Boundary Controllability of Linearized Compressible Navier-Stokes System and Related Equations

A thesis submitted in partial fulfillment of the requirements for the award of the degree of

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Submitted by

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Under the guidance of

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to the

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To my parents and my elder brother

Abstract

In this thesis, we first study the controllability properties of the one dimensional linearized compressible Navier-Stokes equations for both barotropic and non-barotropic fluids using only one boundary control. In the barotropic case, the linearized system (around (Q_0, V_0) with $Q_0, V_0 > 0$) consists of a transport equation (satisfied by the density of the fluid) coupled with a parabolic equation (satisfied by the velocity of the fluid) with first-order coupling. We consider three types of boundary conditions:

(i) **Periodic**, where the control acts on the density (resp. velocity) component and is given by the difference of the values of the solution at both ends;

(ii) **Dirichlet**, where the control acts on the density part through Dirichlet condition at the left end;

(iii) **A mixed-type**, where we study two cases, one when the control acts on the density part through the difference of the solution at both ends with homogeneous Dirichlet conditions on the velocity, and the second when the control acts on the velocity part through Dirichlet condition at the left end with homogeneous Dirichlet condition on density.

In all of the above cases, we have proved optimal null controllability results for the linearized system with respect to the regularity of initial states for the velocity case and with respect to time in the density case. More precisely, in the periodic setup, we prove null controllability of the linearized system at time $T > \frac{2\pi}{V_0}$ in $(\dot{L}^2(0, 2\pi))^2$ (when the control is acting only on density) and in $\dot{H}^1_{\text{per}}(0, 2\pi) \times \dot{L}^2(0, 2\pi)$ (when the control is acting only on velocity), under a necessary and sufficient condition on the coefficients appearing in the system. Further, we prove that null controllability of the system fails when $0 < T < \frac{2\pi}{V_0}$ in $(\dot{L}^2(0, 2\pi))^2$ in the density case and at any $T > 0$ in $\dot{H}^s_{\text{per}}(0, 2\pi) \times \dot{L}^2(0, 2\pi)$ with $0 \leq s < 1$ in the velocity case. The proofs of these controllability results are included in Chapter 3.

Whereas, in the mixed case, we prove null controllability of the linearized system at time $T > 1$ in $\dot{L}^2(0, 1) \times L^2(0, 1)$ (when the control is acting only on density) and in $\dot{H}^{1/2}(0, 1) \times L^2(0, 1)$ (when the control is acting only on velocity), under some sufficient condition on the coefficients. Moreover, null controllability fails when $0 < T < 1$ in $L^2(0, 1) \times L^2(0, 1)$ in the density case and at any $T > 0$ in $\dot{H}^s(0, 1) \times L^2(0, 1)$ with $0 \leq s < \frac{1}{2}$ in the velocity case. As a consequence of these results, we prove null controllability of the linearized system at time $T > 1$ in $\dot{L}^2(0, 1) \times L^2(0, 1)$ using a Dirichlet control acting on density, under the same assumption on the coefficients mentioned above.

Moreover, in all of the above cases, we obtain approximate controllability of the above systems at large time by using the null controllability and backward uniqueness property of the corresponding systems. We have included all these controllability results in Chapter 4.

On the other hand, for non-barotropic fluids, the linearized system (around (Q_0, V_0, ψ_0) with $Q_0, V_0, \psi_0 > 0$) consists of a transport equation (satisfied by the density of the fluid) coupled with two parabolic equations (satisfied by the velocity and temperature) with the first-order couplings. Here, we consider only the periodic boundary conditions onto the system and study the null and approximate controllability properties using only one control acting either on density, velocity or temperature. More precisely, when the control acts only on the density part, we prove null controllability of the linearized system at time $T > \frac{2\pi}{V_0}$ in $(\dot{L}^2(0, 2\pi))^3$ under two assumptions:

(i) Eigenvalues of the associated non-self-adjoint operator have geometric multiplicity 1

(ii) The (irrational) coefficients appearing in the system have a good approximation by rational numbers (called the Diophantine approximation).

Further, in this case, we also prove that null controllability of this system fails when the time is small, that is, when $0 < T < \frac{2\pi}{V_0}$, in the space $(L^2(0,2\pi))^3$. Also, when a boundary control acts either on the velocity or temperature part, we prove null controllability of the linearized system at time $T > \frac{2\pi}{V_0}$ in $\dot{H}^1_{\text{per}}(0,2\pi) \times (\dot{L}^2(0,2\pi))^2$ under the same two hypotheses mentioned above. Moreover, we prove that null controllability of these systems fails at any $T > 0$ in the space $\dot{H}^s_{\text{per}}(0,2\pi) \times (\dot{L}^2(0,2\pi))^2$ with $0 \leq s < 1$.

Similar to the barotropic case, we obtain approximate controllability of the above systems at large time by using the null controllability and backward uniqueness property of the corresponding systems. All these controllability results are included in Chapter 3.

Finally, in Chapter 5, we have considered a coupled system consisting of two nonlinear parabolic equations with square, product and non-local nonlinearities. In the system, a Neumann boundary control is applied to only one state while the other satisfies homogeneous Neumann boundary condition at the left end. On the other hand, at the right end of the interval, the states are coupled in terms of “equality condition of their normal derivatives” and a combined Robin-type condition. In this setup, we prove small-time local null controllability of the system in the space $(L^2(0,1))^2$ by applying the so called “source term method”.

Our proofs of null controllability results rely on the method of moments and an application of the Ingham-type inequalities. The spectral analysis of the associated adjoint operator plays a crucial role in this analysis and we will use this throughout the thesis. We also prove a new Ingham-type inequality in Chapter 4, which generalizes the earlier related results available in the literature. We prove all the controllability results presented in Chapter 3 using this newly obtained Ingham-type inequality, whereas, in Chapters 4, we use both the method of moments and the Ingham-type inequality.

Furthermore, in Chapter 1, we give a brief overview of our main controllability results, and in Chapter 2, we present a detailed study of the basic results on controllability including the transport, heat and some nonlinear heat equations.

List of Symbols

Symbol	Description
$\mathbb{R}^n$	The n-dimensional Euclidean space
$\mathbb{R}$	The real line $(-\infty, \infty)$
∇f	The gradient of the function $f : \mathbb{R}^n \to \mathbb{R}^n$
Δf	The Laplacian of the function $f : \mathbb{R}^n \to \mathbb{R}$
$\Vert\cdot\Vert_X$	The norm of a normed linear space X
X'	The dual of the space X
$\langle u, v\rangle_H$	The inner product of two elements $u, v \in H$ (a Hilbert space)
$\langle u, v\rangle_{X,X'}$	The duality product of the elements $u \in X$ and $v \in X'$
A^*	The adjoint of the operator A
$C^0(\bar{\Omega})$	$\left\{f : \bar{\Omega} \subset \mathbb{R}^n \to \mathbb{R} \ : \ f \text{ is continuous on } \bar{\Omega}\right\}$
$C^k(\bar{\Omega})$	$\left\{f : \bar{\Omega} \subset \mathbb{R}^n \to \mathbb{R} \ : \ f \text{ is } k\text{-times differentiable and } f^{(k)} \text{ is continuous on } \bar{\Omega}\right\}$
$C^\infty(\Omega)$	$\{f : \Omega \subset \mathbb{R}^n \to \mathbb{R} \ : \ f \text{ is infinitely differentiable on } \Omega\}$
$C_c^\infty(\Omega)$	$\{f \in C^\infty(\Omega) \ : \ f \text{ has compact support in } \Omega\}$
$\mathcal{D}'(\Omega)$	The space of all distributions on $\Omega \subset \mathbb{R}^n$ (the dual of $C_c^\infty(\Omega)$)
$L^p(\Omega), 1 \le p < \infty$	$\left\{f : \Omega \subset \mathbb{R}^n \to \mathbb{R} \ : \ f \text{ is Lebesgue measurable and } \Vert f\Vert_{L^p(\Omega)} := \left(\int_\Omega \vert f\vert^p\right)^{\frac{1}{p}} < \infty\right\}$
$L^\infty(\Omega)$	$\left\{f : \Omega \subset \mathbb{R}^n \to \mathbb{R} \ : \ f \text{ is Lebesgue measurable and } \Vert f\Vert_{L^\infty(\Omega)} := \text{ess sup}_\Omega \vert f\vert < \infty\right\}$
$H^k(\Omega)$	The Sobolev space $\left\{u : \Omega \subset \mathbb{R}^n \to \mathbb{R} \ : \ u \in L^2(\Omega) \text{ and its weak derivatives } u', \ldots, u^{(k)} \in L^2(\Omega)\right\}$
$H_0^1(\Omega)$	The closure of $C_c^\infty(\Omega)$ in $H^1(\Omega) = \left\{f \in H^1(\Omega) \ : \ f\vert_{\partial\Omega} = 0\right\}$
$H_{\text{per}}^s(0, 2\pi),\ s > 0$	$\left\{\varphi \ : \ \varphi(x) = \sum_{n\in\mathbb{Z}} c_n e^{inx}, \text{ and } \Vert\varphi\Vert_{H_{\text{per}}^s(0,2\pi)} := \left(\sum_{n\in\mathbb{Z}} \left(1 + \vert n\vert^2\right)^s \vert c_n\vert^2\right)^{\frac{1}{2}} < \infty\right\}$
$\dot{L}^2(\Omega)$	$\left\{f \in L^2(\Omega) \ : \ \int_\Omega f dx = 0\right\}$
$H_{\{z\}}^1(\Omega)$	$\left\{f \in H^1(\Omega) \ : \ f(z) = 0\right\}$
$\dot{H}^k(\Omega)$	$\left\{f \in H^k(\Omega) \ : \ \int_\Omega f dx = 0\right\}$
$C^0([0, T]; X)$	$\{f : [0, T] \to X \ : \ f \text{ is continuous on } [0, T]\}$
$C^k([0, T]; X)$	$\{f : [0, T] \to X \ : \ f \text{ is } k\text{-times continuously differentiable on } [0, T]\}$
$L^p(0, T; X), 1 \le p < \infty$	$\left\{f : [0, T] \to X \ : \ f \text{ is strongly measurable and } \int_0^T \Vert f(t)\Vert_X^p dt < \infty\right\}$
$L^\infty(0, T; X)$	$\left\{f : [0, T] \to X \ : \ f \text{ is strongly measurable and ess sup}_{[0,T]} \Vert f(t)\Vert_X < \infty\right\}$
$u'(t)$ or $u_t(t)$	The derivative of the function $u : [0, T] \to X$

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Chapter **1**

Introduction

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An area of mathematics and engineering known as the control theory of partial differential equations (PDEs) focuses on the dynamical systems that are governed by PDEs. It is one of the most interdisciplinary research area where several areas play a major role, in particular, *functional analysis*, *spectral theory*, *complex analysis*, *non-harmonic Fourier analysis*, *number theory* and *geometry*. The purpose of control theory is to identify control inputs that can impact the evolution of a system's state variables, directing them toward a desired target or trajectory. The control problem gets more difficult when the system dynamics are characterized by PDEs due to the infinite-dimensional structure of the underlying state space. The transport equation, the heat equation, the wave equation, the Korteweg-de Vries (KdV) equation, and the system of thermoelasticity (wave-heat coupled system) are some examples of such partial differential equations. We refer to [Cor07, Ros97, LZ98] for a detail study on controllability of these equations. In this thesis, we study controllability of the linearized compressible Navier-Stokes equations (which is a transport-heat coupled system) and a nonlinear system coupling two parabolic equations in one dimension.

Controllability of PDEs has many applications in various fields, including aerospace engineering, chemical processes, structural mechanics, heat transfer, medical imaging, fluid dynamics, and electromagnetics, among others. The features of the PDE system's controllability and observability (two important concepts in control theory) come under scrutiny in this investigation. The ability to steer the system from any initial state to any desired final state within a finite time using a control input is referred to as controllability and we say the system is controllable. On the other hand, observability refers to the ability to infer the whole state of the system based on the measurements that are currently available. In other words, we say the system is observable if the entire state can be determined by observing only the (partial) information of the output(s). These characteristics are extremely important in evaluating whether or not a control strategy can be implemented and how successful it will be.

Controllability of systems described by PDEs involve applying control inputs either at the boundaries of the system or distributed throughout the spatial domain. The control input(s) applied at the boundaries of the system is referred as "boundary control", whereas the control input(s) acting in the whole domain or some part of it is called "distributed/ internal control" (see the figure below). For example, we can control the temperature of a rod by controlling only the endpoint of it, giving the boundary controllability of the system. On the other hand, applications of distributed controllability includes controlling the temperature of a room by applying heat sources in one/multiple places in the room. In practical situations, both boundary and distributed control strategies have their advantages and limitations, and the choice between them depends on factors such as the nature of the system, the control objectives, practical considerations, and the available control resources. In many cases, a combination of both boundary and distributed control may be used to achieve the desired control performance effectively.

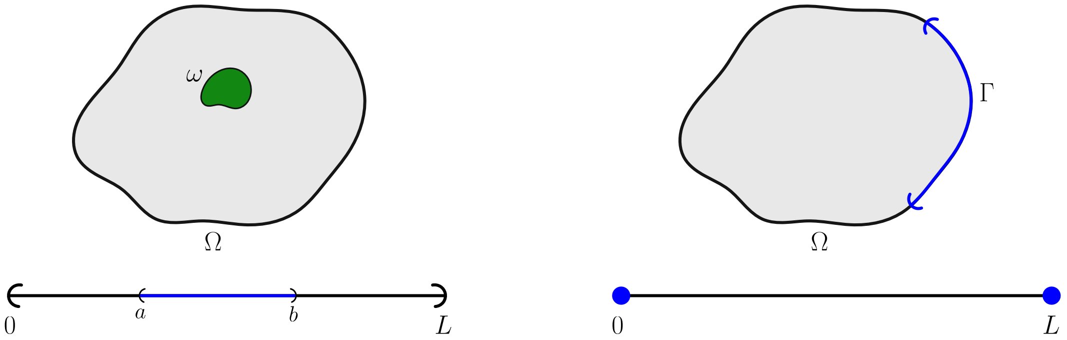

Figure 1.1: Distributed control (left) vs boundary control (right).

