined by (8.12),

$$\eta(x) = (e^{m_2}-e^{m_3})e^{m_1x} + (e^{m_3}-e^{m_1})e^{m_2x} + (e^{m_1}-e^{m_2})e^{m_3x}, \quad \forall x\in(0,1). \tag{8.25}$$

We now compute the first and second derivatives of η which will let us obtain the other component ξ of the set of equations (8.1). We see

$$\eta'(x) = m_1(e^{m_2}-e^{m_3})e^{m_1x} + m_2(e^{m_3}-e^{m_1})e^{m_2x} + m_3(e^{m_1}-e^{m_2})e^{m_3x},$$

$$\eta''(x) = m_1^2(e^{m_2}-e^{m_3})e^{m_1x} + m_2^2(e^{m_3}-e^{m_1})e^{m_2x} + m_3^2(e^{m_1}-e^{m_2})e^{m_3x}.$$

Now, from equation (8.1), one can obtain

$$\eta''(x) + (1-b^2)\eta'(x) + b\lambda\xi(x) = \lambda\eta(x),$$

and therefore, (writing $\mu = -\lambda$)

$$\begin{aligned}
\xi(x) =& \frac{\eta''(x) + (1-b^2)\eta'(x) + \mu\eta(x)}{b\mu} \\
=& \Big(\frac{m_1^2 + (1-b^2)m_1 + \mu}{b\mu}\Big)(e^{m_2} - e^{m_3})e^{m_1 x} + \Big(\frac{m_2^2 + (1-b^2)m_2 + \mu}{b\mu}\Big)(e^{m_3} - e^{m_1})e^{m_2 x} \\
&+ \Big(\frac{m_3^2 + (1-b^2)m_3 + \mu}{b\mu}\Big)(e^{m_1} - e^{m_2})e^{m_3 x}.
\end{aligned} \tag{8.26}$$

Set of eigenfunctions associated with λ_k^p. For the set of eigenvalues $\{\lambda_k^p\}_{k\geq k_0}$ given by (8.22), we denote the eigenfunctions by $\Phi_{\lambda_k^p}$, $\forall k \geq k_0$, where we shall use the notation

$$\Phi_{\lambda_k^p}(x) = \begin{pmatrix} \xi_{\lambda_k^p}(x) \\ \eta_{\lambda_k^p}(x) \end{pmatrix}, \quad \forall k \geq k_0. \tag{8.27}$$

Computing $\eta_{\lambda_k^p}$. Let us recall the values of m_1, m_2 and m_3 from (8.11) and observe that $O(\mu_k^{-1/2}) = O(k^{-1})$. In what follows, we have their explicit expressions for all $k \geq k_0$ large enough, given by

$$\begin{cases}
m_1 = -k^2\pi^2 - 2c_k k\pi - 2id_k k\pi + O(1), \\
m_2 = -\dfrac{1}{2} + d_k - i(k\pi + c_k) + O(k^{-1}), \\
m_3 = -\dfrac{1}{2} - d_k + i(k\pi + c_k) + O(k^{-1}).
\end{cases} \tag{8.28}$$

where we have used the expression of $\mu = \mu_k$ from (8.22).

Recall the values of m_1, m_2, m_3, given by (8.28) and from the expression (8.25), we get that

$$\begin{aligned}
\eta_{\lambda_k^p}(x) =& \left(e^{-\frac{1}{2}+d_k - i(k\pi+c_k)+O(k^{-1})} - e^{-\frac{1}{2}-d_k+i(k\pi+c_k)+O(k^{-1})}\right) e^{x\left(-k^2\pi^2 - 2c_k k\pi - 2id_k k\pi + O(1)\right)} \\
&+ \left(e^{-\frac{1}{2}-d_k+i(k\pi+c_k)+O(k^{-1})} - e^{-k^2\pi^2-2c_k k\pi - 2id_k k\pi+O(1)}\right) e^{x\left(-i(k\pi+c_k)-\frac{1}{2}+d_k+O(k^{-1})\right)} \\
&+ \left(e^{-k^2\pi^2-2c_k k\pi-2id_k k\pi+O(1)} - e^{-\frac{1}{2}+d_k-i(k\pi+c_k)+O(k^{-1})}\right) e^{x\left(i(k\pi+c_k)-\frac{1}{2}-d_k+O(k^{-1})\right)},
\end{aligned} \tag{8.29}$$

for all $x \in (0,1)$ and for all $k \geq k_0$ large enough.

Computing $\xi_{\lambda_k^p}$. By using the values of m_1, m_2, m_3 from (8.28), we have

$$\begin{cases}
m_1^2 = k^4\pi^4 + 4c_k k^3\pi^3 + 4id_k k^3\pi^3 + O(k^2), \\
m_2^2 = -k^2\pi^2 - 2c_k k\pi + ik\pi - 2id_k k\pi + O(1), \\
m_3^2 = -k^2\pi^2 - 2c_k k\pi - ik\pi - 2id_k k\pi + O(1),
\end{cases}$$

for all $k \geq k_0$ large enough.

Also recall that, $\mu_k = -\lambda_k^p = k^2\pi^2 + 2c_k k\pi + 2id_k k\pi + O(1)$, using which we find

$$\frac{m_1^2 + (1-b^2)m_1 + \mu_k}{b\mu_k} = \frac{1}{b}k^2\pi^2 + O(k), \tag{8.30}$$

$$\frac{m_2^2 + (1-b^2)m_2 + \mu_k}{b\mu_k} = \frac{ib}{k\pi} + O(k^{-2}), \tag{8.31}$$

$$\frac{m_3^2 + (1-b^2)m_3 + \mu_k}{b\mu_k} = -\frac{ib}{k\pi} + O(k^{-2}), \tag{8.32}$$

for all $k \geq k_0$ large enough.

Now, by using the quantities (8.30), (8.31) and (8.32) in the expression (8.26), we obtain

$$\begin{aligned}\xi_{\lambda_k^p}(x) &= \left(\frac{k^2\pi^2}{b} + O(k)\right)\left(e^{-i(k\pi+c_k)-\frac{1}{2}+d_k+O(k^{-1})} - e^{i(k\pi+c_k)-\frac{1}{2}-d_k+O(k^{-1})}\right)\\ &\qquad \times e^{x\left(-k^2\pi^2-2c_k k\pi-2id_k k\pi+O(1)\right)}\\ &+ \left(\frac{ib}{k\pi}+O(\frac{1}{k^2})\right)\left(e^{i(k\pi+c_k)+O(k^{-1})-\frac{1}{2}-d_k} - e^{-k^2\pi^2-2c_k k\pi-2id_k k\pi+O(1)}\right)e^{x\left(-i(k\pi+c_k)-\frac{1}{2}+d_k+O(k^{-1})\right)}\\ &- \left(\frac{ib}{k\pi}+O(\frac{1}{k^2})\right)\left(e^{-k^2\pi^2-2c_k k\pi-2id_k k\pi+O(1)} - e^{-i(k\pi+c_k)-\frac{1}{2}+d_k+O(k^{-1})}\right)e^{x\left(i(k\pi+c_k)-\frac{1}{2}-d_k+O(k^{-1})\right)}.\end{aligned} \tag{8.33}$$

Set of eigenfunctions associated with λ_k^h. For the set of eigenvalues $\{\lambda_k^h\}_{|k|\geq k_0}$ given by (8.23), we denote the eigenfunctions by $\Phi_{\lambda_k^h}$, where we shall use the notation

$$\Phi_{\lambda_k^h}(x) = \begin{pmatrix}\xi_{\lambda_k^h}(x)\\ \eta_{\lambda_k^h}(x)\end{pmatrix}, \quad \forall |k|\geq k_0. \tag{8.34}$$

Computing $\eta_{\lambda_k^h}$. Recall that $\tilde{\mu}_k = -\lambda_k^h = b^2 + \alpha_{1,k} + i(2k\pi + \alpha_{2,k})$, for all $|k|\geq k_0$, so that we get

$$\tilde{\mu}_k^{1/2} = \sqrt{|k\pi|} + i\,\mathrm{sgn}(k)\sqrt{|k\pi|} + O(|k|^{-\frac{1}{2}}), \quad \forall |k|\geq k_0, \tag{8.35}$$

(the sign function sgn has been defined by (3.12)).

Then, using the characteristic roots m_1, m_2, m_3, given by (8.11), we get that

$$\begin{cases} m_1 = -\alpha_{1,k} - i(2k\pi+\alpha_{2,k}) + O(|k|^{-1}),\\ m_2 = -\dfrac{1}{2} + \mathrm{sgn}(k)\sqrt{|k\pi|} - i\sqrt{|k\pi|} + O(|k|^{-\frac{1}{2}}),\\ m_3 = -\dfrac{1}{2} - \mathrm{sgn}(k)\sqrt{|k\pi|} + i\sqrt{|k\pi|} + O(|k|^{-\frac{1}{2}}),\end{cases} \tag{8.36}$$

for all $|k|\geq k_0$ large enough.

Using the above information, we now write the expression of $\eta_{\lambda_k^h}(x)$ (we take the formulation after dividing by $k\pi e^{\sqrt{|k\pi|}+\frac{1}{\sqrt{|k|}}}$), given by

$$\begin{aligned}\eta_{\lambda_k^h}(x) &= \frac{1}{k\pi e^{\sqrt{|k\pi|}+\frac{1}{\sqrt{|k|}}}}\left(e^{\mathrm{sgn}(k)\sqrt{|k\pi|}-\frac{1}{2}-i\sqrt{|k\pi|}+O(|k|^{-\frac{1}{2}})} - e^{-\mathrm{sgn}(k)\sqrt{|k\pi|}-\frac{1}{2}+i\sqrt{|k\pi|}+O(|k|^{-\frac{1}{2}})}\right)\\ &\qquad \times e^{-x\left(\alpha_{1,k}+i(2k\pi+\alpha_{2,k})+O(|k|^{-1})\right)}\\ &+ \frac{1}{k\pi e^{\sqrt{|k\pi|}+\frac{1}{\sqrt{|k|}}}}\left(e^{-\mathrm{sgn}(k)\sqrt{|k\pi|}-\frac{1}{2}+i\sqrt{|k\pi|}+O(|k|^{-\frac{1}{2}})} - e^{-\alpha_{1,k}-i(2k\pi+\alpha_{2,k})+O(|k|^{-1})}\right)\\ &\qquad \times e^{x\left(\mathrm{sgn}(k)\sqrt{|k\pi|}-\frac{1}{2}-i\sqrt{|k\pi|}+O(|k|^{-\frac{1}{2}})\right)}\\ &+ \frac{1}{k\pi e^{\sqrt{|k\pi|}+\frac{1}{\sqrt{|k|}}}}\left(e^{-\alpha_{1,k}-i(2k\pi+\alpha_{2,k})+O(|k|^{-1})} - e^{\mathrm{sgn}(k)\sqrt{|k\pi|}-\frac{1}{2}-i\sqrt{|k\pi|}+O(|k|^{-\frac{1}{2}})}\right)\\ &\qquad \times e^{x\left(-\mathrm{sgn}(k)\sqrt{|k\pi|}-\frac{1}{2}+i\sqrt{|k\pi|}+O(|k|^{-\frac{1}{2}})\right)},\end{aligned} \tag{8.37}$$

for all $x\in(0,1)$ and for all $|k|\geq k_0$.

Computing $\xi_{\lambda_k^h}$. By using the values of m_1, m_2, m_3 from (8.36), we calculate the following quantities for all $|k|\geq k_0$ large enough, namely

$$\begin{cases} m_1^2 = -4k^2\pi^2 + 4ik\pi\alpha_{1,k} + O(k),\\ m_2^2 = -\mathrm{sgn}(k)\sqrt{|k\pi|} - 2i\,\mathrm{sgn}(k)|k\pi| + i\sqrt{|k\pi|} + O(1),\\ m_3^2 = \mathrm{sgn}(k)\sqrt{|k\pi|} - 2i\,\mathrm{sgn}(k)|k\pi| - i\sqrt{|k\pi|} + O(1).\end{cases}$$

Next, we compute the following: for all $|k| \geq k_0$ large enough,

$$\frac{m_1^2 + (1-b^2)m_1 + \tilde{\mu}_k}{b\tilde{\mu}_k} = -\frac{\alpha_{1,k}}{b} + \frac{2ik\pi}{b} + O(1), \tag{8.38}$$

$$\frac{m_2^2 + (1-b^2)m_2 + \tilde{\mu}_k}{b\tilde{\mu}_k} = \operatorname{sgn}(k)\frac{b}{2\sqrt{|k\pi|}} + \frac{ib}{2\sqrt{|k\pi|}} + O\Big(\frac{1}{|k|}\Big), \tag{8.39}$$

$$\frac{m_3^2 + (1-b^2)m_3 + \tilde{\mu}_k}{b\tilde{\mu}_k} = -\operatorname{sgn}(k)\frac{b}{2\sqrt{|k\pi|}} - \frac{ib}{2\sqrt{|k\pi|}} + O\Big(\frac{1}{|k|}\Big). \tag{8.40}$$

Using the quantities (8.38), (8.39) and (8.40) in the expression (8.26), we obtain the component $\xi_{\lambda_k^h}(x)$, for all $|k| \geq k_0$ (upon a division by $k\pi e^{\sqrt{|k\pi|}+\frac{1}{\sqrt{|k|}}}$),

$$\begin{aligned}
\xi_{\lambda_k^h}(x) = &\Big(e^{\operatorname{sgn}(k)\sqrt{|k\pi|}-\frac{1}{2}-i\sqrt{|k\pi|}+O(|k|^{-\frac{1}{2}})} - e^{-\operatorname{sgn}(k)\sqrt{|k\pi|}-\frac{1}{2}+i\sqrt{|k\pi|}+O(|k|^{-\frac{1}{2}})}\Big) \\
&\times \frac{(-\alpha_{1,k} + 2ik\pi + O(1))}{bk\pi e^{\sqrt{|k\pi|}+\frac{1}{\sqrt{|k|}}}} \times e^{-x\left(\alpha_{1,k}+i(2k\pi+\alpha_{2,k})+O(|k|^{-1})\right)} \\
&+ \Big(e^{-\operatorname{sgn}(k)\sqrt{|k\pi|}-\frac{1}{2}+i\sqrt{|k\pi|}+O(|k|^{-\frac{1}{2}})} - e^{-\alpha_{1,k}-i(2k\pi+\alpha_{2,k})+O(|k|^{-1})}\Big) \\
\times \frac{1}{k\pi e^{\sqrt{|k\pi|}+\frac{1}{\sqrt{|k|}}}} &\Big(\operatorname{sgn}(k)\frac{b}{2\sqrt{|k\pi|}} + \frac{ib}{2\sqrt{|k\pi|}} + O\Big(\frac{1}{|k|}\Big)\Big) \times e^{x\left(\operatorname{sgn}(k)\sqrt{|k\pi|}-\frac{1}{2}-i\sqrt{|k\pi|}+O(|k|^{-\frac{1}{2}})\right)} \\
&+ \Big(e^{-\alpha_{1,k}-i(2k\pi+\alpha_{2,k})+O(|k|^{-1})} - e^{\operatorname{sgn}(k)\sqrt{|k\pi|}-\frac{1}{2}-i\sqrt{|k\pi|}+O(|k|^{-\frac{1}{2}})}\Big) \\
\times \frac{1}{k\pi e^{\sqrt{|k\pi|}+\frac{1}{\sqrt{|k|}}}} &\Big(-\operatorname{sgn}(k)\frac{b}{2\sqrt{|k\pi|}} - \frac{ib}{2\sqrt{|k\pi|}} + O\Big(\frac{1}{|k|}\Big)\Big) \times e^{x\left(-\operatorname{sgn}(k)\sqrt{|k\pi|}-\frac{1}{2}+i\sqrt{|k\pi|}+O(|k|^{-\frac{1}{2}})\right)},
\end{aligned} \tag{8.41}$$

We can now prove the last part of Lemma 3.2.

8.3. **Proof of Lemma 3.2.** We have already proved the existence of eigenvalues $\{\lambda_k^p\}_{k\geq k_0}$ (parabolic part) and $\{\lambda_k^h\}_{|k|\geq k_0}$ (hyperbolic part) by (8.22) and (8.23) respectively, which is the first part of Lemma 3.2.

It lefts to show the asymptotic properties of the sequences $\{c_k\}_{k\geq k_0}$, $\{d_k\}_{k\geq k_0}$ and $\{\alpha_{1,k}\}_{|k|\geq k_0}$, $\{\alpha_{1,k}\}_{|k|\geq k_0}$.

- Let us use the form of μ_k (i.e., of $-\lambda_k^p$) in the eigenvalue equation (8.17). Then, for large k, it is easy to observe that

$$\begin{aligned}
F(\mu_k) &= 2\sin(k\pi + c_k + id_k) + O(k^{-1}) \\
&= 2(-1)^k \sin(c_k + id_k) + O(k^{-1}).
\end{aligned}$$

  But μ_k is a root of F and thus

$$\sin(c_k + id_k) = O(k^{-1}), \quad \text{for large } k \geq k_0. \tag{8.42}$$

  Now, since $|\sin(c_k + id_k)|^2 = \sin^2(c_k) + \sinh^2(d_k)$, we can write

$$\sin^2(c_k),\ \sinh^2(d_k) \leq \frac{C}{k^2}, \quad \forall k \geq k_0 \text{ large}.$$

  Therefore, $|c_k|^2$, $|d_k|^2 \leq \frac{C}{k^2}$, $\forall k \geq k_0$, that is to say,

$$c_k,\ d_k = O(k^{-1}), \quad \text{for large } k \geq k_0,$$

  which gives the asymptotic formulation (3.2a) of λ_k^p given in Lemma 3.2.
- For the hyperbolic part $\{\lambda_k^h\}_{|k|\geq k_0}$, using the property $\xi_{\lambda_k^h}(0) = \xi_{\lambda_k^h}(1)$ ($\xi_{\lambda_k^h}$ is defined by (8.41)), we obtain that

$$\Big(1 - e^{-\alpha_{1,k}-i2k\pi-i\alpha_{2,k}+O(|k|^{-1})}\Big) + O(|k|^{-1}) = 0,$$

that is,

$$e^{-\alpha_{1,k}-i\alpha_{2,k}} = 1 + O(|k|^{-1}), \quad \text{for large } |k| \geq k_0. \tag{8.43}$$

that is, there exists a $C > 0$ such that

$$\left|e^{-\alpha_{1,k}-i\alpha_{2,k}}\right| \leq \left(1 + \frac{C}{|k|}\right), \quad \forall\, |k| \geq k_0 \text{ large.}$$

As a consequence,

$$e^{-\alpha_{1,k}-i\alpha_{2,k}} \to 1, \quad \text{as } |k| \to +\infty.$$

But both $\alpha_{1,k}$ and $\{\alpha_{2,k}\}$ are bounded, which implies

$$\alpha_{1,k},\ \alpha_{2,k} \to 0, \quad \text{as } |k| \to \infty. \tag{8.44}$$

Since $\left|e^{-\alpha_{1,k}-i\alpha_{2,k}}\right| = e^{-\alpha_{1,k}}$, we have $|\alpha_{1,k}| \leq \dfrac{C}{|k|}$, $\forall\, |k| \geq k_0$ large and that is

$$\alpha_{1,k} = O(k^{-1}), \quad \text{for large } |k| \geq k_0.$$

Using the above result, we get

$$e^{-i\alpha_{2,k}} = 1 + O(k^{-1}), \quad \text{for large } |k| \geq k_0.$$

But, one has $\left|e^{-i\alpha_{2,k}} - 1\right| = 2|\sin(\alpha_{2,k}/2)|$ and therefore,

$$|\alpha_{2,k}| \leq \frac{C}{|k|}, \quad \text{for large } |k| \geq k_0.$$

that is, $\alpha_{2,k} = O(|k|^{-1})$. This yields the asymptotic formulation (3.2b) of λ_k^h given in Lemma 3.2.