There are three types of controllability notions appear in a system such as exact, null and approximate controllability. Exact controllability refers to the ability to steer the state of a system from any given initial state to any given final state in finite time using control input(s). If the system can be steered from any given initial state to the origin, then we say the system is null controllable. On the other hand, approximate controllability means we can steer the state of a system from any given initial state to arbitrarily close to a desired final state using control input(s), rather than reaching the exact final state. It is easy to see that exact controllability always imply null and approximate controllability and in the case of finite dimensional linear (time invariant) systems, all these controllability notions are equivalent; see Section 2.2.1 for more details in this matter. However, for linear systems posed in infinite dimension, there are equations which are null controllable but not exactly/ approximately controllable; for example, the heat equation in bounded interval is null and approximately controllable at any time but not exactly controllable at that time (see Section 2.4 for instance). In contrast to this, we refer to the article [CRR12] where it has been shown that the one dimensional compressible Navier-Stokes system linearized around $(Q_0, 0)$ (with $Q_0 > 0$) is approximately controllable at any time but not null controllable by using a localized distributed control or a boundary control. Furthermore, for the finite dimensional linear systems, controllability at some time will imply controllability at any time, thanks to the famous Kalman rank condition. This phenomena might not necessarily true in the infinite dimensional linear systems or even finite dimensional nonlinear systems. For example, the transport equation posed in a bounded interval is null and approximately controllable at large time but not in small time by using any control (boundary or distributed), see Section 2.3 for details.

In this thesis, we mainly concentrate on controllability of systems involving transport and heat equation(s) or in some cases only heat equations, by using one boundary control. In the next chapter, we will give some highlights on the controllability properties of ODEs, the transport and heat equations, together with some important concepts (related to this thesis) such as the Riesz basis, biorthogonal families, the method of moments and the Ingham's inequalities. In Chapters 3 and 4, we will focus on the linearized Navier-Stokes equations for compressible fluids (barotropic and non-barotropic) and prove null controllability at large time in optimal spaces by using a boundary control. In chapter 5, we deal with some nonlinear heat equations and prove small time local null controllability using a Neumann control. Finally, we conclude the thesis with some future directions in Chapter 6 and with proofs of some well-posedness results in Appendix A.

1.1 Compressible Navier-Stokes system

The compressible Navier-Stokes system is a set of partial differential equations that models the motion of viscous compressible fluid substances such as liquids and gases. The basic rules of mass (continuity equation) and linear momentum conservation (Newton's second law of motion) are utilized to derive the Navier-Stokes equations. They convey both the balance of linear momentum and the conservation of mass for a fluid element. Sometimes they consist of a state equation coupling pressure, temperature, and density. The compressible Navier-Stokes system is very important because they have a wide range of practical uses, for example, they are used in modeling weather, ocean currents, water flow in a pipe and airflow, in particular, in the design of aircraft and cars. The compressible Navier-Stokes equations are also of great interest in a purely mathematical sense. These equations can be represented in $\mathbb{R}^n$ as follows:

Let $I = (0, +\infty)$ be the time interval and $\Omega \subset \mathbb{R}^n$ be a spatial domain. For a viscous compressible, isentropic (barotropic) fluid, that is, when the pressure depends only on the density and the temperature is constant, the Navier-Stokes system in $I \times \Omega$ consists of an equation of continuity

$$\rho_t(t,x) + \operatorname{div}(\rho(t,x)\boldsymbol{u}(t,x)) = 0,$$

and the momentum equation

$$\rho(t,x)[\boldsymbol{u}_t(t,x) + (\boldsymbol{u}(t,x)\cdot\nabla)\boldsymbol{u}(t,x)] + \nabla p(t,x) - \mu\Delta\boldsymbol{u}(t,x) - (\lambda+\mu)\nabla[\operatorname{div}\boldsymbol{u}(t,x)] = 0.$$

Here $\rho := \rho(t,x)$ denotes the density of the fluid and $\boldsymbol{u} := \boldsymbol{u}(t,x) = (u_1(t,x), u_2(t,x), \dots, u_n(t,x))$ is the velocity vector in $\mathbb{R}^n$. The constants λ, μ are called the viscosity coefficients that satisfy the thermodynamic restrictions $\mu > 0, \lambda + \mu \geq 0$ and the pressure $p := p(t,x)$ satisfies the following constitutive equation in $I \times \Omega$

$$p(\rho) = a\rho^\gamma, \quad \text{for } a > 0,\ \gamma \geq 1.$$

In the case of non-barotropic fluids, that is, when the pressure is a function of both density and temperature of the fluid, the Navier-Stokes system consists of an equation of continuity, the momentum equation, and an additional thermal energy equation

$$\begin{aligned} c_v\rho(t,x)[\theta_t(t,x) + \boldsymbol{u}(t,x)\cdot\nabla\theta(t,x)] + \theta(t,x)p_\theta(t,x)\operatorname{div}\boldsymbol{u}(t,x) \\ - \kappa\Delta\theta(t,x) - \lambda(\operatorname{div}\boldsymbol{u}(t,x))^2 - 2\mu\sum_{i,j=1}^{n}\frac{1}{4}\left[(u_i)_{x_j} + (u_j)_{x_i}\right]^2 = 0, \end{aligned}$$

where θ is the temperature of the fluid, c_v is the specific heat constant, and κ is the heat conductivity constant. For an ideal gas, Boyles law gives the pressure $p(t,x) = R\rho(t,x)\theta(t,x)$ in $I \times \Omega$ with R as the universal gas constant. We refer to the book by Feireisl [Fei04] for more insights on the compressible flows; see also the books [Lio98] by Lions, [NS04] by Novotný and Straškraba, and the survey paper [Fei18] by Feireisl.

In the first part of this thesis, we consider the linearized versions of the above systems in one dimension. Here, we will state the associated results (both existing and the results obtained by us) and the details will be given in subsequent chapters.

1.1.1 The barotropic case

Let $T, L > 0$. The Navier-Stokes equations for compressible barotropic fluids in the interval $(0, L)$ reads as

$$\begin{cases} \rho_t + (\rho u)_x = 0, & \text{in } (0,T)\times(0,L), \\ \rho(u_t + uu_x) + a\gamma\rho^{\gamma-1}\rho_x - (\lambda+2\mu)u_{xx} = 0, & \text{in } (0,T)\times(0,L). \end{cases} \tag{1.1}$$

This is a model for a fluid flowing in a thin tube or a narrow channel and it can be viewed as one dimensional approximations of two or three dimensional models (see the figure below). In this thesis, we want to study the linearized system associated to (1.1) around some steady states, but before that, we first define the concept of steady states.

Definition 1.1.1 (Steady states)**.** *We say a function $(\xi, \eta) \in C^2([0,L]\times[0,L])$ is a steady state of the system (1.1) if it satisfies the following stationary problem:*

$$\begin{cases} (\xi\eta)_x = 0, & \textit{in } [0,L], \\ \xi\eta\eta_x + a\gamma\xi^{\gamma-1}\xi_x - (\lambda+2\mu)\eta_{xx} = 0, & \textit{in } [0,L]. \end{cases}$$

Figure 1.2: Water flow in a narrow channel

We refer to the article [MZ19] for the existence of a steady state of the nonlinear system (1.1). With this definition, we note here that any constants of the form (Q_0, V_0) with $Q_0 > 0$ and $V_0 \geq 0$ are steady states of the system (1.1). We linearize the nonlinear system (1.1) around this constant steady state (Q_0, V_0) as follows:

The linearization of the term $(\rho u)_x = \rho_x u + \rho u_x$ around (Q_0, V_0) is $V_0\rho_x + Q_0 u_x$. Similarly, linearization of the terms $\rho(u_t + uu_x) = \rho u_t + \rho u u_x$ and $a\gamma\rho^{\gamma-1}\rho_x$ around (Q_0, V_0) are respectively $Q_0 u_t + Q_0 V_0 u_x$ and $a\gamma Q_0^{\gamma-1}\rho_x$. Thus, we arrive at the system

$$\begin{cases} \rho_t(t,x) + V_0\rho_x(t,x) + Q_0 u_x(t,x) = 0, & \text{in } (0,T)\times(0,L), \\ u_t(t,x) - \dfrac{\lambda+2\mu}{Q_0} u_{xx}(t,x) + V_0 u_x(t,x) + a\gamma Q_0^{\gamma-2}\rho_x(t,x) = 0, & \text{in } (0,T)\times(0,L). \end{cases} \tag{1.2}$$

To prove the existence of a unique solution (well-posedness) of the system (1.2), we need to impose the initial and boundary conditions into the system. Let us take the initial condition as

$$\rho(0,x) = \rho_0(x), \quad u(0,x) = u_0(x), \quad \text{for } x \in (0,L). \tag{1.3}$$

Note that, the second equation of (1.2) is of parabolic types, so we can consider any one of the following boundary conditions on u:

$$\begin{cases} u(t,0) = 0, \quad u(t,L) = 0, & \text{for } t \in (0,T), \\ u_x(t,0) = 0, \quad u_x(t,L) = 0, & \text{for } t \in (0,T), \\ u(t,0) = u(t,L), \quad u_x(t,0) = u_x(t,L), & \text{for } t \in (0,T). \end{cases}$$

On the other hand, if $V_0 = 0$, then the first equation is only a ODE in ρ and therefore we don't need to consider any boundary conditions on ρ in this case. However, if $V_0 > 0$, we get a transport equation in ρ and therefore one can consider the following boundary conditions on ρ:

$$\begin{cases} \rho(t,0) = 0, & \text{for } t \in (0,T), \\ \rho(t,0) = \rho(t,L), & \text{for } t \in (0,T). \end{cases}$$

Together, we write the boundary conditions on ρ and u as follows:

- **Control on density:**

  ◇ $\rho(t,0) = \rho(t,L) + p_1(t), \quad u(t,0) = u(t,L), \quad u_x(t,0) = u_x(t,L),$ (1.4)

  ◇ $\rho(t,0) = p_2(t), \quad u(t,0) = 0, \quad u(t,L) = 0,$ (1.5)

  ◇ $\rho(t,0) = \rho(t,L) + p_3(t), \quad u(t,0) = 0, \quad u(t,L) = 0,$ (1.6)

  for $t \in (0,T)$.

- **Control on velocity:**

  ◇ $\rho(t,0) = \rho(t,L), \quad u(t,0) = u(t,L) + q_1(t), \quad u_x(t,0) = u_x(t,L),$ (1.7)

  ◇ $\rho(t,0) = \rho(t,L), \quad u(t,0) = 0, \quad u(t,L) = q_2(t),$ (1.8)

  for $t \in (0,T)$.

Here $p_i, i = 1,2,3$ and q_i for $i = 1,2$ are the control inputs (unknowns) belonging to some Hilbert space.

Remark 1.1.1. *We note here that proving controllability of the linearized system (1.2) using a Dirichlet boundary control $q_3 \in L^2(0,T)$ acting on velocity and with homogeneous Dirichlet condition on ρ is very challenging and still an open problem.*

In this thesis, our first aim is to study the null and approximate controllability properties of the linearized system (1.2) (around (Q_0, V_0) with $Q_0, V_0 > 0$) with the initial states (1.3) and any one of the boundary conditions (1.4)–(1.8). Before going into the details, let us first define the notions of controllability for the system (1.2).

Definition 1.1.2. *Let H be a Hilbert space. We say the system (1.2) with initial state (1.3) and one of the boundary conditions (1.4)–(1.8) is*

- ***null controllable*** *at time $T > 0$ in the space H if, for any given $(\rho_0, u_0) \in H$ there exists a control $p_1 \in L^2(0,T)$ (resp. $p_2, p_3, q_1, q_2 \in L^2(0,T)$) such that the associated solution (ρ, u) satisfies*

$$(\rho(T), u(T)) = (0,0).$$

- ***approximately controllable*** *at time $T > 0$ in the space H if, for any given $(\rho_0, u_0), (\rho_T, u_T) \in H$ and given $\epsilon > 0$ there exists a control $p_{1,\epsilon} \in L^2(0,T)$ (resp. $p_{2,\epsilon}, p_{3,\epsilon}, q_{1,\epsilon}, q_{2,\epsilon} \in L^2(0,T)$) such that the associated solution $(\rho_\epsilon, u_\epsilon)$ satisfies*

$$\|(\rho_\epsilon(T,\cdot), u_\epsilon(T,\cdot)) - (\rho_T, u_T)\|_H \leq \epsilon.$$

Remark 1.1.2. *We mention here that approximate controllability of (1.2) (around (Q_0, V_0) with $Q_0, V_0 > 0$) with the initial states (1.3) and any one of the boundary conditions (1.4)–(1.8) follows from null controllability due to the backward uniqueness property of the corresponding systems; see Proposition 2.2.1. For this reason, we will concentrate only on the null controllability of these systems.*

Control on density: We first consider the case when only one boundary control is acting on the density component through the condition (1.4). More precisely, for given $T > 0$, we consider the following control problem:

$$\begin{cases} \rho_t + V_0 \rho_x + Q_0 u_x = 0, & \text{in } (0,T) \times (0,2\pi), \\ u_t - \mu_0 u_{xx} + V_0 u_x + b \rho_x = 0, & \text{in } (0,T) \times (0,2\pi), \\ \rho(t,0) = \rho(t,2\pi) + p_1(t), & \text{for } t \in (0,T), \\ u(t,0) = u(t,2\pi), \quad u_x(t,0) = u_x(t,2\pi), & \text{for } t \in (0,T), \\ \rho(0,x) = \rho_0(x), \quad u(0,x) = u_0(x), & \text{in } (0,2\pi), \end{cases} \tag{1.9}$$

with $\mu_0 := \frac{\lambda+2\mu}{Q_0}$ and $b := a\gamma Q_0^{\gamma-2}$. Here p_1 is a control input (unknown) and we take $L = 2\pi$ for simplicity. In this setup, we wish to study the null controllability properties of this system (1.9) at given time $T > 0$ in the space $(L^2(0,2\pi))^2$. Suppose that this is true, that means for any given initial state $(\rho_0, u_0) \in (L^2(0,2\pi))^2$, we can find a control $p_1 \in L^2(0,T)$ such that the solution (ρ, u) of (1.9) satisfies $(\rho(T), u(T)) = (0,0)$. Then, integrating both equations of (1.9) in the interval $(0,T) \times (0,2\pi)$, we get a compatibility condition on the initial states

$$\int_0^{2\pi} \rho_0(x) dx = -V_0 \int_0^T p_1(t) dt, \quad \int_0^{2\pi} u_0(x) dx = -b \int_0^T p_1(t) dt.$$

Since every initial state (ρ_0, u_0) in $(L^2(0,2\pi))^2$ will not satisfy this compatibility condition, we will work on the Hilbert space $(\dot{L}^2(0,2\pi))^2$ to avoid this difficulty, where

$$\dot{L}^2(0,2\pi) := \left\{ f \in L^2(0,2\pi) \ : \ \int_0^{2\pi} f dx = 0 \right\}.$$

In this setup, there is no controllability results known in the literature. The only known results available in the case when an interior control is acting in the density equation. More precisely, in [BKLB20], the authors proved (distributed) null controllability of the linearized system in the space $L^2(0,2\pi)\times \dot{L}^2(0,2\pi)$ at large time T. Moreover, they also proved that null controllability fails in the space $L^2(0,2\pi)\times \dot{L}^2(0,2\pi)$ when the time is small. In the first part of this thesis, we prove similar null controllability results of the system (1.9) using a boundary control. In addition, we derive the necessary and sufficient conditions on the coefficients such that the system (1.9) is null controllable at time T, large enough.

Before writing the main results, we first define the operator $(A,\mathcal{D}(A))$ associated to the system (1.9) as

$$A := \begin{pmatrix} -V_0\partial_x & -Q_0\partial_x \\ -b\partial_x & \mu_0\partial_{xx} - V_0\partial_x \end{pmatrix} \tag{1.10}$$

with the domain $\mathcal{D}(A) := H^1_{\text{per}}(0,2\pi)\times H^2_{\text{per}}(0,2\pi)$, where we denote the Sobolev space

$$H^s_{\text{per}}(0,2\pi) := \left\{ \varphi \ : \ \varphi(x) = \sum_{n\in\mathbb{Z}} c_n e^{inx},\ x\in(0,2\pi),\ \text{and} \ \sum_{n\in\mathbb{Z}} |n|^{2s}|c_n|^2 < \infty \right\},$$

endowed with the norm

$$\|\varphi\|_{H^s_{\text{per}}(0,2\pi)} := \left(\sum_{n\in\mathbb{Z}} \left(1+|n|^2\right)^s |c_n|^2\right)^{\frac{1}{2}}$$

for any $s>0$. Then, we write our first main result concerning the null controllability of the system (1.9) as follows:

Theorem 1.1.1. *The following statements hold:*

(i) The system (1.9) is null controllable at any time $T>\frac{2\pi}{V_0}$ in $(\dot{L}^2(0,2\pi))^2$ if and only if

$$\frac{2\sqrt{bQ_0 - V_0^2}}{\mu_0} \notin \mathbb{N}.$$

(ii) If $0<T<\frac{2\pi}{V_0}$, the system (1.9) cannot be null controllable at any time T in the space $(\dot{L}^2(0,2\pi))^2$.

We note here that null controllability at the optimal time $T=\frac{2\pi}{V_0}$ is inconclusive and there is no controllability results at this optimal time are available in the literature for this system. Also, we must mention here that, if the coefficients Q_0, V_0, μ_0 and b satisfy $\frac{2\sqrt{bQ_0-V_0^2}}{\mu_0}\in\mathbb{N}$, then the associated adjoint operator of A (defined by (1.10)) admits an eigenvalue with algebraic multiplicity and geometric multiplicity both are equal to 2, failing the unique continuation property (see Chapter 3 for details). However, if $\frac{2\sqrt{bQ_0-V_0^2}}{\mu_0}\notin\mathbb{N}$, then all the eigenvalues of A^* have geometric multiplicity 1 and in this case, we can achieve null controllability of the system (1.9) by using one boundary control acting only on density component.

Before proceeding to the next results, we first consider the change of variables:

$$\rho(t,x)\to\alpha\rho(\beta t,\delta x), \quad u(t,x)\to u(\beta t,\delta x), \quad \text{for } (t,x)\in(0,T)\times(0,L),$$

with the choices of $\alpha,\beta,\delta>0$ as

$$\alpha := \left(a\gamma Q_0^{\gamma-3}\right)^{-1/2}, \quad \beta := \frac{Q_0V_0^2}{\lambda+2\mu}, \quad \delta := \frac{Q_0V_0}{\lambda+2\mu}.$$

Then, the system of equations (1.2) reduces to

$$\begin{cases} \rho_t + \rho_x + cu_x = 0, & \text{in } (0,T)\times(0,\delta L), \\ u_t - u_{xx} + u_x + c\rho_x = 0, & \text{in } (0,T)\times(0,\delta L), \end{cases} \tag{1.11}$$

with $c = \frac{Q_0}{V_0}\left(a\gamma Q_0^{\gamma-3}\right)^{1/2}$. Here, we mention that the whole analysis in this case will be performed in the space domain $(0,1)$, which is mainly for the simplicity of computations. The same can be done in the interval $(0,\delta L)$. The system is given below:

$$\begin{cases} \rho_t + \rho_x + cu_x = 0, & \text{in } (0,T)\times(0,1), \\ u_t - u_{xx} + u_x + c\rho_x = 0, & \text{in } (0,T)\times(0,1), \\ \rho(t,0) = p_2(t), & \text{for } t\in(0,T), \\ u(t,0)=0, \quad u(t,1)=0, & \text{for } t\in(0,T), \\ \rho(0,x)=\rho_0(x), \quad u(0,x)=u_0(x), & \text{in } (0,1), \end{cases} \tag{1.12}$$

where $p_2 \in L^2(0,T)$ is the control input (unknown). Similar to the above, there is no controllability result known for this system (1.12). Further, when a distributed control is acting in the density equation, then also no controllability results are known in the literature. In the next part of this thesis, we prove null controllability of this system (1.12) at large time T in the space $\dot{L}^2(0,1)\times L^2(0,1)$ by using a boundary control $p_2 \in L^2(0,T)$, which is stated below.

Theorem 1.1.2. *The following statements hold:*

(i) Let us assume that $c^4+8c^2+5<4\pi^2$. Then, the system (1.13) is null controllable at any time $T>1$ in the space $\dot{L}^2(0,1)\times L^2(0,1)$.

(ii) Let $c>0$ be given. Then, the system (1.13) cannot be null controllable at small time $0<T<1$ in the space $\dot{L}^2(0,1)\times L^2(0,1)$.

We must mention here that finding a complete set of eigenfunctions of the associated adjoint operator is very difficult due to the Dirichlet boundary conditions. This difficulty arises because of the fact that the operator $\frac{d}{dx}$ on $H^1_{\{0\}}(0,1)$ do not have any non-trivial spectrum, where

$$H^1_{\{0\}}(0,1) := \left\{\varphi \in H^1(0,1) \ : \ \varphi(0)=0\right\}.$$

Thus, we cannot deal with the system (1.12) directly to prove the controllability results. However, to prove Theorem 1.1.2, we will consider the following control problem:

$$\begin{cases} \rho_t + \rho_x + cu_x = 0, & \text{in } (0,T)\times(0,1), \\ u_t - u_{xx} + u_x + c\rho_x = 0, & \text{in } (0,T)\times(0,1), \\ \rho(t,0) = \rho(t,1) + p_3(t), & \text{for } t\in(0,T), \\ u(t,0)=0, \quad u(t,1)=0, & \text{for } t\in(0,T), \\ \rho(0,x)=\rho_0(x), \quad u(0,x)=u_0(x), & \text{in } (0,1). \end{cases} \tag{1.13}$$

Here $p_3 \in L^2(0,T)$ is the boundary control acting as the difference between the values at $x=0$ and $x=1$. If we prove null controllability of the system (1.13) using a control $p_3 \in L^2(0,T)$, then we can define the control $p(t) := \rho(t,1)+p_3(t)$ for $t\in(0,T)$, which will be a null control for the system (1.12) once we prove $\rho(\cdot,1)\in L^2(0,T)$. Similarly, we can prove null controllability of (1.12) by assuming the same for the system (1.13). Thus, null controllability of (1.12) is equivalent to that for the system (1.13). So, our next goal is to study the null controllability of the system (1.13). In this context, we mention here that the condition on c mentioned in Theorem 1.1.2 arises while proving the controllability results related to the system (1.13), as explained below.

Theorem 1.1.3. *The following statements hold:*

(i) Let us assume that $c^4+8c^2+5<4\pi^2$. Then, the system (1.13) is null controllable at any time $T>1$ in the space $\dot{L}^2(0,1)\times L^2(0,1)$.

(ii) Let $c>0$ be given. Then, the system (1.13) cannot be null controllable at small time $0<T<1$ in the space $\dot{L}^2(0,1)\times L^2(0,1)$.

As mentioned before, if the system (1.13) is null controllable at time T, then we get a similar compatibility condition on ρ_0 (obtained by integrating the first equation in (1.12)) as

$$\int_0^1 \rho_0(x)dx = \int_0^T p_2(t)dt,$$

which is the main reason for obtaining the null controllability space as $\dot{L}^2(0,1) \times L^2(0,1)$.

Remark 1.1.3. *The condition on c is required to prove that all the eigenvalues of the associated adjoint operator are geometrically simple. However, characterization of all such c for which the system* (1.13) *satisfy the null controllability criterion is still unknown.*

As a consequence of this result, together with the fact $\rho(\cdot,1) \in L^2(0,T)$, we can conclude null controllability of the system (1.12) at time $T > 1$ in $\dot{L}^2(0,1)\times L^2(0,1)$ under the assumption $c^4+8c^2+5 < 4\pi^2$, that is, Theorem 1.1.2. This kind of techniques has been applied in many places, for instance in [CC09a, CHO16].

Control on velocity: We next consider the case when there is a boundary control acting in the velocity component through the condition (1.7). The system is given below:

$$\begin{cases} \rho_t + V_0\rho_x + Q_0 u_x = 0, & \text{in } (0,T)\times(0,2\pi), \\ u_t - \mu_0 u_{xx} + V_0 u_x + b\rho_x = 0, & \text{in } (0,T)\times(0,2\pi), \\ \rho(t,0) = \rho(t,2\pi), & \text{for } t \in (0,T), \\ u(t,0) = u(t,2\pi) + q_1(t), \quad u_x(t,0) = u_x(t,2\pi), & \text{for } t \in (0,T), \\ \rho(0,x) = \rho_0(x), \quad u(0,x) = u_0(x), & \text{in } (0,2\pi). \end{cases} \tag{1.14}$$

Here $q_1 \in L^2(0,T)$ is a control input (unknown). In this case also, we want to study the controllability properties of this system (1.14) at a given time $T > 0$ in the space $L^2(0,2\pi)\times L^2(0,2\pi)$. Similar to the density case, if the system (1.14) is null controllable at time T, then we get a compatibility condition on the initial states

$$\int_0^{2\pi} \rho_0(x)dx = -Q_0\int_0^T q_1(t)dt, \quad \int_0^{2\pi} u_0(x)dx = -V_0\int_0^T q_1(t)dt.$$

For this reason, we will work on the Hilbert space $\dot{L}^2(0,2\pi)\times\dot{L}^2(0,2\pi)$. Before stating our controllability results, let us first mention some known results for the system (1.14). In [CM15, Theorem 1.6], it is known that the system (1.14) is null controllable at any time $T > \frac{2\pi}{V_0}$ in the space $\dot{H}^{s+1}_{\text{per}}(0,2\pi)\times\dot{H}^{s}_{\text{per}}(0,2\pi)$ with $s > \frac{9}{2}$ using a boundary control $q_1 \in L^2(0,T)$ acting in the velocity part. The proof of this result was inspired by the work of Martin, Rosier and Rouchon [MRR13]. On the other hand, when an interior control is acting only in the velocity equation, it is known in [CMRR14] that the system is (distributed) null controllable at time $T > \frac{2\pi}{V_0}$ in the space $\dot{H}^1_{\text{per}}(0,2\pi)\times L^2(0,2\pi)$. Moreover, the space $\dot{H}^1_{\text{per}}(0,2\pi)\times L^2(0,2\pi)$ is optimal in the sense that if we take initial states from the space $\dot{H}^s_{\text{per}}(0,2\pi)\times L^2(0,2\pi)$ with $0 \le s < 1$, then the system cannot be null controllable at any time $T > 0$ by using a localized distributed control. Further, lack of null controllability at small time $0 < T < \frac{2\pi}{V_0}$ is shown for the system (1.14) in [Mai15] by constructing some Gaussian beam solutions. However, there are no controllability results for the system (1.14) at large time are available in the literature when the initial states belong to the space $\dot{H}^{s+1}_{\text{per}}(0,2\pi)\times\dot{H}^{s}_{\text{per}}(0,2\pi)$ with $s \le \frac{9}{2}$. The next result gives a complete answer to this question in terms of the regularity of the initial states.

Theorem 1.1.4. *The following statements hold:*

(i) The system (1.14) *is null controllable at any time* $T > \frac{2\pi}{V_0}$ *in* $\dot{H}^1_{\text{per}}(0,2\pi)\times\dot{L}^2(0,2\pi)$ *if and only if*

$$\frac{2\sqrt{bQ_0 - V_0^2}}{\mu_0} \notin \mathbb{N}.$$

(ii) If $0 \leq s < 1$, the system (1.14) cannot be null controllable at any time $T > 0$ in the space $\dot{H}^s_{\text{per}}(0, 2\pi) \times \dot{L}^2(0, 2\pi)$.

As mentioned in the density case, we have necessary and sufficient condition on the coefficients for null controllability in this case also. Moreover, we mention here that null controllability of the system (1.14) is inconclusive at the optimal time $T = \frac{2\pi}{V_0}$.

In the barotropic case, we finally consider the following system (see system (1.13)):

$$\begin{cases} \rho_t + \rho_x + cu_x = 0, & \text{in } (0,T) \times (0,1), \\ u_t - u_{xx} + u_x + c\rho_x = 0, & \text{in } (0,T) \times (0,1), \\ \rho(t,0) = \rho(t,1), \quad u(t,0) = 0, \quad u(t,1) = q_2(t), & \text{for } t \in (0,T), \\ \rho(0,x) = \rho_0(x), \quad u(0,x) = u_0(x), & \text{in } (0,1), \end{cases} \tag{1.15}$$

with $q_2 \in L^2(0,T)$ as the boundary control. We will work on the Hilbert space $\dot{L}^2(0,1) \times L^2(0,1)$ due to the compatibility condition on ρ_0:

$$\int_0^1 \rho_0(x) dx = c \int_0^1 q_2(t) dt.$$

Like the system (1.12), our aim is to study controllability properties under homogeneous Dirichlet condition on ρ and with a boundary control acting on velocity through Dirichlet condition. However, this is a very challenging problem and still no result is available in the literature (even in the distributed case). For this reason, we will work on the above system (1.15) and study the controllability properties. More precisely, we have the following result:

Theorem 1.1.5. *The following statements hold:*

(i) Let us assume that $c^4 + 8c^2 + 5 < 4\pi^2$. Then there exists a countable set $\mathcal{N}$ such that for chosen $c \notin \mathcal{N}$, the system (1.15) is null controllable at any time $T > 1$ in $\dot{H}^{\frac{1}{2}}(0,1) \times L^2(0,1)$.