Finally, we recall that the existence of lower frequencies of eigenvalues are already given in Section 3.3.

Thus, the proof of Lemma 3.2 is complete.

8.4. **Proof of Proposition 3.3–Part 1.** In this portion, we shall simplify the expressions of the eigenfunctions (for large frequencies) using the properties of $c_k, d_k, \alpha_{1,k}, \alpha_{2,k}$ obtained in Section 8.3.

– *The parabolic part.* Recall the component $\xi_{\lambda_k^p}$ given by (8.33). By using the condition $\xi_{\lambda_k^p}(0) = \xi_{\lambda_k^p}(1)$, one can deduce that

$$\left(e^{-i(k\pi+c_k)-\frac{1}{2}+d_k+O(k^{-1})} - e^{i(k\pi+c_k)-\frac{1}{2}-d_k+O(k^{-1})}\right) = O\left(\frac{1}{k^3}\right), \quad \text{for large } k \geq k_0.$$

We further observe that (since c_k, d_k are of $O(1/k)$)

$$\begin{aligned}
&e^{i(1-x)(k\pi+c_k+id_k)+O(k^{-1})} - e^{-i(1-x)(k\pi+c_k+id_k)+O(k^{-1})} \\
&= 2i\sin((1-x)(k\pi+c_k+id_k)) + O(k^{-1}) \\
&\sim_{+\infty} 2i\sin(k\pi(1-x)) + O(k^{-1}).
\end{aligned}$$

Using the above ingredients in the expressions of $\eta_{\lambda_k^p}$ and $\xi_{\lambda_k^p}$ given by (8.29) and (8.33), we conclude that

$$\begin{aligned}
&\eta_{\lambda_k^p}(x) = e^{-\frac{1}{2}(1+x)}\sin(k\pi(1-x)) + O\left(\frac{1}{k}\right), \\
&\xi_{\lambda_k^p}(x) = \frac{ib}{k\pi}e^{-\frac{1}{2}(1+x)}\cos(k\pi(1-x)) + e^{x\left(-k^2\pi^2+O(1)\right)} \times O\left(\frac{1}{k}\right) + O\left(\frac{1}{k^2}\right),
\end{aligned}$$

for all $x \in (0,1)$.

– *The hyperbolic part.* For the hyperbolic part, we simply use the fact: $\alpha_{1,k} = O(|k|^{-1})$, $\alpha_{2,k} = O(|k|^{-1})$ in the expressions (8.37) and (8.41), to obtain the required formulations (3.10) and (3.11).

8.5. **Proof of Lemma 3.5: bounds of the eigenfunctions.** In this section, we shall give the sketch of the estimates for $\xi_{\lambda_k^p}$, $\eta_{\lambda_k^p}$ for $k \geq k_0$ and $\xi_{\lambda_k^h}$, $\eta_{\lambda_k^h}$ for $|k| \geq k_0$. We use the interpolation results of Sobolev spaces to find the $(H^s_\sharp(0,1))'$ and $H^{-s}(0,1)$-norms of the eigen-components.

We present the proof for $0 < s < 1$. In a similar way, one can prove the estimates for $s \geq 1$.

- *The parabolic part.* Recall the expressions of $\xi_{\lambda_k^p}$ and $\eta_{\lambda_k^p}$ from (3.7) and (3.8) respectively. Note that
$$\left\|\xi_{\lambda_k^p}\right\|_{L^2(0,1)} \leq \frac{C}{k} \quad \text{and} \quad \left\|\xi_{\lambda_k^p}\right\|_{(H^1_\sharp(0,1))'} \leq \frac{C}{k^2}, \quad \text{for } k \geq k_0 \text{ large.}$$
Therefore, using the interpolation between $(H^1_\sharp(0,1))'$ and $L^2(0,1)$ spaces, we get for any $0 < s < 1$ (since $-s = s \times (-1) + (1-s) \times 0$),
$$\left\|\xi_{\lambda_k^p}\right\|_{(H^s_\sharp(0,1))'} \leq \left\|\xi_{\lambda_k^p}\right\|^{1-s}_{L^2(0,1)} \left\|\xi_{\lambda_k^p}\right\|^{s}_{(H^1_\sharp(0,1))'} \leq \frac{C}{|k|^{1+s}}, \quad \text{for } k \geq k_0 \text{ large.}$$
We also have
$$\left\|\eta_{\lambda_k^p}\right\|_{L^2(0,1)} \leq C \quad \text{and} \quad \left\|\eta_{\lambda_k^p}\right\|_{H^{-1}(0,1)} \leq \frac{C}{k}, \quad \text{for } k \geq k_0 \text{ large.}$$
Thus, for any $0 < s < 1$, we deduce that
$$\left\|\eta_{\lambda_k^p}\right\|_{H^{-s}(0,1)} \leq \left\|\eta_{\lambda_k^p}\right\|^{1-s}_{L^2(0,1)} \left\|\eta_{\lambda_k^p}\right\|^{s}_{H^{-1}(0,1)} \leq \frac{C}{|k|^{s}}, \quad \text{for } k \geq k_0 \text{ large.}$$
On the other hand, to find the lower bounds, first we observe that
$$\left\|\xi_{\lambda_k^p}\right\|_{L^2(0,1)} \geq \frac{C}{k} \quad \text{and} \quad \left\|\xi_{\lambda_k^p}\right\|_{H^1_\sharp(0,1)} \geq C, \quad \text{for } k \geq k_0 \text{ large.}$$
Now, using the interpolation between $(H^s_\sharp(0,1))'$ for $0 < s < 1$ and $H^1_\sharp(0,1)$, we obtain that (as $0 = \frac{1}{1+s} \times (-s) + \frac{s}{1+s} \times 1$)
$$\left\|\xi_{\lambda_k^p}\right\|_{L^2(0,1)} \leq \left\|\xi_{\lambda_k^p}\right\|^{\frac{1}{1+s}}_{(H^s_\sharp(0,1))'} \left\|\xi_{\lambda_k^p}\right\|^{\frac{s}{1+s}}_{H^1_\sharp(0,1)},$$
and therefore
$$\left\|\xi_{\lambda_k^p}\right\|_{(H^s_\sharp(0,1))'} \geq \left\|\xi_{\lambda_k^p}\right\|^{1+s}_{L^2(0,1)} \left\|\xi_{\lambda_k^p}\right\|^{-s}_{H^1_\sharp(0,1)} \geq \frac{C}{k^{1+s}},$$
for $k \geq k_0$ large enough.
Next, we have
$$\left\|\eta_{\lambda_k^p}\right\|_{L^2(0,1)} \geq C \quad \text{and} \quad \left\|\eta_{\lambda_k^p}\right\|_{H^1_0(0,1)} \geq Ck, \quad \text{for } k \geq k_0 \text{ large,}$$
and thus, by following the similar strategy as previous, we deduce that
$$\left\|\eta_{\lambda_k^p}\right\|_{H^{-s}(0,1)} \geq \left\|\eta_{\lambda_k^p}\right\|^{1+s}_{L^2(0,1)} \left\|\eta_{\lambda_k^p}\right\|^{-s}_{H^1_0(0,1)} \geq \frac{C}{k^s},$$
for $k \geq k_0$ large enough.
- *The hyperbolic part.* The steps will be exactly same as we analysed for the parabolic part. In this case, we have the following estimates:
$$C_1 \leq \left\|\xi_{\lambda_k^h}\right\|_{L^2(0,1)} \leq C_2, \quad \left\|\xi_{\lambda_k^h}\right\|_{(H^1_\sharp(0,1))'} \leq \frac{C}{|k|}, \quad \left\|\xi_{\lambda_k^h}\right\|_{H^1_\sharp(0,1)} \geq C\,|k|,$$
$$\frac{C_1}{|k|} \leq \left\|\eta_{\lambda_k^h}\right\|_{L^2(0,1)} \leq \frac{C_2}{|k|}, \quad \left\|\eta_{\lambda_k^h}\right\|_{H^{-1}(0,1)} \leq \frac{C}{|k|^2}, \quad \left\|\eta_{\lambda_k^h}\right\|_{H^1_0(0,1)} \geq C,$$
for large enough $|k| \geq k_0$.
Then, by following the interpolation arguments as previous, we can determine the required norm-estimates of $\xi_{\lambda_k^h}$ and $\eta_{\lambda_k^h}$, that is (3.15).

This completes the proof of Lemma 3.5.

9. Further remarks and conclusion

In the present work, we have proved the boundary null-controllability of our linearized 1D compressible Navier-Stokes system when a control acting either on the velocity or density part. For the velocity case, we have shown that when the initial states are chosen from the space $\dot{H}^{\frac{1}{2}}_{\sharp}(0,1) \times L^2(0,1)$, the system (1.4) is null-controllable at time $T > 1$. Moreover, for $0 \leq s < \frac{1}{2}$, the system fails to verify the null-controllability at any $T > 0$ in the space $\dot{H}^{s}_{\sharp}(0,1) \times L^2(0,1)$. Thus, the space is $\dot{H}^{\frac{1}{2}}_{\sharp}(0,1) \times L^2(0,1)$ is optimal w.r.t. the null-controllability of the system (1.4).

For the density case, we can even allow the $\dot{L}^2(0,1) \times L^2(0,1)$ initial states for the systems (1.5) and (1.6) to be null-controllable at time $T > 1$. We further proved that for small time, that is when $0 < T < 1$, the system (1.5) is no more null-controllable in the space $L^2(0,1) \times L^2(0,1)$.

In view of the above discussion, one immediate open question is the (non) null-controllability of the velocity case (the system (1.4)) or the full Dirichlet density case (system (1.6)) in small time $0 < T < 1$. We also cannot conclude the (non) null-controllability of the systems (1.4), (1.5) or (1.6) at the optimal time $T = 1$.

Let us make some final remarks related to our work.

- **Backward uniqueness and approximate controllability.** The backward uniqueness property tells that when the solution of a system (without any control) vanishes at some time $T > 0$, then it is identically zero at all time. This property plays an important role in the context of unique continuation and controllability.

  In this regard, we mention that the backward uniqueness is well-known for the cases when the associated operator forms a $\mathcal{C}^0$-group (hyperbolic case), for instance the system

$$\begin{cases} \rho_t + \rho_x = 0, & \text{in } (0,T) \times (0,1), \\ \rho(t,0) = \rho(t,1), & t \in (0,T), \\ \rho(0,x) = \rho_0(x), & x \in (0,1), \end{cases}$$

  or an analytic semigroup (parabolic case), for instance the system

$$\begin{cases} u_t - u_{xx} = 0, & \text{in } (0,T) \times (0,1), \\ u(t,0) = u(t,1) = 0, & t \in (0,T), \\ u(0,x) = u_0(x), & x \in (0,1). \end{cases}$$

  Let us come to our problem. Consider the following system without any control input,

$$\begin{cases} \rho_t + \rho_x + b u_x = 0 & \text{in } (0,T) \times (0,1), \\ u_t - u_{xx} + u_x + b\rho_x = 0 & \text{in } (0,T) \times (0,1), \\ \rho(t,0) = \rho(t,1) & \text{for } t \in (0,T), \\ u(t,0) = 0, \quad u(t,1) = 0 & \text{for } t \in (0,T), \\ \rho(0,x) = \rho_0(x), \quad u(0,x) = u_0(x) & \text{for } x \in (0,1). \end{cases} \tag{9.1}$$

  Since the system (9.1) is of mixed nature (coupling between parabolic and hyperbolic components), the backward uniqueness question is interesting from the mathematical point of view. In fact, it has been indicated in [4, 5] by Avalos and Triggiani, and in [41] by Lasiecka, Renardy and Triggiani, that the backward uniqueness property is a delicate issue for the coupled parabolic-hyperbolic systems.

  But in our case, the advantage is that the (generalized) eigenfunctions of the operator A forms a Riesz basis in $L^2(0,1) \times L^2(0,1)$ (see Remark 3.8). Also, we have that $(A, D(A))$ defines a strongly continuous semigroup in $L^2(0,1) \times L^2(0,1)$. As a result, we have the following: if the solution (ρ, u) to the system (9.1) satisfies

$$\rho(T,\cdot) = u(T,\cdot) = 0 \quad \text{in} \quad (0,1),$$

  then we necessarily have

$$\rho_0 = u_0 = 0, \quad \text{in } (0,1), \quad \text{i.e.,} \quad \rho(t,x) = u(t,x) = 0 \quad \text{in} \quad (0,T) \times (0,1).$$

The above *backward uniqueness property* of (9.1), that is the free system of (1.4) (resp. (1.5)), together with the null-controllability of (1.4) (resp. (1.5)), we deduce the approximate controllability of the system (1.4) (resp. (1.5)) at time $T > 1$ in the space $\dot{H}^{\frac{1}{2}}_{\sharp}(0,1) \times L^2(0,1)$ (resp. $\dot{L}^2(0,1) \times L^2(0,1)$).

Finally, the approximate controllability of the system (1.6) at time $T > 1$ in the space $\dot{L}^2(0,1) \times L^2(0,1)$ follows from the null-controllability result Theorem 1.6 and the backward uniqueness of the free system associated to (1.6) (as proved by Renardy in [51]).

- **Growth bound of the semigroup and a stability result when** $(\rho_0, u_0) \in \dot{L}^2(0,1) \times L^2(0,1)$**.** Recall the space

$$\dot{L}^2(0,1) := \Big\{ \phi \in L^2(0,1) : \int_0^1 \phi = 0 \Big\}.$$

We shall point out some stability result associated with the system (9.1) (that is, without any control) when the initial data $(\rho_0, u_0) \in \dot{L}^2(0,1) \times L^2(0,1)$.

In this case, the operator A with its formal expression (1.7) has the domain

$$\mathcal{D}(A) = \Big\{ \Phi = (\xi, \eta) \in \dot{H}^1(0,1) \times H^2(0,1) : \xi(0) = \xi(1),\ \eta(0) = \eta(1) = 0 \Big\}, \tag{9.2}$$

where $\dot{H}^1(0,1)$ contains all the functions in $H^1(0,1)$ with mean zero. Similarly, A^* has its formal expression as (1.9) with the same domain $\mathcal{D}(A^*) = \mathcal{D}(A)$ as of (9.2).

It is enough to obtain the growth bound of the semigroup $\{S^*(t)\}_{t\geq 0}$ generated by $(A^*, \mathcal{D}(A^*))$ in $L^2(0,1) \times L^2(0,1)$. Then, using the fact $\|S(t)\| = \|S^*(t)\|$ we can deduce the growth of the semigroup $\{S(t)\}_{t\geq 0}$ generated by $(A, \mathcal{D}(A))$ (in $L^2(0,1) \times L^2(0,1)$).

We first ensure that $\lambda = 0$ cannot be an eigenvalue of A^* (or A) with the domain (9.2). If yes, then the associated eigenfunction will be $(1,0)$, but this is not possible since $(1,0) \notin \mathcal{D}(A^*)$. Also, observe that the first component of the eigenfunction of A^* (or A) corresponding to any eigenvalue has mean zero (in the light of Remark 8.1). As a consequence, in this case we can prove that the set of eigenfunctions of A^* (or A) with the domain given by (9.2) forms a Riesz basis for $\dot{L}^2(0,1) \times L^2(0,1)$ (using Theorem 3.6). So, $(A^*, \mathcal{D}(A^*))$ (or $(A, \mathcal{D}(A))$) is indeed a Riesz-spectral operator since there is no accumulation point of the set of eigenvalues of A^* (or A), see the book of Curtain and Zwart [24, Chapter 3].

Now in one hand, since $\lambda \neq 0$, all the eigenvalues of A^* with domain (9.2) have negative real parts (see (8.5)), i.e.,

$$\mathrm{Re}(\lambda) < 0, \quad \forall \lambda \in \sigma(A^*).$$

On the other hand, thanks to Lemma 3.2, the set of parabolic and hyperbolic branches of the eigenvalues of A^* with domain (9.2) have the following asymptotics properties:

$$\begin{aligned} \lambda_k^p &= -k^2\pi^2 + O(1), && \text{for large } k \geq k_0, \\ \lambda_k^h &= -b^2 - 2ik\pi + O(|k|^{-1}), && \text{for large } |k| \geq k_0. \end{aligned}$$

Thus, there exists some $\omega_0 \in [-b^2, 0)$ such that

$$\omega_0 = \sup \big\{ \mathrm{Re}(\lambda) : \lambda \in \sigma(A) \big\} < 0.$$

Now recall that $(A^*, \mathcal{D}(A^*))$ is a Riesz-spectral operator and so the semigroup $\{S^*(t)\}_{t\geq 0}$ generated by $(A^*, \mathcal{D}(A^*))$ has the following growth

$$\|S^*(t)\| \leq Ce^{\omega_0 t}, \quad \forall t \geq 0.$$

But, $\|S(t)\| = \|S^*(t)\|$ and therefore

$$\|S(t)\| \leq Ce^{\omega_0 t}, \quad \forall t \geq 0.$$

with $-b^2 \leq \omega_0 < 0$, which gives the exponential stability of the system (9.1) with initial data $(\rho_0, u_0) \in \dot{L}^2(0,1) \times L^2(0,1)$.

- **Characterization of the coefficient** b**.** We have proved the null-controllability of linearized compressible Navier-Stokes systems (1.4), (1.5) and (1.6) at a large time provided the coefficient b is small, in particular $b^4 + 8b^2 + 5 < 4\pi^2$. This condition ensures that all the eigenvalues of A^* has geometric multiplicity 1, thanks to Proposition 3.1-Part (iv). However, this is not a necessary condition for achieving null-controllability of the systems (1.4), (1.5) and (1.6). To be more precise, characterization of all $b > 0$ such that the systems (1.4), (1.5) and (1.6) are null-controllable at a large time is not obtained and it is a very difficult problem due to the complicated cubic polynomial (4.7). Equivalently, one can say that finding all $b > 0$ such that all the eigenvalues of A^* are geometrically simple is unknown.

- **A Dirichlet-Dirichlet system with control on velocity.** Recall that, when we exerted a Dirichlet boundary control on velocity, we have considered the condition $\rho(t,0) = \rho(t,1)$ for the density part. It would be really interesting to deal with the full Dirichlet case when a control q acts on the velocity, that is the following system

$$\begin{cases} \rho_t + \rho_x + bu_x = 0 & \text{in } (0,T)\times(0,1), \\ u_t - u_{xx} + u_x + b\rho_x = 0 & \text{in } (0,T)\times(0,1), \\ \rho(t,0) = 0 & \text{for } t \in (0,T), \\ u(t,0) = 0, \ u(t,1) = q(t) & \text{for } t \in (0,T), \\ \rho(0,x) = \rho_0(x), \ u(0,x) = u_0(x) & \text{for } x \in (0,1). \end{cases} \tag{9.3}$$

This is really a challenging open problem to handle because of the difficulty in analyzing the spectral properties of the associated adjoint operator. This can be considered as a future work.

Appendix A. Proof of the well-posedness results

This section is devoted to prove the well-posedness of the solution to our control system (1.5). More precisely, we shall prove Lemma 2.1 and Theorem 2.6.

A.1. **Existence of semigroup: proof of Lemma 2.1.** The proof is divided into several parts. Recall the operator $(A, D(A))$ given by (1.7)–(1.8) and denote $\mathbf{Z} = L^2(0,1)\times L^2(0,1)$ over the field $\mathbb{C}$.