(ii) If $0 \leq s < \frac{1}{2}$, the system (1.15) cannot be null controllable at any time $T > 0$ in the space $\dot{H}^s(0,1) \times L^2(0,1)$.

Remark 1.1.4. *Like the previous case, we cannot obtain any controllability results when the time is small, that is when $0 < T \leq \frac{2\pi}{V_0}$, by following the proof in the density case. Also, the condition on c is required to prove eigenvalues of the associated linear operator have geometric multiplicity 1, as mentioned before in the density case. Moreover, the set $\mathcal{N}$ appears in the above result while proving the Fattorini-Hautus criterion. However, the complete characterization of this (possible) critical set $\mathcal{N}$ is still not known.*

1.1.2 The non-barotropic case

Let $L > 0$. We next consider the Navier-Stokes system for compressible non-barotropic fluids in $(0, L)$:

$$\begin{cases} \rho_t + (\rho u)_x = 0, & \text{in } (0,T) \times (0,L), \\ \rho(u_t + uu_x) + R(\rho\theta)_x - (\lambda + 2\mu)u_{xx} = 0, & \text{in } (0,T) \times (0,L), \\ c_v\rho[\theta_t + u\theta_x] + R\rho\theta u_x - \kappa\theta_{xx} - (\lambda + 2\mu)u_x^2 = 0, & \text{in } (0,T) \times (0,L). \end{cases} \tag{1.16}$$

Similar to the barotropic case, we want to study controllability properties of the linearized system around some constant steady state (Q_0, V_0, ψ_0) with $Q_0, V_0, \psi_0 > 0$. Note that (Q_0, V_0, ψ_0) are solutions of the following stationary problem:

$$\begin{cases} (\xi\eta)_x = 0, & \text{in } [0,L], \\ \xi\eta\eta_x + R(\xi\zeta)_x - (\lambda + 2\mu)\eta_{xx} = 0, & \text{in } [0,L], \\ c_v\xi\eta\zeta_x + R\xi\eta_x\zeta - \kappa\zeta_{xx} - (\lambda + 2\mu)\eta_x^2 = 0, & \text{in } [0,L], \end{cases}$$

where $(\xi,\eta,\zeta)\in C^2([0,L]\times[0,L]\times[0,L])$. Note that, linearization of the terms $(\rho u)_x, \rho(u_t+uu_x), (\rho\theta)_x$, $\rho(\theta_t+u\theta_x), \rho\theta u_x$ and u_x^2 around (Q_0,V_0,ψ_0) are respectively $V_0\rho_x+Q_0u_x, Q_0(u_t+V_0u_x), \psi_0\rho_x+Q_0\theta_x$, $Q_0(\theta_t+V_0\theta_x), Q_0\psi_0u_x$ and 0. Thus, the system linearized around (Q_0,V_0,ψ_0) with $Q_0,V_0,\psi_0>0$ is given by

$$\begin{cases}\rho_t+V_0\rho_x+Q_0u_x=0, & \text{in } (0,T)\times(0,L),\\ u_t-\dfrac{\lambda+2\mu}{Q_0}u_{xx}+\dfrac{R\psi_0}{Q_0}\rho_x+V_0u_x+R\theta_x=0, & \text{in } (0,T)\times(0,L),\\ \theta_t-\dfrac{\kappa}{Q_0c_v}\theta_{xx}+\dfrac{R\psi_0}{c_v}u_x+V_0\theta_x=0, & \text{in } (0,T)\times(0,L).\end{cases} \tag{1.17}$$

We take the initial condition as

$$\rho(0,x)=\rho_0(x),\quad u(0,x)=u_0(x),\quad \theta(0,x)=\theta_0(x),\quad x\in(0,L). \tag{1.18}$$

In this case, we will consider any one of the following boundary conditions on the system (1.17).

- **Control on density**
  - $\rho(t,0)=\rho(t,L)+p(t),\ u(t,0)=u(t,L),\ u_x(t,0)=u_x(t,L),\ \theta(t,0)=\theta(t,L),\ \theta_x(t,0)=\theta_x(t,L),$ (1.19)
- **Control on velocity**
  - $\rho(t,0)=\rho(t,L),\ u(t,0)=u(t,L)+q(t),\ u_x(t,0)=u_x(t,L),\ \theta(t,0)=\theta(t,L),\ \theta_x(t,0)=\theta_x(t,L),$ (1.20)
- **Control on temperature**
  - $\rho(t,0)=\rho(t,L),\ u(t,0)=u(t,L),\ u_x(t,0)=u_x(t,L),\ \theta(t,0)=\theta(t,L)+r(t),\ \theta_x(t,0)=\theta_x(t,L),$ (1.21)

for $t\in(0,T)$, where p,q,r are boundary controls. In this setup, we first define the controllability notions:

Definition 1.1.3. *Let H be a Hilbert space. We say the system (1.17) with initial state (1.18) and boundary condition (1.19) (resp. (1.20), (1.21)) is*

- ***null controllable*** *at time $T>0$ in the space H if, for any given $(\rho_0,u_0,\theta_0)\in H$, there exists a control $p\in L^2(0,T)$ (resp. $q,r\in L^2(0,T)$) such that the associated solution (ρ,u,θ) of (1.17) satisfies*
$$(\rho(T),u(T),\theta(T))=(0,0,0).$$
- ***approximately controllable*** *at time $T>0$ in the space H if, for given (ρ_0,u_0,θ_0),(ρ_T,u_T,θ_T) $\in H$ and any $\epsilon>0$, there exists a control $p_\epsilon\in L^2(0,T)$ (resp. $q_\epsilon,r_\epsilon\in L^2(0,T)$) such that the associated solution $(\rho_\epsilon,u_\epsilon,\theta_\epsilon)$ of (1.17) satisfies*
$$\|(\rho_\epsilon(T),u_\epsilon(T),\theta_\epsilon(T))-(\rho_T,u_T,\theta_T)\|_H\le\epsilon.$$

We study mainly the null controllability of the system (1.17) at a given time $T>0$ starting from the initial condition (1.18) and with one of the boundary conditions (1.19), (1.20) and (1.21). There is no controllability results known in the literature for the system (1.17) at large time in the boundary control case. However, in [Mai15], a lack of null controllability result is known at small time (under Dirichlet boundary conditions) by using three localized interior controls acting in density, velocity and temperature, or using a boundary control acting only on the velocity part. In this thesis, we prove null controllability of the system (1.17) at large time in optimal spaces by using one boundary control mentioned above. Further, we also prove a lack of null controllability result at small time in the density case (like the barotropic case). Before stating our results, we first denote the (positive) coefficients appearing in the non-barotropic system (1.17) as

$$\lambda_0:=\frac{\lambda+2\mu}{Q_0},\quad \kappa_0:=\frac{\kappa}{Q_0c_v}, \tag{1.22}$$

and define the set

$$\mathcal{S}:=\left\{(\lambda_0,\kappa_0)\ :\ \sqrt{\frac{\lambda_0}{\kappa_0}}\notin\mathbb{Q}\right\}. \tag{1.23}$$

Note that, we have introduce a new notation λ_0 to denote the constant $\frac{\lambda+2\mu}{Q_0}$ instead of μ_0 to separate it from the barotropic case. We also define the operator $(A, \mathcal{D}(A))$ associated to the system (1.17) as

$$A := \begin{pmatrix} -V_0\partial_x & -Q_0\partial_x & 0 \\ -\frac{R\psi_0}{Q_0}\partial_x & \lambda_0\partial_{xx} - V_0\partial_x & -R\partial_x \\ 0 & -\frac{R\psi_0}{c_v}\partial_x & \kappa_0\partial_{xx} - V_0\partial_x \end{pmatrix} \tag{1.24}$$

with the domain $\mathcal{D}(A) := H^1_{\text{per}}(0, 2\pi) \times (H^2_{\text{per}}(0, 2\pi))^2$. In this setup, we will state our main results which concerns null controllability of the system (1.17). We take $L = 2\pi$ for simplicity of the computations.

Theorem 1.1.6. *Let us assume that $(\lambda_0, \kappa_0) \in \mathcal{S}$ be such that there exists a $M > 0$ with the property that*

$$\left| \sqrt{\frac{\lambda_0}{\kappa_0}} - \frac{a}{b} \right| > \frac{1}{b^M} \tag{1.25}$$

holds for all rational numbers $\frac{a}{b}$. We further assume that all the eigenvalues of the adjoint operator of A (defined by (1.24)) have geometric multiplicity equal to 1. Then,

(i) the system (1.17)-(1.18)-(1.19) is null controllable at any time $T > \frac{2\pi}{V_0}$ in the space $(\dot{L}^2(0, 2\pi))^3$.

(ii) the systems (1.17)-(1.18)-(1.20) and (1.17)-(1.18)-(1.21) are null controllable at any time $T > \frac{2\pi}{V_0}$ in the space $\dot{H}^1_{\text{per}}(0, 2\pi) \times (\dot{L}^2(0, 2\pi))^2$.

Proving the property that eigenvalues have geometric multiplicity 1 is not straightforward like the system (1.9), due to the complicated cubic characteristics polynomial associated to the operator A. Also, like the system (1.13) or (1.15), we do not have any characterization of the coefficients λ_0 and κ_0 for which the system will necessarily be null controllable. We refer to Chapter 3 for more insights in this matter.

The next results shows that in the density case, the system (1.17) fails to satisfy null controllability when the time is small. Moreover, we can achieve optimal space for null controllability in the velocity and temperature control case. These results are similar to those obtained in the barotropic case.

Proposition 1.1.1. *The following statements hold:*

(i) The system (1.17)-(1.18)-(1.19) is not null controllable at small time $0 < T < \frac{2\pi}{V_0}$ in $(\dot{L}^2(0, 2\pi))^3$.

(ii) The systems (1.17)-(1.18)-(1.20) and (1.17)-(1.18)-(1.21) are not null controllable at any time $T > 0$ in the space $\dot{H}^s_{\text{per}}(0, 2\pi) \times (\dot{L}^2(0, 2\pi))^2$ for any $0 \le s < 1$.

Similar to the barotropic case, lack of null controllability of the system (1.17)-(1.18)-(1.20) or (1.17)-(1.18)-(1.21) is open when the time is small, in particular, when $0 < T < \frac{2\pi}{V_0}$. Moreover, null controllability of the system (1.17) at time $T = \frac{2\pi}{V_0}$ is inconclusive in all cases, whether there is a control act in density, velocity or temperature.

In the non-barotropic case, we finally write the following result, which shows that the restriction $(\lambda_0, \kappa_0) \in \mathcal{S}$ is not sufficient to conclude null controllability of (1.17).

Proposition 1.1.2. *There exist constants $(\lambda_0, \kappa_0) \in \mathcal{S}$ and $Q_0, V_0, \psi_0, R, c_v > 0$ for which the systems (1.17)-(1.18)-(1.19), (1.17)-(1.18)-(1.20) and (1.17)-(1.18)-(1.21) are not null controllable at any time $T > 0$ in the space $(L^2(0, 2\pi))^3$.*

To prove all the aforementioned controllability results, we will use mainly two techniques; the method of moments and an application of parabolic-hyperbolic Ingham-type inequalities. In the next chapter, we will briefly introduce these notions and show some applications in the case of 1d heat equation. Here, we will write the new Ingham-type inequality that will be very crucial to prove the some of the above null controllability results.

Proposition 1.1.3 (A combined Ingham-type inequality). *Let $(\lambda_n)_{n\in\mathbb{N}}$ and $(\gamma_n)_{n\in\mathbb{Z}}$ be sequences of complex numbers satisfying the following properties: there is $N \in \mathbb{N}$ such that*

(i) $\lambda_n \neq \lambda_m$ *for all* $m, n \in \mathbb{N}$ *with* $m \neq n$, *and* $\gamma_k \neq \gamma_l$ *for all* $k, l \in \mathbb{Z}$ *with* $k \neq l$,

(ii) $\frac{-\mathrm{Re}(\lambda_n)}{|\mathrm{Im}(\lambda_n)|} \geq \hat{c}$ *for some* $\hat{c} > 0$ *and all* $n \geq N$,

(iii) there exist $r > 1$ *and* $\delta > 0$ *such that* $|\lambda_n - \lambda_m| \geq \delta\, |n^r - m^r|$ *for all* $m, n \geq N$ *with* $m \neq n$ *and*

(iv) there exist $A_0 \geq 0, B_0 \geq \delta$ *and* $\epsilon > 0$ *such that* $\epsilon(A_0 + B_0 n^r) \leq |\lambda_n| \leq A_0 + B_0 n^r$ *for all* $n \geq N$,

(v) $\gamma_n = \beta + \tau i n + e_n$ *for all* $|n| \geq N$, *where* $\beta \in \mathbb{C}, \tau > 0$ *and* $(e_n)_{|n|\geq N} \in \ell_2$.

(vi) $\{\lambda_n \ : \ n \in \mathbb{N}\} \cap \{\gamma_n \ : \ n \in \mathbb{Z}\} = \emptyset$.

Then, for any time $T > \frac{2\pi}{\tau}$, *there exists a positive constant* C *such that*

$$\int_0^T \left| \sum_{n\in\mathbb{N}} a_n e^{\lambda_n t} + \sum_{n\in\mathbb{Z}} b_n e^{\gamma_n t} \right|^2 dt \geq C \left(\sum_{n\in\mathbb{N}} |a_n|^2 \, e^{2\mathrm{Re}(\lambda_n)T} + \sum_{n\in\mathbb{Z}} |b_n|^2 \right) \tag{1.26}$$

holds for all sequences $(a_n)_{n\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{Z}}$ *in* ℓ_2.

This result is a generalization of the previously obtained Ingham-type inequalities, including [ZZ03a, ZZ03b, ZZ04, KT15]. Our proof is based on a decoupling technique as mentioned in [Zua16] and [CMRR14]. In fact, our proof works with more general assumptions on the sequences $(\lambda_n)_{n\in\mathbb{N}}$ and $(\gamma_n)_{n\in\mathbb{Z}}$ for which each of the individual parabolic and hyperbolic Ingham inequalities hold; see the works [You01, LZ02, KL05, MZ04] for a variations of these individual inequalities. We refer to Chapter 4 for more details in this regard.

We conclude this section with some known results for the compressible Navier-Stokes system. Ervedoza, Glass, Guerrero and Puel in [EGGP12] proved a local exact controllability result for the 1D compressible (linear and nonlinear) Navier-Stokes system for barotropic fluids in a bounded domain $(0, L)$ for regular initial data in $H^3(0, L) \times H^3(0, L)$ with two boundary controls, when time is large enough. This result has been improved by Ervedoza and Savel in [ES18] by choosing the initial data from $H^1(0, L) \times H^1(0, L)$; see also a generalized result [EGG16] by Ervedoza, Glass and Guerrero for dimensions 2 and 3. In this thesis, we have proved null and approximate controllability of the linear system using only one boundary control and our method of proving the controllability results are independent of the works mentioned above. On the other hand, for non-barotropic fluids, we mention the work of [Mol19], where the author proved local null controllability of the nonlinear system, in dimensions 1, 2 and 3, at large time in the space $H^2(\Omega)\times H^2(\Omega)\times H^2(\Omega)$ using three controls acting on velocity and temperature on the whole boundary and density on the inflow boundary. Moreover, in one dimension, this result has been improved by choosing the initial state from $H^1(0, L)\times H^1(0, L)\times H^1(0, L)$.

1.2 A nonlinear two-parabolic system

Recall that, the linearized Navier-Stokes system for a compressible barotropic fluid consists of a transport equation coupled with a parabolic equation. In the case of non-barotropic fluids, the linearized compressible Navier-Stokes system consists of a transport equation coupled with two parabolic equations. To study the controllability properties of these systems with Dirichlet or Neumann boundary conditions, the "vanishing viscosity method" might be useful, where we add a small viscosity term to view it as a parabolic equation. More precisely, for $\varepsilon > 0$ small enough, we consider the following parabolic equation corresponding to the first equation of (1.2) (or (1.17)):

$$\rho_t - \varepsilon \rho_{xx} + V_0 \rho_x + Q_0 u_x = 0, \quad \text{in } (0, T) \times (0, L). \tag{1.27}$$

This method was first studied by Coron and Guerrero in [CG05], where they proved boundary null controllability of the transport equation ($Q_0 = 0$ in (1.27)) by studying a one parameter family of parabolic equations (1.27) and then taking the parameter ε tend to 0. Since then, this method has been widely applied to prove controllability of many systems, see for instance the works [GL16, CnG15, CnG16, GG08, GL07, Gla10, CW24] and the references therein.

With this new equation (1.27), our system (1.2) (resp. (1.17)) now consists of two (resp. three) coupled parabolic equations. Thus, studying controllability results for these new systems with an estimate on the control (depending on ϵ) will be very useful to conclude some controllability results for the systems (1.2) and (1.17). On the other hand, to prove some local controllability results for the nonlinear systems (1.1) and (1.16) using only one boundary control, this method might be useful. However, due to the presence of complicated nonlinearity in each systems (1.1) and (1.16), proving local controllability of these systems is very challenging. For this reason, we consider a simplified system consisting of two parabolic equations coupled with square, product and non-local nonlinearities, and study the boundary null-controllability result by means of one Neumann boundary control. More precisely, for given finite time $T > 0$, we consider the following system:

$$\begin{cases} y_t - y_{xx} = f\big(y, z, \int_0^1 y, \int_0^1 z\big), & \text{in } (0,T)\times(0,1), \\ z_t - z_{xx} = g\big(y, z, \int_0^1 y, \int_0^1 z\big), & \text{in } (0,T)\times(0,1), \\ y_x(t,0) = q(t), \quad z_x(t,0) = 0, & \text{for } t \in (0,T), \\ y_x(t,1) = z_x(t,1), & \text{for } t \in (0,T), \\ y(t,1) + z(t,1) + \alpha y_x(t,1) = 0, & \text{for } t \in (0,T), \\ y(0,x) = y_0(x), \quad z(0,x) = z_0(x), & \text{in } (0,1). \end{cases} \tag{1.28}$$

Here, $\alpha \geq 0$ is some real parameter and (y_0, z_0) is the given initial data which we choose from the space $[L^2(0,1)]^2$. The function $q \in L^2(0,T)$ is the control input acting at $x = 0$ through the Neumann condition. At the point $x = 1$, the states y and z are coupled in terms of the "equality condition of their normal derivatives" and a "combined Robin-type condition". In the literature, this kind of combined conditions is typically called the δ'-type condition, see for instance [BK13, p. 26, Chapter 1.4.4] or [Exn96]. In fact, it has been addressed in [Exn96] that the wavefunction of a quantum mechanical particle living on a graph often satisfies the δ'-type boundary conditions at the junction points.

The nonlinear functions f and g in (1.28) are given by

$$\begin{cases} f\big(y, z, \int_0^1 y, \int_0^1 z\big) = -yz + ay^2 + bz^2 + r_1(t)y, \\ g\big(y, z, \int_0^1 y, \int_0^1 z\big) = yz + cy^2 + dz^2 + r_2(t)z, \end{cases} \tag{1.29}$$

where a, b, c, d are $L^\infty((0,T)\times(0,1))$ functions and

$$\begin{cases} r_1(t) = \alpha_1 \displaystyle\int_0^1 \Big(\psi_{1,1}(x)y(t,x) + \psi_{2,1}(x)z(t,x)\Big)\, dx, \\ r_2(t) = \alpha_2 \displaystyle\int_0^1 \Big(\psi_{1,2}(x)y(t,x) + \psi_{2,2}(x)z(t,x)\Big)\, dx, \end{cases} \tag{1.30}$$

with α_1, α_2 are real constants and $\psi_{1,j}, \psi_{2,j} \in L^\infty(0,1)$ for $j = 1, 2$. In this setup, we want to study the small time local null controllability of the system (1.28) in the space $(L^2(0,1))^2$. First, we define this notion:

Definition 1.2.1. *We say the system* (1.28) *is* ***small-time locally null controllable*** *around the equilibrium* $(0,0)$ *in* $(L^2(0,1))^2$ *if, for any given* $T > 0$ *there is a* $\delta > 0$ *such that for chosen initial state* $(y_0, z_0) \in (L^2(0,1))^2$ *with* $\|(y_0, z_0)\|_{(L^2(0,1))^2} \leq \delta$*, there exists a control* $q \in L^2(0,T)$ *satisfying*

$$(y(T), z(T)) = (0,0).$$

We want to mention here that the nonlinear model (1.28) is a reaction-diffusion system which often describes several biological phenomenon or chemical reactions, commonly known as "Lotka-Volterra" model with diffusion (without any boundary conditions and control for the moment, let say), that sometimes characterize the dynamics of a biological system where two species *prey* and *predator* interact between each other; see for instance [Per15, Jos14, Mur02]. In this regard, we refer the very detailed work [RBZ22], where several results concerning the controllability of reaction-diffusion systems in biology and social sciences have been addressed. In our model, we consider that the two species are interacting in the reference domain (through the nonlinear functions f, g) as well as at one boundary end (through the coupled conditions at $x = 1$). Then, our goal is to put an external control force only on one species from the other boundary end to locally control both the species at a given time T. More precisely, we prove the following result:

Theorem 1.2.1. *Let f and g be given by (1.29) and $\alpha \geq 0$. Then, the nonlinear system (1.28) is small-time locally null-controllable around the equilibrium $(0,0)$ in the space $(L^2(0,1))^2$.*

The proof of this result involves several steps, which we listed below; the detailed proof is given in Chapter 5.

Step 1. First, we prove the null controllability result of the associated linear model to (1.28) around the equilibrium $(0,0)$ using the method of moments with an estimation of the control cost, precisely $Me^{M/T}$, where M is independent in T.

Step 2. Next, by applying the source term method introduced in [LTT13], we prove a null controllability result of the linearized model with additional source terms in $L^2(0,T;(L^2(0,1))^2)$ which are exponentially decreasing as $t \to T^-$, and in this step, we notably use the precise control cost as prescribed earlier.

Step 3. Finally, we use the Banach fixed-point theorem to obtain the local null-controllability for our nonlinear system (5.1).

Chapter **2**

Preliminaries

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The goal of this chapter is to present an overview of the basic controllability results and tools that are closely related to this thesis. We first describe the basic functional tools including the concepts of biorthogonal sequence, the method of moments and some variations of Ingham's inequalities. Next, we will explain how these notions are used to derive controllability of finite and infinite dimensional linear systems, in particular, for transport and heat equations in one dimension. Finally, we make some remarks about nonlinear systems.

2.1 Functional Tools

In this section, we describe the tools that will be used throughout the thesis. We mostly state the results without proof as these are well-known and proofs of these results can be found in any functional analysis and PDE books, see for instance [Bre11, Kes89, Kes09, Eva10]. However, we give proofs of some of the important results (related to this thesis) and refer to the articles/books in others.

Let $\mathbb{F}$ be a field, which is either $\mathbb{R}$ or $\mathbb{C}$ depending on the situations. A **topological vector space** is a vector space V, over the field $\mathbb{F}$, with a Hausdorff topology and with the property that the following maps are continuous:

$$(u, v) \mapsto u + v, \quad (\alpha, u) \mapsto \alpha u$$

for all $u, v \in V$ and $\alpha \in \mathbb{F}$. A **normed linear space** is a topological vector space V that is endowed with a norm $\|\cdot\|_V$. If V is complete with respect to this norm, we say V is a **Banach space**. Further, if this norm on V is induced from an inner product, that is $\|u\|_V = \sqrt{\langle u, u\rangle_V}$, we say V is a **Hilbert space**.

Let X and Y be Banach spaces. A linear operator from X into Y is an ordered pair $(A, \mathcal{D}(A))$ such that $\mathcal{D}(A)$ is a subspace of X and the map $A : \mathcal{D}(A) \subset X \to Y$ is linear. We say A is **bounded** if there exists a positive constant C such that $\|Au\|_Y \leq C\|u\|_X$ for all $u \in \mathcal{D}(A)$. The operator A is **unbounded** if it is not bounded. Moreover, A is **densely defined** if $\mathcal{D}(A)$ is dense in X and A is **closed** if the graph of A, defined as $\mathcal{G}(A) := \{(u, Au) \ : \ u \in \mathcal{D}(A)\}$ is closed in $X \times Y$. We now provide some examples of unbounded linear operators which are relevant to this thesis.

Example 2.1.1.

(a) Let us consider $X = (C^0[0,1], \|\cdot\|_\infty)$, $\mathcal{D}(A) = C^1[0,1]$ *and* $A : \mathcal{D}(A) \subset X \to X$ *is defined by* $Au = u'$*. Then* A *is an unbounded operator. In fact, for the sequence* $u_n(t) = t^n \in \mathcal{D}(A)$*, we have* $\|u_n\|_X = 1$ *for all* $n \in \mathbb{N}$ *but* $\|Au_n\|_X \to \infty$ *as* $n \to \infty$*.*

(b) Let $L > 0$*. The operator* $A : \mathcal{D}(A) \subset L^2(0,L) \to L^2(0,L)$ *defined by* $Au = u'$ *is unbounded on* $L^2(0,L)$ *in each of the following domains:*

$$(i) \quad \mathcal{D}(A) = H^1(0,L), \quad (ii) \quad \mathcal{D}(A) = H^1_0(0,L), \quad (iii) \quad \mathcal{D}(A) = H^1_{\{0\}}(0,L), \quad (iv) \quad \mathcal{D}(A) = H^1_{\text{per}}(0,L).$$

In fact, we have for $u_n(x) = \sin(\frac{nx}{L}) \in \mathcal{D}(A)$, $\|Au_n\|_{L^2(0,L)} \to \infty$ *but* $(u_n)_{n\in\mathbb{N}}$ *is bounded in* $L^2(0,L)$*.*

(c) Let $L > 0$*. Then, it is easy to see that the operator* $A : \mathcal{D}(A) \subset L^2(0,L) \to L^2(0,L)$ *defined by* $Au = u''$ *is unbounded on* $L^2(0,L)$ *in each of the following domains:*

$$(i) \quad \mathcal{D}(A) = H^2(0,L) \cap H^1_0(0,L), \quad (ii) \ \mathcal{D}(A) = H^2_{\text{per}}(0,L).$$

Let X be a Banach space. The dual of X is denoted by X' and defined as the space of all bounded linear functionals on X, that is

$$X' := \{f : X \to \mathbb{R} \ : \ f \text{ is a bounded linear operator}\}.$$

For the Hilbert space, we have a characterization of its dual in terms of the following famous result:

Theorem 2.1.1 (Riesz Representation Theorem)**.** *Let* H *be a Hilbert space and* $f \in H'$*. Then there exists a unique* $v \in H$ *such that*

$$f(u) = \langle u, v \rangle_H$$

for all $u \in H$*. Moreover, we have* $\|f\|_H = \|v\|_H$*.*

As a consequence of this result, the map $v \mapsto f_v$ is a linear isometry of H onto H'. Thus, we can identify a Hilbert space H with its own dual via this Riesz isometry map. However, this characterization might not be possible to every space under consideration at a time; as explained below:

Remark 2.1.1. *Let* V *be a dense subspace of a Hilbert space* H *which is continuously embedded in* H*, that is, there exists a* $C > 0$ *such that* $\|u\|_H \leq C\|u\|_V$ *for all* $u \in V$*. Then one can prove the following relation:*

$$V \subset H \cong H' \subset V'.$$

Here the later inclusion (from H' *into* V'*) is also dense. This relation shows that we cannot simultaneously identify* V *with* V' *and* H *with* H'*. In this situation, we say* H *is the pivot space (identified with its dual via Riesz isometry) and* V' *is the dual of* V *with respect to the pivot space* H*. We give some examples below which shows that this situation can arise in the case of Sobolev spaces.*

Example 2.1.2. *We present here some of the examples of Hilbert spaces and their duals with respect to some pivot spaces. These examples can be found, for instance, in the books [Bre11, Kes89, Kes09].*

(i) We take $V = H^1_0(0,L)$ *and* $H = L^2(0,L)$*. We identify the space* $L^2(0,L)$ *with its dual and denote the dual of* $H^1_0(0,L)$ *by* $H^{-1}(0,L)$*. Moreover, we have the following inclusion:*

$$H^1_0(0,L) \subset L^2(0,L) \cong (L^2(0,L))' \subset H^{-1}(0,L).$$

(ii) *We take* $V = H^1_{\text{per}}(0, L)$ *and* $H = L^2(0, L)$*. Then, as before, we identify the space* $L^2(0, L)$ *with its dual and we have the following inclusion:*

$$H^1_{\text{per}}(0, L) \subset L^2(0, L) \cong (L^2(0, L))' \subset (H^1_{\text{per}}(0, L))'.$$