Part 1. *The operator A is dissipative.* We check that, all $\mathbf{U} = (\rho, u) \in D(A)$

$$\begin{aligned} \operatorname{Re}\langle A\mathbf{U}, \mathbf{U}\rangle_{\mathbf{Z}} &= \operatorname{Re}\left\langle \begin{pmatrix} -\rho_x - bu_x \\ -b\rho_x + u_{xx} - u_x \end{pmatrix}, \begin{pmatrix} \rho \\ u \end{pmatrix} \right\rangle_{\mathbf{Z}} \\ &= \operatorname{Re}\left(-\int_0^1 \bar{\rho}\rho_x dx - b\int_0^1 \bar{\rho}u_x dx - b\int_0^1 \rho_x \bar{u} dx + \int_0^1 \bar{u}u_{xx}dx - \int_0^1 \bar{u}u_x dx \right) \\ &= -\frac{1}{2}\int_0^1 \frac{d}{dx}(|\rho|^2)dx - \int_0^1 \bar{u}_x u_x dx - \frac{1}{2}\int_0^1 \frac{d}{dx}(|u|^2)dx \\ &= -\int_0^1 |u_x|^2\, dx \leq 0, \end{aligned}$$

Part 2. *The operator A is maximal.* This is equivalent to the following. For any $\lambda > 0$ and any $\begin{pmatrix} f \\ g \end{pmatrix} \in \mathbf{Z}$ we can find a $\begin{pmatrix} \rho \\ u \end{pmatrix} \in D(A)$ such that

$$(\lambda I - A)\begin{pmatrix} \rho \\ u \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \tag{A.1}$$

that is

$$\begin{aligned} \lambda\rho + \rho_x + bu_x &= f, \\ \lambda u + b\rho_x - u_{xx} + u_x &= g. \end{aligned}$$

Let $\varepsilon > 0$. Instead of solving the above problem, we will solve the following regularized problem

$$\begin{aligned} \lambda\rho + \rho_x + bu_x - \varepsilon\rho_{xx} &= f, \\ \lambda u + b\rho_x + u_x - u_{xx} &= g, \end{aligned} \tag{A.2}$$

with the following boundary conditions

$$\rho(0) = \rho(1), \quad \rho_x(0) = \rho_x(1), \quad u(0) = 0, \quad u(1) = 0.$$

We now proceed through the following steps.

Step 1. We consider the space V, given by

$$V = \left\{(\rho, u) \in H^1(0,1) \times H^1(0,1) \ : \ \rho(0) = \rho(1), \quad u(0) = 0, \quad u(1) = 0\right\}.$$

Using Lax-Milgram theorem, we first prove that the system (A.2) has a unique solution in V. Define the operator $B : V \times V \to \mathbb{C}$ by

$$B\left(\begin{pmatrix}\rho\\u\end{pmatrix}, \begin{pmatrix}\sigma\\v\end{pmatrix}\right) = \varepsilon \int_0^1 \rho_x \bar{\sigma}_x dx + b\int_0^1 u_x \bar{\sigma} dx + \int_0^1 \rho_x \bar{\sigma} dx + \lambda \int_0^1 \rho\bar{\sigma} dx$$
$$+ \int_0^1 u_x \bar{v}_x dx + \int_0^1 u_x \bar{v} dx + b\int_0^1 \rho_x \bar{v} dx + \lambda\int_0^1 u\bar{v} dx,$$

for all $\begin{pmatrix}\rho\\u\end{pmatrix}, \begin{pmatrix}\sigma\\v\end{pmatrix} \in V$. Then, one can show that B is continuous and coercive. Thus, by Lax-Milgram theorem, for every $\varepsilon > 0$, there exists a unique solution $(\rho^\varepsilon, u^\varepsilon) \in V$ such that

$$B\left(\begin{pmatrix}\rho^\varepsilon\\u^\varepsilon\end{pmatrix}, \begin{pmatrix}\sigma\\v\end{pmatrix}\right) = F\left(\begin{pmatrix}\sigma\\v\end{pmatrix}\right), \quad \forall \begin{pmatrix}\sigma\\v\end{pmatrix} \in V,$$

where $F : V \to \mathbb{C}$ is the linear functional given by

$$F\left(\begin{pmatrix}\sigma\\v\end{pmatrix}\right) := \int_0^1 f\bar{\sigma} dx + \int_0^1 g\bar{v} dx.$$

Step 2. Now, observe that

$$\text{Re}\left(B\left(\begin{pmatrix}\rho^\varepsilon\\u^\varepsilon\end{pmatrix}, \begin{pmatrix}\rho^\varepsilon\\u^\varepsilon\end{pmatrix}\right)\right) \leq \int_0^1 |f\overline{\rho^\varepsilon}| + \int_0^1 |g\overline{u^\varepsilon}| \leq \frac{1}{2}\int_0^1 \left(|f|^2 + |\overline{\rho^\varepsilon}|^2\right) + \frac{1}{2}\int_0^1 \left(|g|^2 + |\overline{u^\varepsilon}|^2\right),$$

which yields

$$\varepsilon \int_0^1 |\rho_x^\varepsilon|^2 + \frac{\lambda}{2}\int_0^1 |\rho^\varepsilon|^2 + \int_0^1 |u_x^\varepsilon|^2 + \frac{\lambda}{2}\int_0^1 |u^\varepsilon|^2 \leq \frac{1}{2}\int_0^1 |f|^2 + \frac{1}{2}\int_0^1 |g|^2$$

This shows that $(u^\varepsilon)_{\varepsilon \geq 0}$ is bounded in $H^1(0,1)$, $(\rho^\varepsilon)_{\varepsilon\geq 0}$ is bounded in $L^2(0,1)$ and $(\sqrt{\varepsilon}\rho_x^\varepsilon)_{\varepsilon\geq 0}$ is bounded in $L^2(0,1)$. Since the spaces $H^1(0,1)$ and $L^2(0,1)$ are reflexive, there exist subsequences, still denoted by $(u^\varepsilon)_{\varepsilon\geq 0}$, $(\rho^\varepsilon)_{\varepsilon\geq 0}$, and functions $\rho \in L^2(0,1)$ and $u \in H^1(0,1)$, such that

$$u^\varepsilon \rightharpoonup u \text{ in } H^1(0,1), \text{ and } \rho^\varepsilon \rightharpoonup \rho \text{ in } L^2(0,1).$$

Furthermore, we have

$$\int_0^1 |\varepsilon\rho_x^\varepsilon|^2 = \varepsilon\int_0^1 \left|\sqrt{\varepsilon}\rho^\varepsilon\right|^2 \to 0, \text{ as } \varepsilon \to 0.$$

Now, since $B\left(\begin{pmatrix}\rho^\varepsilon\\u^\varepsilon\end{pmatrix}, \begin{pmatrix}\sigma\\v\end{pmatrix}\right) = F\left(\begin{pmatrix}\sigma\\v\end{pmatrix}\right)$, for all $\begin{pmatrix}\sigma\\v\end{pmatrix} \in V$, we may take $\begin{pmatrix}\sigma\\0\end{pmatrix} \in V$, so that we obtain

$$\varepsilon\int_0^1 \rho_x^\varepsilon\bar{\sigma}_x + b\int_0^1 u_x^\varepsilon\bar{\sigma} + \int_0^1 \rho_x^\varepsilon\bar{\sigma} + \lambda\int_0^1 \rho^\varepsilon\bar{\sigma} = \int_0^1 f\bar{\sigma}. \tag{A.3}$$

Similarly, by taking $\begin{pmatrix}0\\v\end{pmatrix} \in V$, we get

$$\int_0^1 u_x^\varepsilon\bar{v}_x + \int_0^1 u_x^\varepsilon\bar{v} + b\int_0^1 \rho_x^\varepsilon\bar{v} + \lambda\int_0^1 u^\varepsilon\bar{v} = \int_0^1 g\bar{v}. \tag{A.4}$$

Integrating by parts, we get from equation (A.3) that,

$$\varepsilon\int_0^1 \rho_x^\varepsilon\bar{\sigma}_x + b\int_0^1 u_x^\varepsilon\bar{\sigma} - \int_0^1 \rho^\varepsilon\bar{\sigma}_x + \lambda\int_0^1 \rho^\varepsilon\bar{\sigma} = \int_0^1 f\bar{\sigma}.$$

Then, passing to the limit $\varepsilon \to 0$, we obtain

$$b\int_0^1 u_x\bar{\sigma} - \int_0^1 \rho\bar{\sigma}_x + \lambda\int_0^1 \rho\bar{\sigma} = \int_0^1 f\bar{\sigma},$$

and the above relation is true $\forall \sigma \in \mathcal{C}_c^\infty(0,1)$. As a consequence,

$$bu_x + \rho_x + \lambda\rho = f, \tag{A.5}$$

in the sense of distribution and therefore $\rho_x = f - bu_x - \lambda\rho \in L^2(0,1)$; in other words, $\rho \in H^1(0,1)$.

Step 3. We now show $u(0) = u(1) = 0$. Since the inclusion map $i : H^1(0,1) \to \mathcal{C}^0([0,1])$ is compact and $u^\varepsilon \rightharpoonup u$ in $H^1(0,1)$, we obtain

$$u^\varepsilon \to u \quad \text{in } \mathcal{C}^0([0,1]).$$

Thus, $(u^\varepsilon(0), u^\varepsilon(1)) \to (u(0), u(1))$. Since $u^\varepsilon(0) = u^\varepsilon(1) = 0$ for all $\varepsilon > 0$, we have

$$u(0) = u(1) = 0.$$

Similarly from the identity (A.4), one can deduce that

$$-u_{xx} + u_x + b\rho_x + \lambda u = g, \tag{A.6}$$

in the sense of distribution and therefore $u_{xx} \in L^2(0,1)$, that is $u \in H^2(0,1)$.