(iii) *Let us take* $H = \ell_2(\mathbb{R})$ *and* $V \subset H$ *be defined by*

$$V := \left\{ x = (x_n)_{n\in\mathbb{N}} \ : \ \sum_{n\in\mathbb{N}} n^2 |x_n|^2 < \infty \right\}.$$

Then V *is a Hilbert space endowed with the inner product* $\langle x, y\rangle_V := \sum_{n\in\mathbb{N}} n^2 x_n y_n$ *for* $x = (x_n)_{n\in\mathbb{N}}$ *and* $y = (y_n)_{n\in\mathbb{N}} \in V$*. We identify* H *with its dual and the dual of* V *is identified as*

$$V' := \left\{ u = (u_n)_{n\in\mathbb{N}} \ : \ \sum_{n\in\mathbb{N}} \frac{1}{|n|^2} |u_n|^2 < \infty \right\}.$$

Moreover, we have the following relation:

$$V \subset H \cong H' \subset V'.$$

We now define the adjoint of a linear operator. Let X, Y be Banach spaces and $A : \mathcal{D}(A) \subset X \to Y$ be a densely defined linear operator. The adjoint of A is an operator $(A^*, \mathcal{D}(A^*))$ on X' defined as follows: Define

$$\mathcal{D}(A^*) := \Big\{ f \in X' \ : \ \exists\, C > 0 \text{ such that } |f(Au)| \le C \|u\|_X, \ \forall u \in \mathcal{D}(A) \Big\}.$$

Take $f \in \mathcal{D}(A^*)$. Let us define $g_f : \mathcal{D}(A) \to \mathbb{R}$ by $g_f(u) = f(Au)$ for all $u \in \mathcal{D}(A)$. Then g_f is a bounded linear functional on $\mathcal{D}(A)$. Thus, there exists a unique extension $\tilde{g}_f$ of g_f on $\overline{\mathcal{D}(A)} = X$. This implies $\tilde{g}_f \in X'$. We define $A^* f = \tilde{g}_f$. Note that for all $u \in \mathcal{D}(A)$, $A^* f(u) = \tilde{g}_f(u) = g_f(u) = f(Au)$.

For a Hilbert space H, we define the adjoint operator A^* as the unique vector $f \in \mathcal{D}(A^*)$ such that $\langle A^* f, u\rangle_H = \langle f, Au\rangle_H$ for all $u \in \mathcal{D}(A)$ and we say A is self-adjoint if $(A, \mathcal{D}(A)) = (A^*, \mathcal{D}(A^*))$.

We now define a family of bounded linear operators which will be very crucial for infinite dimensional linear control systems.

Definition 2.1.1 (Semigroup)**.** *Let* X *be a Banach space and* $\{S(t)\}_{t\ge 0}$ *be a family of bounded linear operators on* V*. We say* $\{S(t)\}_{t\ge 0}$ *is a* C^0*-Semigroup if*

(i) $S(0) = I$*, where* I *is the identity map on* V*.*

(ii) $S(t + s) = S(t)S(s)$ *for all* $t, s \ge 0$*. (Semigroup property)*

(iii) *For every* $u \in X$, $S(t)u \to u$ *as* $t \to 0_+$*. (Continuity property)*

Further, if $\|S(t)\| \le 1$ *for all* $t \ge 0$*, we say that* $\{S(t)\}_{t\ge 0}$ *is a contraction semigroup.*

Example 2.1.3. *We write the following examples of semigroups which will be used throughout this thesis.*

(a) *If* A *is a bounded linear operator on a Banach space* X*, then the family* $\{S(t)\}_{t\ge 0}$ *defined by* $S(t) = e^{tA}$ *for* $t \ge 0$ *is a* C^0*-semigroup on* X*.*

(b) *Let* $X := \big\{ f : [0, \infty) \to \mathbb{R} \ : \ f$ *is bounded and uniformly continuous on* $[0, \infty)\big\}$*. It is easy to verify that* X *is a Banach space with respect to the sup norm* $\|\cdot\|_\infty$*. Let us define* $S(t) : X \to X$ *by*

$$(S(t)f)(s) = f(t + s) \text{ for all } t, s \ge 0.$$

Then $\{S(t)\}_{t\ge 0}$ *is a* C^0*-semigroup on* X*.*

(c) Let $X := \ell_2(\mathbb{C}) = \left\{x = (x_n)_{n\in\mathbb{N}} \ : \ \sum_{n\in\mathbb{N}} |x_n|^2 < \infty\right\}$ *and let* $(\lambda_n)_{n\in\mathbb{N}}$ *be a sequence of non-negative real numbers. Let us define* $S(t) : X \to X$ *by*

$$S(t)x := \left(e^{-\lambda_n t} x_n\right)_{n\in\mathbb{N}} \quad \text{for } x = (x_n)_{n\in\mathbb{N}} \in X, \ \text{and } t \geq 0.$$

Then $\{S(t)\}_{t\geq 0}$ *defines a* C^0*-semigroup on* X.

(d) Let H *be a separable Hilbert space and* $(\varphi_n)_{n\in\mathbb{N}}$ *be an orthonormal basis of* H*. Also, let* $(\lambda_n)_{n\in\mathbb{N}}$ *be a sequence of non-negative real numbers. Then the family* $\{S(t)\}_{t\geq 0}$*, where* $S(t) : H \to H$ *is defined by*

$$S(t)u = \sum_{n\in\mathbb{N}} e^{-\lambda_n t} \langle u, \varphi_n\rangle_H \varphi_n, \ \text{for } u \in H,$$

is a C^0*-semigroup on* H.

Definition 2.1.2 (Infinitesimal Generator)**.** *Let* X *be a Banach space and* $\{S(t)\}_{t\geq 0}$ *be a* C^0*-semigroup on* X*. Then the operator* $(A, \mathcal{D}(A))$ *defined by*

$$\begin{cases} \mathcal{D}(A) = \left\{u \in X \ : \ \lim_{t\to 0_+} \frac{S(t)u-u}{t} \ \text{exists}\right\}, \\ Au = \lim_{t\to 0_+} \frac{S(t)u-u}{t} \quad \text{for all } u \in \mathcal{D}(A) \end{cases}$$

is called an infinitesimal generator of the semigroup $\{S(t)\}_{t\geq 0}$.

Example 2.1.4. *We will consider only the examples of semigroups considered above (Example 2.1.3) and write the corresponding infinitesimal generators.*

(a) The bounded operator A *considered in Example 2.1.3-(a) is the infinitesimal generator of the semigroup* $\{e^{tA}\}_{t\geq 0}$ *on* X.

(b) In Example 2.1.3-(b), the generator is given by $(A, \mathcal{D}(A))$ *where*

$$\begin{cases} \mathcal{D}(A) := \left\{f \in X \ : \ f' \in X\right\}, \\ Af := f', \quad f \in \mathcal{D}(A). \end{cases}$$

(c) The semigroup defined in 2.1.3-(c) has the generator $(A, \mathcal{D}(A))$ *where*

$$\begin{cases} \mathcal{D}(A) := \left\{x = (x_n)_{n\in\mathbb{N}} \in X \ : \ (\lambda_n x_n)_{n\in\mathbb{N}} \in X\right\}, \\ Ax := (\lambda_n x_n)_{n\in\mathbb{N}}, \quad x \in \mathcal{D}(A). \end{cases}$$

(d) Finally, in Example 2.1.3-(d), the generator is given by $(A, \mathcal{D}(A))$ *where*

$$\begin{cases} \mathcal{D}(A) := \left\{u \in X \ : \ \sum_{n\in\mathbb{N}} \left|\lambda_n \langle u, \varphi_n\rangle_H\right|^2 < \infty\right\}, \\ Au := \sum_{n\in\mathbb{N}} \lambda_n \langle u, \varphi_n\rangle_H \varphi_n, \quad u \in \mathcal{D}(A). \end{cases}$$

We leave the details here and refer to the book [Vra03], which contains several examples of semigroups and its generators including the above.

We now write some important properties of a C^0-semigroup on a Banach space.

Theorem 2.1.2 (Properties of a C^0-Semigroup)**.** *Let* X *be a Banach space and let* $\{S(t)\}_{t\geq 0}$ *be a* C^0*-semigroup on* X*. Then the following statements hold:*

(i) There exist $M \geq 1$ *and* $\omega \in \mathbb{R}$ *such that* $\|S(t)\| \leq Me^{\omega t}$*, for all* $t \geq 0$.

(ii) *For all* $u \in X$ *and* $t > 0$, $\int_0^t S(\tau)u d\tau \in \mathcal{D}(A)$ *and* $A\left(\int_0^t S(\tau)u d\tau\right) = S(t)u - u$.

(iii) *For all* $u \in \mathcal{D}(A)$ *and* $t \geq 0$, $S(t)u \in \mathcal{D}(A)$, $\frac{d}{dt}S(t)u = AS(t)u = S(t)Au$, *and the mapping* $t \in [0, \infty) \mapsto S(t)u \in X$ *is* C^1.

Corollary 2.1.1. *Let* $\tau > 0$ *be given. If* A *generates a* C^0*-semigroup* $\{S(t)\}_{t \geq 0}$ *in a Banach space* X*, then for given* $u_0 \in \mathcal{D}(A)$ *and* $f \in C^1([0, \tau]; X)$*, the equation*

$$\begin{cases} u'(t) = Au(t) + f(t), & t \in (0, \tau), \\ u(0) = u_0 \end{cases} \tag{2.1}$$

admits a unique solution u *in the space* $C^1([0, \tau]; X) \cap C([0, \tau]; \mathcal{D}(A))$*. Moreover,* u *has the expression*

$$u(t) = S(t)u_0 + \int_0^t S(t - s)f(s)ds, \quad \text{for all } t \in [0, \tau].$$

We refer to the book [CZ95, Theorem 3.1.3, page 103] for a proof of the above result; see also the book [BDPDM07]. In view of this result, it is enough to find the corresponding semigroup to guarantee the existence of a solution to linear systems. In this context, we write the following result which gives necessary and sufficient condition on the operator A for generating a C^0-semigroup.

Theorem 2.1.3 (Hille-Yosida)**.** *A linear operator* $(A, \mathcal{D}(A))$ *on a Banach space* X *generates a* C^0*-semigroup of contractions* $\{S(t)\}_{t \geq 0}$ *on* X *if, and only if,*

(i) $(A, \mathcal{D}(A))$ *is closed,*

(ii) $(A, \mathcal{D}(A))$ *is densely defined,*

(iii) *for every* $\lambda > 0$, $(\lambda I - A)^{-1} : X \to \mathcal{D}(A)$ *is a bounded linear operator and* $\left\|(\lambda I - A)^{-1}\right\| \leq \frac{1}{\lambda}$.

Property (iii) of the above Theorem might be difficult to prove for unbounded linear operators in arbitrary Banach spaces. However, in Hilbert space, we can find equivalent results that are relatively easy compared to the above result. To state these results, we need the following notions:

Definition 2.1.3 (Maximal Dissipative)**.** *Let* H *be a Hilbert space. A linear operator* $(A, \mathcal{D}(A))$ *on* H *is said to be*

(i) *dissipative if* $\langle Au, u \rangle_H \leq 0$ *for all* $u \in \mathcal{D}(A)$.

(ii) *maximal dissipative if* A *is dissipative and* $\text{Range}(I - A) = H$.

With the help of these two properties, we now write the following results which we will use throughout this thesis to prove the existence of a unique solution to the linear systems. For a proof of these results, we refer to the book [Paz83, Section 1.4, Page 13]; see also [Kes89, Section 4.5, Page 188].

Theorem 2.1.4 (Lumer-Philips)**.** *An operator* $(A, \mathcal{D}(A))$ *on a Hilbert space* H *generates a* C^0*-semigroup of contractions if, and only if,* $(A, \mathcal{D}(A))$ *is maximal dissipative.*

Corollary 2.1.2. *Let* $(A, \mathcal{D}(A))$ *be a closed and densely defined linear operator on a Hilbert space* H*. Then,* $(A, \mathcal{D}(A))$ *generates a* C^0*-semigroup of contractions if both* $(A, \mathcal{D}(A))$ *and* $(A^*, \mathcal{D}(A^*))$ *are dissipative.*

From the Lumer-Philips theorem, the existence of a unique solution to the linear system (2.1) in a Hilbert space is equivalent to proving A is maximal dissipative and to prove an operator is maximal, the following result plays an important role. This result is a generalization of the famous Riesz representation theorem (Theorem 2.1.1); see the books [DL88, DL90] for a proof of this result.

Theorem 2.1.5 (Lax-Milgram Theorem)**.** *Let* H *be a Hilbert space. Let* $B : H \times H \to \mathbb{R}$ *be a bilinear mapping such that* *(i) there exists a positive constant* $\alpha > 0$ *such that* $|B(u, v)| \le \alpha \|u\|_H \|v\|_H$ *for all* $u, v \in H$,

(ii) there exists a positive constant $\beta > 0$ *such that* $B(u, u) \ge \beta \|u\|_H^2$ *for all* $u \in H$.

Then, for every bounded linear functional $f : H \to \mathbb{R}$ *there exists a unique* $u \in H$ *such that* $B(u, v) = \langle f, v \rangle_H$ *for all* $v \in H$.

The next important part in this thesis involves the spectrum of a linear operator. Let X be a Banach space and $A : \mathcal{D}(A) \subset X \to X$ be a linear operator. For given $\lambda \in \mathbb{C}$, we define the operator $A_\lambda : \mathcal{D}(A) \subset X \to X$ by $A_\lambda := (\lambda I - A)$. The inverse of A_λ, that is the operator $(\lambda I - A)^{-1}$ (if exists) is called the **resolvent operator** of A. Further, the **resolvent set** of A is denoted by $\rho(A)$ and is defined as

$$\rho(A) = \left\{\lambda \in \mathbb{C} \;:\; (\lambda I - A)^{-1} \text{ is a densely defined bounded linear operator on } X\right\}.$$

The complement of the resolvent set is called the **spectrum** of the operator A, that is, $\sigma(A) := \mathbb{C} \backslash \rho(A)$. Moreover, the spectrum of A can be partitioned into the following disjoint sets:

(a) The **point spectrum** of **discrete spectrum** is denoted by $\sigma_p(A)$ and is defined as

$$\sigma_p(A) := \{\lambda \in \mathbb{C} \;:\; (\lambda I - A) \text{ is not invertible}\}.$$

The element $\lambda \in \sigma_p(A)$ is called an **eigenvalue** of the operator A and the elements $u \in \ker(A_\lambda)$ are called **eigenvectors/ eigenfunctions** of A corresponding to this eigenvalue λ. Moreover, the dimension of $\ker(A_\lambda)$ is called the **geometric multiplicity** of the eigenvalue λ.