We now show $\rho(0) = \rho(1)$. Recall that, $bu_x + \rho_x + \lambda\rho = f$ and therefore

$$b\int_0^1 u_x\bar{\sigma} + \int_0^1 \rho_x\bar{\sigma} + \lambda\int_0^1 \rho\bar{\sigma} = \int_0^1 f\bar{\sigma}.$$

Integrating by parts, we get

$$b\int_0^1 u_x\bar{\sigma} - \int_0^1 \rho\bar{\sigma}_x + \rho\bar{\sigma}|_0^1 + \lambda\int_0^1 \rho\bar{\sigma} = \int_0^1 f\bar{\sigma}. \tag{A.7}$$

From (A.3), we deduce

$$\varepsilon\int_0^1 \rho_x^\varepsilon\bar{\sigma}_x + b\int_0^1 u_x^\varepsilon\bar{\sigma} - \int_0^1 \rho^\varepsilon\bar{\sigma}_x + \lambda\int_0^1 \rho^\varepsilon\bar{\sigma} = \int_0^1 f\bar{\sigma}. \tag{A.8}$$

Taking $\varepsilon \to 0$, we get

$$b\int_0^1 u_x\bar{\sigma} - \int_0^1 \rho\bar{\sigma}_x + \lambda\int_0^1 \rho\bar{\sigma} = \int_0^1 f\bar{\sigma}. \tag{A.9}$$

Comparing (A.7) and (A.9), one has $\rho(0)\bar{\sigma}(0) = \rho(1)\bar{\sigma}(1)$. But $\sigma(0) = \sigma(1)$, and thus

$$\rho(0) = \rho(1).$$

So, we get $\begin{pmatrix}\rho\\ u\end{pmatrix} \in D(A)$. Hence, the operator A is maximal.

A.2. **Solution by transposition: proof of Theorem 2.6.** In this section, we are going to proof the existence of solution to our control problem (1.5), more precisely Theorem 2.6. We omit the proof for Theorem 2.5, when a control acts on the velocity part.

Step 1. We first consider system (1.5) with zero initial data and nonhomogeneous boundary conditions, that is,

$$\begin{cases} \rho_t + \rho_x + bu_x = 0 & \text{in } (0,T)\times(0,1), \\ u_t - u_{xx} + u_x + b\rho_x = 0 & \text{in } (0,T)\times(0,1), \\ \rho(t,0) = \rho(t,1) + p(t) & \text{for } t \in (0,T), \\ u(t,0) = 0, \quad u(t,1) = 0 & \text{for } t \in (0,T), \\ \rho(0,x) = u(0,x) = 0 & \text{for } x \in (0,1), \end{cases} \tag{A.10}$$

with $p \in L^2(0,T)$.

We now prove the existence of solution to the new system (A.10).

Theorem A.1. *For a given $p \in L^2(0,T)$, the system (A.10) has a unique solution $(\tilde{\rho}, \tilde{u})$ belonging to the space $L^2(0,T;L^2(0,1)) \times L^2(0,T;L^2(0,1))$ in the sense of transposition. Moreover, the operator:*

$$p \mapsto (\tilde{\rho}, \tilde{u}),$$

is linear and continuous from $L^2(0,T)$ into $L^2(0,T;L^2(0,1)) \times L^2(0,T;L^2(0,1))$.

Proof. Existence: Let us define a map $\Lambda_1 : L^2(0,T;L^2(0,1)) \times L^2(0,T;L^2(0,1)) \to L^2(0,T)$,

$$\Lambda_1(f,g) = \sigma(t,1), \tag{A.11}$$

where (σ, v) is the unique solution to the adjoint system (2.1) with given source term (f,g) and $(\sigma_T, v_T) = (0,0)$. The map Λ_1 is well-defined because of the hidden regularity as mentioned in Appendix B, Corollary B.2.

Now, thanks to Proposition 2.3, the map

$$(f,g) \mapsto (\sigma, v)$$

is linear and continuous from $L^2(0,T;L^2(0,1)) \times L^2(0,T;L^2(0,1))$ to $L^2(0,T;L^2(0,1)) \times L^2(0,T;H^1_0(0,1))$, which implies that the map Λ_1 given by (A.11) is linear and continuous (Corollary B.2).

So, we can define the adjoint to Λ_1 as follows

$$\Lambda_1^* : L^2(0,T) \to L^2(0,T;L^2(0,1)) \times L^2(0,T;L^2(0,1)), \tag{A.12}$$

which is also linear and continuous.

Let us denote $\Lambda_1^*(p) = (\tilde{\rho}, \tilde{u})$. Then, for $(\tilde{\rho}, \tilde{u})$, we have

$$\begin{aligned} \int_0^T \int_0^1 \tilde{\rho}(t,x) f(t,x) dx dt + \int_0^T \int_0^1 \tilde{u}(t,x) g(t,x) dx dt &= \langle \Lambda_1^* p, (f,g) \rangle \\ &= \langle p, \Lambda_1(f,g) \rangle \\ &= \int_0^T p(t) \sigma(t,1) dt, \end{aligned}$$

for every (f,g) in $L^2(0,T;L^2(0,1)) \times L^2(0,T;L^2(0,1))$. Hence for any $p \in L^2(0,T)$, $(\tilde{\rho}, \tilde{u})$ is the solution to the system (A.10) in the sense of transposition and

$$\begin{aligned} \|(\tilde{\rho}, \tilde{u})\|_{L^2(L^2) \times L^2(L^2)} &= \|\Lambda_1^*(p)\|_{L^2(L^2) \times L^2(L^2)} \\ &\leq \|\Lambda_1^*\| \, \|p\|_{L^2(0,T)}. \end{aligned} \tag{A.13}$$

Uniqueness: If $p = 0$ on $(0,T)$, we have

$$\int_0^T \int_0^1 \rho(t,x) f(t,x) dx dt + \int_0^T \int_0^1 u(t,x) g(t,x) dx dt = 0,$$

for all $(f,g) \in L^2(0,T;L^2(0,1)) \times L^2(0,T;L^2(0,1))$, which gives $(\rho, u) = (0,0)$ and therefore the solution to the system (A.10) is unique. □

Step 2. We now consider the system (1.5) with non-zero initial data and homogeneous boundary conditions and check the existence, uniqueness of solution. The system reads as

$$\begin{cases} \rho_t + \rho_x + b u_x = 0 & \text{in } (0,T) \times (0,1), \\ u_t - u_{xx} + u_x + b\rho_x = 0 & \text{in } (0,T) \times (0,1), \\ \rho(t,0) = \rho(t,1) & \text{for } t \in (0,T), \\ u(t,0) = 0, \quad u(t,1) = 0 & \text{for } t \in (0,T), \\ \rho(0,x) = \rho_0(x), \ u(0,x) = u_0(x) & \text{for } x \in (0,1), \end{cases} \tag{A.14}$$

with $(\rho_0, u_0) \in L^2(0,1) \times L^2(0,1)$.

Theorem A.2. *For any $(\rho_0, u_0) \in L^2(0,1) \times L^2(0,1)$, the system (A.14) has a unique solution $(\check{\rho}, \check{u})$ belonging to the space $L^2(0,T;L^2(0,1)) \times L^2(0,T;L^2(0,1))$ in the sense of transposition. Moreover, the operator:*

$$(\rho_0, u_0) \mapsto (\check{\rho}, \check{u}),$$

is linear and continuous from $L^2(0,1) \times L^2(0,1)$ into $L^2(0,T;L^2(0,1)) \times L^2(0,T;L^2(0,1))$.

Proof. Existence: Let us define a map $\Lambda_2 : L^2(0,T;L^2(0,1)) \times L^2(0,T;L^2(0,1)) \to L^2(0,1) \times L^2(0,1)$,

$$\Lambda_2(f,g) = (\sigma(0,\cdot), v(0,\cdot)), \tag{A.15}$$

where (σ, v) is the unique solution to the adjoint system (2.1) with given source term (f,g) and $(\sigma_T, v_T) = (0,0)$.

Now, thanks to Proposition 2.3, the map

$$(f,g) \mapsto (\sigma, v)$$

is linear and continuous from $L^2(0,T;L^2(0,1)) \times L^2(0,T;L^2(0,1))$ to the space $\mathcal{C}([0,T];L^2(0,1)) \times [\mathcal{C}([0,T];L^2(0,1)) \cap L^2(0,T;H_0^1(0,1))]$, which implies that the map Λ_2 given by (A.15) is linear and continuous.

So, we can define the adjoint to Λ_2 as follows

$$\Lambda_2^* : L^2(0,1) \times L^2(0,1) \to L^2(0,T;L^2(0,1)) \times L^2(0,T;L^2(0,1)), \tag{A.16}$$

which is also linear and continuous.

Let us denote $\Lambda_2^*(\rho_0,u_0) = (\check{\rho},\check{u})$. Then, for $(\check{\rho},\check{u})$, we have

$$\begin{aligned}\int_0^T\int_0^1 \check{\rho}(t,x)f(t,x)dxdt + \int_0^T\int_0^1 \check{u}(t,x)g(t,x)dxdt &= \langle \Lambda_2^*(\rho_0,u_0),(f,g)\rangle \\ &= \langle (\rho_0,u_0),\Lambda_2(f,g)\rangle \\ &= \langle (\rho_0,u_0),(\sigma(0,\cdot),v(0,\cdot))\rangle ,\end{aligned}$$

for every (f,g) in $L^2(0,T;L^2(0,1)) \times L^2(0,T;L^2(0,1))$. Hence for any $(\rho_0,u_0) \in L^2(0,1)\times L^2(0,1)$, $(\check{\rho},\check{u})$ is the solution to the system (A.10) and

$$\begin{aligned}\|(\check{\rho},\check{u})\|_{L^2(L^2)\times L^2(L^2)} &= \|\Lambda_2^*(\rho_0,u_0)\|_{L^2(L^2)\times L^2(L^2)} \\ &\leq \|\Lambda_2^*\| \ \|(\rho_0,u_0)\|_{L^2(0,1)\times L^2(0,1)}.\end{aligned} \tag{A.17}$$

Uniqueness: Let the system (A.14) has two solutions (ρ_1,u_1) and (ρ_2,u_2). Introduce

$$(\rho,u) = (\rho_1,u_1) - (\rho_2,u_2).$$

Then one can show that the only possibility is $(\rho,u) = (0,0)$, using the initial and boundary conditions: $\rho(0,x) = u(0,x) = 0$ for all $x \in (0,1)$ and $\rho(t,0) = \rho(t,1)$, $u(t,0) = u(t,1) = 0$ for all $t \in (0,T)$. ☐

Proof of Theorem 2.6. We now recall the system (1.5) with given boundary data $p \in L^2(0,T)$ and initial data $(\rho_0,u_0) \in L^2(0,1)\times L^2(0,1)$. Then, thanks to Theorem A.1 & A.2,

$$(\rho,u) := (\tilde{\rho},\tilde{u}) + (\check{\rho},\check{u}),$$

is the unique solution to (1.5).

It remains to prove the continuity estimate of the solution (ρ,u). Let $H : L^2(0,1)\times L^2(0,1)\times L^2(0,T) \to L^2(0,T;L^2(0,1)) \times L^2(0,T;L^2(0,1))$ be defined by

$$H(\rho_0,u_0,p) = (\rho,u). \tag{A.18}$$

Then H is linear. Furthermore, using (A.13) and (A.17), we get

$$\begin{aligned}&\|H(\rho_0,u_0,p)\|_{L^2(0,T;L^2(0,1))\times L^2(0,T;L^2(0,1))} = \|(\tilde{\rho},\tilde{u}) + (\check{\rho},\check{u})\|_{L^2(0,T;L^2(0,1))\times L^2(0,T;L^2(0,1))} \\ &\leq \|\Lambda_1^*\| \, \|p\|_{L^2(0,T)} + \|\Lambda_2^*\| \, \|(\rho_0,u_0)\|_{L^2(0,1)\times L^2(0,1)} \\ &\leq C\left(\|p\|_{L^2(0,T)} + \|\rho_0\|_{L^2(0,1)} + \|u_0\|_{L^2(0,1)}\right).\end{aligned}$$

Finally, the required regularity result (2.3)–(2.4) can be obtained by applying the usual regularity of parabolic equation (with homogeneous boundary data) and then using that, the regularity of transport part follows immediately.

The proof is complete. ☐

Appendix B. A hidden regularity result

Consider the following system

$$\begin{cases} \rho_t + \rho_x + bu_x = 0 & \text{in } (0,T)\times(0,1), \\ u_t - u_{xx} + u_x + b\rho_x = 0 & \text{in } (0,T)\times(0,1), \\ \rho(t,0) = \rho(t,1) + p(t) & \text{for } t\in(0,T), \\ u(t,0) = 0, \ u(t,1) = 0 & \text{for } t\in(0,T), \\ \rho(0,x) = \rho_0(x), \ u(0,x) = u_0(x) & \text{for } x\in(0,1), \end{cases} \tag{B.1}$$

where $(\rho_0,u_0) \in L^2(0,1)\times L^2(0,1)$ and $p \in L^2(0,T)$ are given data. Then, one has the following result.

Lemma B.1. *For any $(\rho_0,u_0) \in L^2(0,1)\times L^2(0,1)$ and $p\in L^2(0,T)$, the density component ρ to the system (B.1) satisfies $\rho(\cdot,1) \in L^2(0,T)$.*

Proof. The proof is split into two steps. First, recall Theorem 2.6 so that one has

$$(\rho,u)\in \mathcal{C}^0([0,T];L^2(0,1))\times[\mathcal{C}^0([0,T];L^2(0,1))\cap L^2(0,T;H^1_0(0,1))].$$

Step 1. Let us take the initial state $\rho_0\in H^1_\sharp(0,1)$ (i.e., $\rho_0\in H^1(0,1)$ with $\rho_0(0)=\rho_0(1)$), $u_0\in H^1_0(0,1)$ and the boundary data $p\in H^1_{\{0\}}(0,T)$. Then one can prove that the solution (ρ,u) to system (B.1) lies in the space $[H^1(0,T;L^2(0,1))\cap L^2(0,T;H^1(0,1))]\times[L^2(0,T;H^2(0,1)\cap H^1_0(0,1))\cap H^1(0,T;L^2(0,1))]$, see for instance [19]. Therefore, $u_x\in L^2(0,T;H^1(0,1))$ and so the integration by parts are justified. Multiplying the first equation of (B.1) by $x\rho$, we get

$$\int_0^T\int_0^1 x\rho\rho_t dxdt+\int_0^T\int_0^1 x\rho\rho_x dxdt+b\int_0^T\int_0^1 x\rho u_x dxdt=0.$$

Integrating by parts and using the boundary conditions, we obtain

$$\frac{1}{2}\int_0^1 x(\rho^2(T,x)-\rho_0^2(x))dx+\frac{1}{2}\int_0^T\rho^2(t,1)dt-\frac{1}{2}\int_0^T\int_0^1\rho^2dxdt+b\int_0^T\int_0^1 x\rho u_x dxdt=0. \tag{B.2}$$

Therefore

$$\begin{aligned}\int_0^T\rho^2(t,1)dt&=-\int_0^1 x(\rho^2(T,x)-\rho_0^2(x))dx+\int_0^T\int_0^1\rho^2dxdt-2b\int_0^T\int_0^1 x\rho u_x dxdt\\&\leq(1+b)\int_0^T\int_0^1\rho^2dxdt+b\int_0^T\int_0^1 u_x^2dxdt+\int_0^1\rho_0^2(x)dx.\end{aligned}$$

Using the continuity estimate (2.4), we obtain

$$\int_0^T\rho^2(t,1)dt\leq C\left(\int_0^1\rho_0^2(x)dx+\int_0^1 u_0^2(x)dx+\int_0^T p^2(t)dt\right). \tag{B.3}$$

Step 2. Let $(\rho_0,u_0)\in L^2(0,1)\times L^2(0,1)$ and $p\in L^2(0,T)$. By density, there exists sequences $\rho_0^n\in H^1_\sharp(0,1)$, $u_0^n\in H^1_0(0,1)$ and $p^n\in H^1_{\{0\}}(0,T)$ such that $\rho_0^n\to\rho$, $u_0^n\to u_0$ in $L^2(0,1)$ and $p^n\to p$ in $L^2(0,T)$. Let (ρ^n,u^n) be the solution to (B.1) corresponding to the initial state (ρ_0^n,u_0^n) and boundary data p^n. Using (B.3) from Step 1, we have

$$\int_0^T(\rho^n)^2(t,1)dt\leq C\left(\int_0^1(\rho_0^n)^2(x)dx+\int_0^1(u_0^n)^2(x)dx+\int_0^T(p^n)^2(t)dt\right).$$

We first observe that

$$\int_0^1(\rho_0^n)^2(x)dx+\int_0^1(u_0^n)^2(x)dx+\int_0^T(p^n)^2(t)dt\to\int_0^1\rho_0^2(x)dx+\int_0^1u_0^2(x)dx+\int_0^T p^2(t)dt,$$

as $n\to+\infty$. Therefore, the sequence $\left(\int_0^T(\rho^n)^2(t,1)dt\right)_n$ is indeed a Cauchy sequence and hence convergent. Then, by the uniqueness of solution to (B.1), we can define $\int_0^T\rho^2(t,1)dt:=\lim_{n\to+\infty}\int_0^T(\rho^n)^2(t,1)dt$, which yields

$$\int_0^T\rho^2(t,1)dt\leq C\left(\int_0^1\rho_0^2(x)dx+\int_0^1u_0^2(x)dx+\int_0^T p^2(t)dt\right).$$

This concludes the proof of the lemma. □

Let us now consider the following system

$$\begin{cases}-\sigma_t-\sigma_x-bv_x=f & \text{in } (0,T)\times(0,1),\\ -v_t-v_{xx}-v_x-b\sigma_x=g & \text{in } (0,T)\times(0,1),\\ \sigma(t,0)=\sigma(t,1) & \text{for } t\in(0,T),\\ v(t,0)=v(t,1)=0 & \text{for } t\in(0,T),\\ \sigma(T,x)=0,\quad v(T,x)=0 & \text{for } x\in(0,1),\end{cases} \tag{B.4}$$

with $f,g\in L^2(0,T;L^2(0,1))$. We can similarly conclude the following result.

Corollary B.2. *For any $f, g \in L^2(0,T;L^2(0,1))$, the solution component σ to the adjoint system* (B.4) *satisfies the following estimate.*

$$\|\sigma(\cdot,1)\|_{L^2(0,T)} \leq C\left(\|f\|_{L^2(0,T;L^2(0,1))} + \|g\|_{L^2(0,T;L^2(0,1))}\right). \tag{B.5}$$

Acknowledgements

The work of the first author is partially supported by the French Government research program "Investissements d'Avenir" through the IDEX-ISITE initiative 16-IDEX-0001 (CAP 20-25) and the Praemium Academiae of Š. Nečasová. He also thanks the Department of Mathematics & Statistics, IISER Kolkata for the hospitality and support during his visits in July-August, 2023 and March-April, 2024. S. Chowdhury acknowledge the MATRICS Research Grant (ref. no. MTR/2021/000561) and R. Dutta acknowledge the DST-INSPIRE grants (ref. no. DST/INSPIRE/04/2015/002388) for the financial supports. The work of J. Kumbhakar is supported by the Prime Minister's Research Fellowship (ref. no. 41-1/2018-TS-1/PMRF), Government of India.

References

[1] S. Ahamed, D. Maity, and D. Mitra. Lack of null controllability of one dimensional linear coupled transport-parabolic system with variable coefficients. *J. Differential Equations*, 320:64–113, 2022.

[2] E. V. Amosova. Exact local controllability for equations of the dynamics of a viscous gas. *Differ. Uravn.*, 47(12):1754–1772, 2011.

[3] H. Arfaoui, F. Ben Belgacem, H. El Fekih, and J.-P. Raymond. Boundary stabilizability of the linearized viscous Saint-Venant system. *Discrete Contin. Dyn. Syst. Ser. B*, 15(3):491–511, 2011.

[4] G. Avalos and R. Triggiani. Backward uniqueness of the s.c. semigroup arising in parabolic-hyperbolic fluid-structure interaction. *J. Differential Equations*, 245(3):737–761, 2008.

[5] G. Avalos and R. Triggiani. Backwards uniqueness of the C_0-semigroup associated with a parabolic-hyperbolic Stokes-Lamé partial differential equation system. *Trans. Amer. Math. Soc.*, 362(7):3535–3561, 2010.