(b) The **continuous spectrum** is denoted by $\sigma_c(A)$ and is defined as

$$\sigma_c(A) := \left\{\lambda \in \mathbb{C} \;:\; (\lambda I - A)^{-1} \text{ is a densely defined unbounded operator on } X\right\}.$$

(c) The **residual spectrum** is denoted by $\sigma_r(A)$ and is defined as

$$\sigma_r(A) := \left\{\lambda \in \mathbb{C} \;:\; (\lambda I - A)^{-1} \text{ exists but not densely defined}\right\}.$$

Note that, if $\lambda \in \sigma_r(A)$, then the operator $(\lambda I - A)^{-1}$ may be bounded or unbounded.

We also mention here that, if X is a Hilbert space and $\lambda \in \sigma(A)$ then $\bar{\lambda} \in \sigma(A^*)$.

Let X, Y be Banach spaces. We say a linear operator $A : X \to Y$ is **compact** if $A(B)$ is relatively compact in Y for every bounded set B in X. In other words, A is compact if and only if, for every bounded sequence $(u_n)_{n\in\mathbb{N}}$ in X, the sequence $(Au_n)_{n\in\mathbb{N}}$ has a convergent subsequence in Y. Moreover, the operator A is compact if and only if A^* is compact.

We now write two important results related to the spectrum and resolvent of a linear operator, which we have used throughout this thesis. The proof can be found in [DS71, Lemma 2, Page 2292].

Theorem 2.1.6. *Let* H *be a Hilbert space and* $A : \mathcal{D}(A) \subset H \to H$ *be a linear operator. Then:*

(i) If the resolvent operator $(\lambda I - A)^{-1}$ *is compact in* H *for some* $\lambda \in \rho(A)$, *then the spectrum of* A *is discrete and contains only the eigenvalues of* A.

(ii) If the resolvent $(\lambda I - A)^{-1}$ *of* A *is compact in* H *for some* $\lambda \in \rho(A)$, *then it is compact for every* $\lambda \notin \sigma(A)$.

We conclude this section with an important result for a compact self-adjoint operator that guarantees the existence of a basis consisting only the eigenvectors of that operator. For the proof of this result, we refer to the book [Bre11, Theorem 6.11, Page Page 167].

Theorem 2.1.7. *Let* H *be a separable Hilbert space and let* $A : \mathcal{D}(A) \subset H \to H$ *be a compact self-adjoint operator. Then there exists an orthonormal basis of* H *consisting of the eigenvectors of* A.

2.1.1 Riesz basis

The concept of Riesz basis comes from functional analysis and operator theory, in particular, in the context of Hilbert spaces. It is a special type of basis that possesses properties related to the standard orthonormal basis but doesn't necessarily consist of orthogonal vectors. We first define the notion of a Riesz basis (see [CZ20, Section 3.2, Page 79] for instance).

Definition 2.1.4. *Let H be a Hilbert space. We say a family $\{\varphi_n \ : \ n \in \mathbb{N}\} \subset H$ is a **Riesz basis** of H if the following conditions hold:*

(i) The set $\{\varphi_n \ : \ n \in \mathbb{N}\}$ is complete in H, that is, $\overline{\operatorname{span}\{\varphi_n \ : \ n \in \mathbb{N}\}} = H$.

(ii) There exist constants $C_1, C_2 > 0$ such that the inequality

$$C_1 \sum_{n=1}^{N} |a_n|^2 \leq \left\| \sum_{n=1}^{N} a_n \varphi_n \right\|_H^2 \leq C_2 \sum_{n=1}^{N} |a_n|^2 \tag{2.2}$$

holds for any given finite sequence $(a_n)_{1 \leq n \leq N} \subset \mathbb{C}$.

We first write the following result which gives a relation between Riesz and orthonormal basis. The proof of this result can be found in many books, for instance in [CZ20, Lemma 3.2.2]; see also the book by Young [You01, Chapter 1].

Theorem 2.1.8. *Let H be a Hilbert space and $\{e_n \ : \ n \in \mathbb{N}\}$ be an orthonormal basis of H. Then, a family $\{\varphi_n \ : \ n \in \mathbb{N}\} \subset H$ is a Riesz basis of H if, and only if, there exists an invertible linear transformation $T : H \to H$ such that $Te_n = \varphi_n$ holds for all $n \in \mathbb{N}$.*

Using the above result, we can prove that a Riesz basis has similar properties to an orthonormal basis. More precisely, we have the following result, the proof of which can be found, for instance, in [CZ20, Lemma 3.2.4, Page 82].

Theorem 2.1.9. *Let H be a Hilbert space and let $\{\varphi_n \ : \ n \in \mathbb{N}\} \subset H$ be a Riesz basis of H. Then there exists a unique family $\{\psi_n \ : \ n \in \mathbb{N}\}$ in H such that every $u \in H$ can be expressed uniquely as*

$$u = \sum_{n \in \mathbb{N}} \langle u, \psi_n \rangle_H \, \varphi_n$$

with

$$C_1 \sum_{n \in \mathbb{N}} \left| \langle u, \psi_n \rangle_H \right|^2 \leq \|u\|_H^2 \leq C_2 \sum_{n \in \mathbb{N}} \left| \langle u, \psi_n \rangle_H \right|^2$$

for some constants $C_1, C_2 > 0$.

We can further characterize the Riesz basis even when we do not have any orthonormal basis. In fact, the following result shows that any independent family that is close to a Riesz basis (in a sense given below) is also a Riesz basis. We refer to the book [You01, Theorem 15, Page 38] and the article of Gohberg and Krein [GK69] for a proof of this result.

Theorem 2.1.10 (Bari)**.** *Let H be a Hilbert space and $\{\varphi_n \ : \ n \in \mathbb{N}\}$ be a Riesz basis of H. Let $\{\psi_n \ : \ n \in \mathbb{N}\}$ be a subset of H with the following properties:*

(i) The set $\{\psi_n \ : \ n \in \mathbb{N}\}$ is ω-linearly independent in H, that is, if there exists $(a_n)_{n \in \mathbb{N}} \subset \mathbb{C}$ such that $\sum_{n \in \mathbb{N}} a_n \psi_n = 0$, then $a_n = 0$ for all $n \in \mathbb{N}$.

(ii) The set $\{\psi_n \ : \ n \in \mathbb{N}\}$ is quadratically close to $\{\varphi_n \ : \ n \in \mathbb{N}\}$ in H, that is, $\sum_{n \in \mathbb{N}} \|\varphi_n - \psi_n\|_H^2 < \infty$.

Then, $\{\psi_n \ : \ n \in \mathbb{N}\}$ is also a Riesz basis of H.

In control theoretic perspective, one requires to find a Riesz basis that consists only the (generalized) eigenvectors of certain linear operators. To do so, one can apply the above result and therefore we need to estimate "high frequencies of eigen-elements" by asymptotic analysis technique. Also, we need to find a sequence of (generalized) eigenvectors $\{\psi_n\}_{n\geq N+1}$ (for some large $N \in \mathbb{N}$) that is quadratically close to a given Riesz basis $\{\phi_n\}_{n\geq N+1}$. Finally, the most difficult part is to show that the number of linearly independent "lower frequencies" of eigenvectors is exactly N. To simplify the last step, we write the following result which includes these lower frequencies of eigenvectors and for the proof of this result, we refer to the book by Singer [Sin70, Corollary 11.4]; see also the article [Guo01].

Proposition 2.1.1. *Let H be a Hilbert space and $\{\varphi_n \,:\, n \in \mathbb{N}\}$ be a Riesz basis of H. Let $\{\psi_n \,:\, n \geq N+1\}$ (for some $N \geq 0$) be a subset of H such that $\sum_{n\geq N+1} \|\varphi_n - \psi_n\|_H^2 < \infty$. Then there exists an $M \geq N$ such that the set $\{\varphi_n \,:\, 1 \leq n \leq M\} \cup \{\psi_n \,:\, n \geq M+1\}$ forms a Riesz basis of H.*

The above result includes only higher frequencies of the elements $\{\psi_n\}$ and lower frequencies of the known basis $\{\phi_n\}$. If $\{\psi_n \,:\, n \in \mathbb{N}\}$ is the set of (generalized) eigenvectors of a linear operator A, then from this result we cannot conclude Riesz basis property of the whole set of eigenvectors $\{\psi_n \,:\, n \in \mathbb{N}\}$, as the set $\{\varphi_n \,:\, 1 \leq n \leq M\}$ might not necessarily be the (generalized) eigenvector of A. To ease this difficulty, we will state the following result of B. Z. Guo (see [Guo01, Theorem 6.3]) which shows that we can obtain a Riesz basis with elements from only the (generalized) eigenvectors of the operator A.

Theorem 2.1.11. *Let H be a Hilbert space and $A : \mathcal{D}(A) \subset H \to H$ be a densely defined linear operator such that the resolvent of A is compact in H. Let $\{\varphi_n \,:\, n \in \mathbb{N}\}$ be a Riesz basis of H. If there exists a family of generalized eigenvectors $\{\psi_n \,:\, n \geq N\} \subset H$ of A (for some $N \geq 0$) such that $\sum_{n\geq N+1} \|\varphi_n - \psi_n\|_H^2 < \infty$, then:*

(i) There exist constant $M > N$ and a finite sequence of generalized eigenvectors $\{\tilde{\psi}_n \,:\, 1 \leq n \leq M\}$ of A such that the family $\{\tilde{\psi}_n \,:\, 1 \leq n \leq M\} \cup \{\psi_n \,:\, n \geq M+1\}$ forms a Riesz basis of H.

(ii) The spectrum of A is $\sigma(A) = \{\lambda_n \,:\, n \in \mathbb{N}\}$, where λ_n is the corresponding eigenvalues of A (counted with algebraic multiplicity).

(iii) If there exists an $M_0 > 0$ such that $\lambda_n \neq \lambda_m$ for all $m, n > M_0$, then there exists an $N_0 > M_0$ such that all the eigenvalues $(\lambda_n)_{n>N_0}$ of A are algebraically simple.

Example 2.1.5. *We now give some examples of Riesz bases in respective Hilbert spaces and refer to the books [You01] and [CZ20], which contains several interesting examples of Riesz bases.*

(a) Every orthonormal basis in a Hilbert space H is a Riesz basis of H.

(b) The families $\left\{\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right) \,:\, n \in \mathbb{N}\right\}$ and $\left\{\frac{1}{\sqrt{L}}e^{\frac{2in\pi x}{L}} \,:\, n \in \mathbb{N}\right\}$ are orthonormal bases of $L^2(0,L)$ and hence form Riesz bases of $L^2(0,L)$.

(c) Let $s > 0$. The families $\left\{n^s \sin\left(\frac{n\pi x}{L}\right) \,:\, n \in \mathbb{N}\right\}$ and $\left\{n^s e^{\frac{2in\pi x}{L}} \,:\, n \in \mathbb{N}\right\}$ are Riesz bases of $(H^s(0,L))'$. This can be proved easily by applying Theorem 2.1.8.

2.1.2 Biorthogonal sequences

Biorthogonal sequences are an important part in various areas of mathematics and in particular, they are used to prove controllability of several dynamical systems. In this section, we define the biorthogonal sequence and state some important results of existence of such sequences. These results play crucial roles throughout this thesis.

Definition 2.1.5. *Let H be a Hilbert space and $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements in H. We say a family $(y_k)_{k\in\mathbb{N}} \subset H$ is biorthogonal to $(x_n)_{n\in\mathbb{N}}$ in H if*

$$\langle x_n, y_k\rangle_H = \delta_n^k \quad \text{for all } n, k \in \mathbb{N}.$$

We first consider the simplest case which guarantees the existence of a biorthogonal sequence of the family $(e^{-n^2\pi^2 t})_{n\in\mathbb{N}}$. Moreover, we obtain some bounds on the biorthogonal sequence, which is very crucial to prove controllability results of the linear systems.

Theorem 2.1.12. *Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of distinct positive reals. Let us further assume that there exist $N \in \mathbb{N}$ large enough and a constant $\delta > 0$ such that*

$$\begin{cases} \displaystyle\sum_{n\geq N} \frac{1}{\lambda_n} \leq \epsilon, & \text{for all } \epsilon > 0, \\ |\lambda_{n+1} - \lambda_n| \geq \delta, & \text{for all } n \in \mathbb{N}. \end{cases}$$

Then, for given $T > 0$, there exists a biorthogonal sequence $(q_k)_{k\in\mathbb{N}} \subset L^2(0,T)$ to the family $(e^{-\lambda_n t})_{n\in\mathbb{N}}$ in the space $L^2(0,T)$ with the estimate

$$\|q_k\|_{L^2(0,T)} \leq Ce^{\epsilon\lambda_k} \quad \text{for all } k \in \mathbb{N}, \tag{2.3}$$

for some constant $C > 0$.

We refer to the work of Fernández-Cara, González-Burgos and Teresa for a proof of this result in a more general setting (see Theorem 2.1.14 below); see also the lecture notes by Boyer [Boy23, Theorem IV.1.10, Page 52] and of Micu and Zuazua [Zua06, Theorem 2.6.2, Page 142]. In fact, the above result is a consequence of the well-known Müntz Theorem, which says that the family $(e^{\lambda_n t})_{n\in\mathbb{N}}$ is complete in $L^2(0,T)$ if and only if $\displaystyle\sum_{n\in\mathbb{N}} \frac{1}{\lambda_n} = \infty$. Moreover, the gap condition is required to obtain the L^2-estimate on the biorthogonal sequence $(q_k)_{k\in\mathbb{N}}$. In this context, we mention that the Müntz Theorem is a generalization of the famous *Weierstrass approximation theorem*, see [Rud87, Section 15.25, Page 312] for more details in this regard.

There are many generalizations of this result available in the literature. We present here some of the results that are relevant to this thesis.