[6] S. A. Avdonin and S. A. Ivanov. *Families of exponentials*. Cambridge University Press, Cambridge, 1995. The method of moments in controllability problems for distributed parameter systems, Translated from the Russian and revised by the authors.

[7] M. Badra, S. Ervedoza, and S. Guerrero. Local controllability to trajectories for non-homogeneous incompressible Navier-Stokes equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 33(2):529–574, 2016.

[8] K. Beauchard, A. Koenig, and K. Le Balc'h. Null-controllability of linear parabolic transport systems. *J. Éc. polytech. Math.*, 7:743–802, 2020.

[9] A. Benabdallah, F. Boyer, M. González-Burgos, and G. Olive. Sharp estimates of the one-dimensional boundary control cost for parabolic systems and application to the N-dimensional boundary null controllability in cylindrical domains. *SIAM J. Control Optim.*, 52(5):2970–3001, 2014.

[10] E. Cerpa and E. Crépeau. Rapid exponential stabilization for a linear Korteweg-de Vries equation. *Discrete Contin. Dyn. Syst. Ser. B*, 11(3):655–668, 2009.

[11] E. Cerpa, C. Montoya, and B. Zhang. Local exact controllability to the trajectories of the Korteweg–de Vries–Burgers equation on a bounded domain with mixed boundary conditions. *J. Differential Equations*, 268(9):4945–4972, 2020.

[12] F. W. Chaves-Silva, L. Rosier, and E. Zuazua. Null controllability of a system of viscoelasticity with a moving control. *J. Math. Pures Appl. (9)*, 101(2):198–222, 2014.

[13] S. Chowdhury. Approximate controllability for linearized compressible barotropic Navier-Stokes system in one and two dimensions. *J. Math. Anal. Appl.*, 422(2):1034–1057, 2015.

[14] S. Chowdhury, R. Dutta, and S. Majumdar. Boundary stabilizability of the linearized compressible Navier-Stokes system in one dimension by backstepping approach. *SIAM J. Control Optim.*, 59(3):2147–2173, 2021.

[15] S. Chowdhury, R. Dutta, and S. Majumdar. Boundary controllability and stabilizability of a coupled first-order hyperbolic-elliptic system. *Evol. Equ. Control Theory*, 12(3):907–943, 2023.

[16] S. Chowdhury, D. Maity, M. Ramaswamy, and J.-P. Raymond. Local stabilization of the compressible Navier-Stokes system, around null velocity, in one dimension. *J. Differential Equations*, 259(1):371–407, 2015.

[17] S. Chowdhury and D. Mitra. Null controllability of the linearized compressible Navier-Stokes equations using moment method. *J. Evol. Equ.*, 15(2):331–360, 2015.

[18] S. Chowdhury, D. Mitra, M. Ramaswamy, and M. Renardy. Null controllability of the linearized compressible Navier Stokes system in one dimension. *J. Differential Equations*, 257(10):3813–3849, 2014.

[19] S. Chowdhury and M. Ramaswamy. Optimal control of linearized compressible Navier-Stokes equations. *ESAIM Control Optim. Calc. Var.*, 19(2):587–615, 2013.

[20] S. Chowdhury, M. Ramaswamy, and J.-P. Raymond. Controllability and stabilizability of the linearized compressible Navier-Stokes system in one dimension. *SIAM J. Control Optim.*, 50(5):2959–2987, 2012.

[21] J.-M. Coron. *Control and nonlinearity*, volume 136 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2007.

[22] J.-M. Coron, L. Hu, and G. Olive. Stabilization and controllability of first-order integro-differential hyperbolic equations. *J. Funct. Anal.*, 271(12):3554–3587, 2016.

[23] J.-M. Coron and P. Lissy. Local null controllability of the three-dimensional Navier-Stokes system with a distributed control having two vanishing components. *Invent. Math.*, 198(3):833–880, 2014.

[24] R. Curtain and H. Zwart. *Introduction to infinite-dimensional systems theory*, volume 71 of *Texts in Applied Mathematics*. Springer, New York, [2020] ©2020. A state-space approach.
[25] J. Edward. Ingham-type inequalities for complex frequencies and applications to control theory. *J. Math. Anal. Appl.*, 324(2):941–954, 2006.
[26] S. Ervedoza, O. Glass, and S. Guerrero. Local exact controllability for the two- and three-dimensional compressible Navier-Stokes equations. *Comm. Partial Differential Equations*, 41(11):1660–1691, 2016.
[27] S. Ervedoza, O. Glass, S. Guerrero, and J.-P. Puel. Local exact controllability for the one-dimensional compressible Navier-Stokes equation. *Arch. Ration. Mech. Anal.*, 206(1):189–238, 2012.
[28] S. Ervedoza and M. Savel. Local boundary controllability to trajectories for the 1D compressible Navier Stokes equations. *ESAIM Control Optim. Calc. Var.*, 24(1):211–235, 2018.
[29] E. Fernández-Cara, M. González-Burgos, and L. de Teresa. Boundary controllability of parabolic coupled equations. *J. Funct. Anal.*, 259(7):1720–1758, 2010.
[30] E. Fernández-Cara, S. Guerrero, O. Yu. Imanuvilov, and J.-P. Puel. Local exact controllability of the Navier-Stokes system. *J. Math. Pures Appl. (9)*, 83(12):1501–1542, 2004.
[31] V. Girinon. *Quelques problèmes aux limites pour les équations de Navier-Stokes*. 2008. DOCTORAT de l'Université Paul Sabatier Toulouse III.
[32] B.-Z. Guo. Riesz basis approach to the stabilization of a flexible beam with a tip mass. *SIAM J. Control Optim.*, 39(6):1736–1747, 2001.
[33] S. W. Hansen. Bounds on functions biorthogonal to sets of complex exponentials; control of damped elastic systems. *J. Math. Anal. Appl.*, 158(2):487–508, 1991.
[34] S. W. Hansen. Boundary control of a one-dimensional linear thermoelastic rod. *SIAM J. Control Optim.*, 32(4):1052–1074, 1994.
[35] A. E. Ingham. Some trigonometrical inequalities with applications to the theory of series. *Math. Z.*, 41(1):367–379, 1936.
[36] S. Jaffard, M. Tucsnak, and E. Zuazua. On a theorem of Ingham. volume 3, pages 577–582. 1997. Dedicated to the memory of Richard J. Duffin.
[37] T. Kato. *Perturbation theory for linear operators*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.
[38] A. Koenig and P. Lissy. Null-controllability of underactuated linear parabolic-transport systems with constant coefficients, https://arxiv.org/abs/2301.00471, 2023.
[39] V. Komornik and P. Loreti. *Fourier series in control theory*. Springer Monographs in Mathematics. Springer-Verlag, New York, 2005.
[40] V. Komornik and G. Tenenbaum. An Ingham-Müntz type theorem and simultaneous observation problems. *Evol. Equ. Control Theory*, 4(3):297–314, 2015.
[41] I. Lasiecka, M. Renardy, and R. Triggiani. Backward uniqueness for thermoelastic plates with rotational forces. *Semigroup Forum*, 62(2):217–242, 2001.
[42] K. Le Balc'h. Approximate controllability of the linearized boussinesq system in a two dimensional channel, https://arxiv.org/abs/1912.12245, 2020.
[43] G. Lebeau and E. Zuazua. Null-controllability of a system of linear thermoelasticity. *Arch. Rational Mech. Anal.*, 141(4):297–329, 1998.
[44] A. López. *Control y perturbaciones singulares de sistemas parabólicos*. PhD thesis, Universidad Complutense de Madrid, 1999.
[45] A. López and E. Zuazua. Uniform null-controllability for the one-dimensional heat equation with rapidly oscillating periodic density. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 19(5):543–580, 2002.
[46] D. Maity. Some controllability results for linearized compressible Navier-Stokes system. *ESAIM Control Optim. Calc. Var.*, 21(4):1002–1028, 2015.
[47] P. Martin, L. Rosier, and P. Rouchon. Null controllability of the structurally damped wave equation with moving control. *SIAM J. Control Optim.*, 51(1):660–684, 2013.
[48] S. Micu and E. Zuazua. An introduction to the controllability of partial differential equations. 2004.
[49] D. Mitra, M. Ramaswamy, and J.-P. Raymond. Largest space for the stabilizability of the linearized compressible Navier-Stokes system in one dimension. *Math. Control Relat. Fields*, 5(2):259–290, 2015.
[50] D. Mitra, M. Ramaswamy, and J.-P. Raymond. Local stabilization of compressible Navier-Stokes equations in one dimension around non-zero velocity. *Adv. Differential Equations*, 22(9-10):693–736, 2017.
[51] M. Renardy. Backward uniqueness for linearized compressible flow. *Evol. Equ. Control Theory*, 4(1):107–113, 2015.
[52] L. Rosier. Exact boundary controllability for the Korteweg-de Vries equation on a bounded domain. *ESAIM Control Optim. Calc. Var.*, 2:33–55, 1997.
[53] R. M. Young. *An introduction to nonharmonic Fourier series*. Academic Press, Inc., San Diego, CA, first edition, 2001.
[54] X. Zhang and E. Zuazua. Control, observation and polynomial decay for a coupled heat-wave system. *C. R. Math. Acad. Sci. Paris*, 336(10):823–828, 2003.
[55] X. Zhang and E. Zuazua. Polynomial decay and control of a 1-d model for fluid-structure interaction. *C. R. Math. Acad. Sci. Paris*, 336(9):745–750, 2003.
[56] X. Zhang and E. Zuazua. Polynomial decay and control of a $1-d$ hyperbolic-parabolic coupled system. *J. Differential Equations*, 204(2):380–438, 2004.
[57] E. Zuazua. Controllability and observability of partial differential equations: some results and open problems. In *Handbook of differential equations: evolutionary equations. Vol. III*, Handb. Differ. Equ., pages 527–621. Elsevier/North-Holland, Amsterdam, 2007.

[58] E. Zuazua. Stable observation of additive superpositions of partial differential equations. *Systems Control Lett.*, 93:21–29, 2016.