Theorem 2.1.13. *Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers with the following properties: There exists $N \in \mathbb{N}$ large enough and constants $\epsilon, \delta, \hat{c}, A_0 > 0$, $r > 1$ and $B_0 \geq \delta$ such that*

(P1) $\lambda_k \neq \lambda_n$ *for all* $k, n \in \mathbb{N}$ *with* $k \neq n$,

(P2) $\frac{\mathrm{Re}(\lambda_n)}{|\mathrm{Im}(\lambda_n)|} \geq \hat{c}$ *for all* $n \geq N$,

(P3) $|\lambda_k - \lambda_n| \geq \delta\,|k^r - n^r|$ *for all* $k, n \geq N$ *with* $k \neq n$,

(P4) $\epsilon(A_0 + B_0 n^r) \leq |\lambda_n| \leq A_0 + B_0 n^r$ *for all* $n \geq N$.

Then there exists a sequence $(q_k)_{k\in\mathbb{N}}$ biorthogonal to $(e^{-\lambda_n t})_{n\in\mathbb{N}}$ in $L^2(0,T)$. Moreover, for given $\epsilon > 0$ there exists a constant $C_\epsilon > 0$ such that

$$\|q_k\|_{L^2(0,T)} \leq C_\epsilon e^{\epsilon \mathrm{Re}(\lambda_k)} \quad \text{for all } k \in \mathbb{N}. \tag{2.4}$$

This result has been proved by Hansen in [Han91]. We note here that, the above result is also valid in the case $r = 1$. This is written in the following result:

Theorem 2.1.14. *Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers with the properties*

$$\begin{cases} \mathrm{Re}(\lambda_n) \geq \delta\,|\lambda_n|, \quad |\lambda_n - \lambda_k| \geq \hat{c}\,|n-k|, \quad \text{for all } n, k \in \mathbb{N}, \\ \displaystyle\sum_{n\in\mathbb{N}} \frac{1}{|\lambda_n|} < \infty \end{cases} \tag{2.5}$$

for some $\delta, \hat{c} > 0$. Then there exists a biorthogonal sequence $(q_k)_{k\in\mathbb{N}}$ to $(e^{-\lambda_n t})_{n\in\mathbb{N}}$ in $L^2(0,T)$. Moreover, for given $\epsilon > 0$, we have the following estimate

$$\|q_k\|_{L^2(0,T)} \leq Ce^{\epsilon \mathrm{Re}(\lambda_k)} \quad \text{for all } k \in \mathbb{N}, \tag{2.6}$$

where $C > 0$ is a constant.

For the proof of this result, we refer to the work [FCGBdT10, Lemma 3.1]. We must mention here that the condition $\sum_{n\in\mathbb{N}} \frac{1}{|\lambda_n|} < \infty$ and $\mathrm{Re}(\lambda_n) \geq \delta |\lambda_n|$ is enough to find the existence of a biorthogonal family. However, to obtain the required bound on the biorthogonal sequence, the gap condition becomes the necessary part.

In all of the above cases, the constant C appearing in the biorthogonal estimates do not have precise dependence on T. In the context of controllability of nonlinear systems, this dependence plays a crucial role (see Chapter 5 for more details). The following result gives an optimal estimates on this constant and in fact this is the more general result currently available in the literature. For the proof of this result, we refer to the lecture note by Boyer [Boy23, Theorem V.4.26 & Corollary V.4.27], see also [ABM21].

Theorem 2.1.15. *Let Λ be a subset of complex numbers satisfying the following properties:*

(i) There exists $\eta > 0$ such that $\Lambda \subset S_\eta$, where

$$S_\eta := \{z \in \mathbb{C} \; : \; \mathrm{Re}(z) > 0, \; \textit{and} \; |\mathrm{Im}(z)| < \sinh(\eta)\,|\mathrm{Re}(z)|\}.$$

(ii) There exists $\kappa > 0$ and $\beta \in (0,1)$ such that

$$N_\Lambda(r) \leq \kappa r^\beta, \quad \textit{for all } r > 0 \tag{2.7}$$

and

$$|N_\Lambda(r) - N_\Lambda(s)| \leq \kappa \left(1 + |r-s|^\beta\right), \quad \textit{for all } r, s > 0, \tag{2.8}$$

where $N_\lambda : [0,\infty) \to \mathbb{N}$ is the counting function defined by

$$N_\Lambda(r) := \#\{\lambda \in \Lambda \; : \; |\lambda| < r\}.$$

(iii) There exists $\gamma > 0$ such that

$$|\lambda - \mu| \geq \gamma, \quad \textit{for all } \lambda, \mu \in \Lambda, \lambda \neq \mu. \tag{2.9}$$

Then, for any given $T > 0$, there exists a family $(q_{\lambda,T})_{\lambda\in\Lambda}$ in $L^2(0,T)$ biorthogonal to $(e^{-\lambda t})_{\lambda\in\Lambda}$, that is,

$$\int_0^T e^{-\lambda t} q_{\mu,T}\, dt = \delta_{\lambda,\mu}, \quad \textit{for all } \lambda, \mu \in \Lambda.$$

Moreover, we have the following estimate

$$\left\|q_{\mu,T}\right\|_{L^2(0,T)} \leq M e^{\frac{T}{2}\mathrm{Re}(\mu) + M(\mathrm{Re}(\mu))^\beta + MT^{-\frac{\beta}{1-\beta}}}, \quad \textit{for all } \mu \in \Lambda, \tag{2.10}$$

for some constant $M > 0$ depending only on κ, β and γ.

2.1.3 The method of moments

One of the important topic of discussion in this thesis is the method of moments, which is very useful for the study of controllability of linear systems (both finite and infinite dimensional). This method can be used, in particular, to prove controllability of ordinary differential equations, the heat equation, wave equation and other partial differential equations whose solutions can be computed using separation of variables, see for instance [FR71, FR75, Rus78]. In Section 2.4, we have explained one application of this method in the case of 1d heat equation and in Chapters 4 and 5, this method is also applied for some coupled linear systems. In the present section, we give a brief description of this method, which can be found, for instance, in the book [DZ06, Section 3.3, Page 36].

Let H be a Hilbert space and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in H. For a given sequence $(a_n)_{n\in\mathbb{N}} \in \ell_2$, the problem of moments is to find a $q \in H$ such that

$$\langle q, x_n \rangle_H = a_n, \; \text{for all } n \in \mathbb{N}. \tag{2.11}$$

To solve the above problem, it is enough to find a biorthogonal family of $(x_n)_{n\in\mathbb{N}}$ in H. In fact, if $q_k \in H$ satisfy $\langle q_k, x_n\rangle_H = \delta_n^k$ for all $n,k \in \mathbb{N}$, then the element $q = \sum_{k\in\mathbb{N}} a_k q_k$ verifies $\langle q, x_n\rangle_H = \sum_{k\in\mathbb{N}} a_k \langle q_k, x_n\rangle_H = \sum_{k\in\mathbb{N}} a_k \delta_n^k = a_n$ for all $n \in \mathbb{N}$. Thus we get a solution for the general problem provided $q \in H$, i.e., $\|q\|_H < \infty$. More precisely, we have the following statement:

Lemma 2.1.1. *Let H be a Hilbert space and let $x_n \in H$ for all $n \in \mathbb{N}$. If $(q_k)_{k\in\mathbb{N}}$ is a biorthogonal sequence to $(x_n)_{n\in\mathbb{N}}$ in H, then for any given sequence $(a_n)_{n\in\mathbb{N}} \in \ell_2$ such that*

$$\sum_{k\in\mathbb{N}} |a_k| \;\; \|q_k\|_H < \infty,$$

there exists a solution $q = \sum_{k\in\mathbb{N}} a_k q_k \in H$ of the moment problem (2.11).

The above result shows that solving a moments problem consists of determining a biorthogonal sequence with appropriate norms. We now give an example which we will describe in detail in the later sections in this thesis.

Example 2.1.6. *Let us consider the Hilbert space $H = L^2(0,T)$. For a sequence of positive real numbers $(\lambda_n)_{n\in\mathbb{N}}$, we define $x_n(t) = e^{-\lambda_n t}$ for $t \in [0,T]$. Then, for given $(a_n)_{n\in\mathbb{N}} \in \ell_2$, the problem of moments is to find a $q \in L^2(0,T)$ such that*

$$\int_0^T q(t)e^{-\lambda_n t}dt = a_n, \quad \forall n \in \mathbb{N}. \tag{2.12}$$

If $(q_k)_{k\in\mathbb{N}}$ is biorthogonal to $(e^{-\lambda_n t})_{n\in\mathbb{N}}$ in $L^2(0,T)$, then the solution of the moment problem (2.12) is given by

$$q(t) = \sum_{n\geq 1} a_n q_n(t), \quad t \in (0,T),$$

provided this series is convergent in $L^2(0,T)$, that is

$$\sum_{n\geq 1} |a_n| \, \|q_n\|_{L^2(0,T)} < \infty.$$

2.1.4 Ingham's inequalities

Apart from the method of moments, we will use the well-known Ingham's inequality and some variations of it to deduce our main controllability results of this thesis. This type of inequality is a fundamental result in harmonic analysis, particularly in the study of Fourier series and Fourier transforms. More precisely, it is a key tool in proving results related to the convergence of Fourier series and the decay properties of Fourier transforms. Further, it is also very useful in proving certain observability inequalities, giving some controllability results for the linear control systems.

We first write the following result known as the original Ingham's inequality, the proof of which was given by Ingham in [Ing36]; see also the lecture note [MZ04, Theorem 2.4.1] by Micu and Zuazua and the book [KL05, Theorem 4.3] by Komornik and Loretti.

Theorem 2.1.16. *Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of real numbers satisfying*

$$\gamma := \inf_{m\neq n} |\lambda_m - \lambda_n| > 0. \tag{2.13}$$

Then, for every bounded interval $[a,b]$ with $b-a > \frac{2\pi}{\gamma}$, there exist constants $C_1, C_2 > 0$ such that

$$C_1 \sum_{n\in\mathbb{N}} |a_n|^2 \leq \int_a^b \left|\sum_{n\in\mathbb{N}} a_n e^{i\lambda_n t}\right|^2 dt \leq C_2 \sum_{n\in\mathbb{N}} |a_n|^2 \tag{2.14}$$

holds for every $(a_n)_{n\in\mathbb{N}} \in \ell_2$.

Remark 2.1.2. *We note here that the above inequality (2.14) generalizes the well-known Parseval's identity.*

Here, the gap condition (2.13) is necessary to prove this inequality. Indeed, if the inequality (2.14) is true then we have for $a_k = 1$ if $k = n, m$ and $a_k = 0$ for all $k \neq m, n$ that

$$
\begin{aligned}
2C_1 \leq \int_a^b \left| e^{i\lambda_n t} - e^{i\lambda_m t} \right|^2 dt &\leq \int_a^b |\cos(\lambda_n t) - \cos(\lambda_m t) + i(\sin(\lambda_n t) - \sin(\lambda_m t))|^2 \, dt \\
&= \int_a^b [2 - 2\cos(\lambda_n - \lambda_m)t] dt \\
&\leq \int_a^b |\lambda_n - \lambda_m|^2 \, t^2 dt = |\lambda_n - \lambda_m|^2 \, \frac{b^3 - a^3}{3}
\end{aligned}
$$

and therefore we get

$$
|\lambda_n - \lambda_m| \geq \sqrt{\frac{6C_1}{b^3 - a^3}} > 0.
$$

Remark 2.1.3. *The positive constants C_1, C_2 appearing in the Ingham inequality depends on a, b and γ, but the explicit expressions of these (optimal) constants are still unknown. However, some estimates of these constants C_1 and C_2 are known. More precisely, for $[a, b] = [0, T]$, it is known in [Ing36] that*

$$
C_1 \geq \frac{2}{\pi}\left(T - \frac{4\pi^2}{\gamma^2 T}\right), \quad \text{and} \quad C_2 \leq \frac{20T}{\min(2\pi, \gamma T)}.
$$

Moreover, if we take $\lambda_n = n$ for $n \in \mathbb{N}$, then we can explicitly compute these constants and is given by

$$
C_1 = \left[\frac{T}{2\pi}\right]\pi, \quad \text{and} \quad C_2 = C_1 + 1,
$$

see for instance [HLP16]. We also mention here that, the Ingham inequality (2.14) is inconclusive in the optimal case, that is when $b - a = \frac{2\pi}{V_0}$.

Note that, the Ingham inequality (2.14) implies that the family $\left\{e^{i\lambda_n t} \ : \ n \in \mathbb{N}\right\}$ forms a Riesz basis in the space $\overline{\operatorname{span}\left\{e^{i\lambda_n t} \ : \ n \in \mathbb{N}\right\}}^{L^2(0,T)}$, see the Definition 2.1.4 of Riesz basis. Following the same idea of proving Theorem 2.1.16, we can allow small perturbations on the sequence $(\lambda_n)_{n\in\mathbb{N}}$ and still obtain the similar inequality, see [CMRR14] for instance. This result is very useful in the case of coupled hyperbolic PDEs or in particular when a perturbation term is present in the equations.

Theorem 2.1.17. *Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of real numbers satisfying*

$$
\gamma := \inf_{m \neq n} |\lambda_m - \lambda_n| > 0. \tag{2.15}
$$

Let $(\epsilon_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers converging to 0. Then, for every bounded interval $[a, b]$ with $b - a > \frac{2\pi}{\gamma}$, there exist constants $C_1, C_2 > 0$ such that

$$
C_1 \sum_{n\in\mathbb{N}} |a_n|^2 \leq \int_a^b \left| \sum_{n\in\mathbb{N}} a_n e^{(i\lambda_n + \epsilon_n)t} \right|^2 dt \leq C_2 \sum_{n\in\mathbb{N}} |a_n|^2 \tag{2.16}
$$

holds for every sequence $(a_n)_{n\in\mathbb{N}} \in \ell_2$.

Corollary 2.1.3. *Let $(\lambda_n)_{n\in\mathbb{Z}}$ be a sequence of real numbers satisfying*

$$
\gamma := \inf_{\substack{m,n\in\mathbb{Z} \\ m \neq n}} |\lambda_m - \lambda_n| > 0. \tag{2.17}
$$

Let $(\epsilon_n)_{n\in\mathbb{Z}}$ be a sequence of complex numbers converging to 0. Then, for every bounded interval $[a, b]$ with $b - a > \frac{2\pi}{\gamma}$, there exist constants $C_1, C_2 > 0$ such that

$$
C_1 \sum_{n\in\mathbb{Z}} |a_n|^2 \leq \int_a^b \left| \sum_{n\in\mathbb{Z}} a_n e^{(i\lambda_n + \epsilon_n)t} \right|^2 dt \leq C_2 \sum_{n\in\mathbb{Z}} |a_n|^2 \tag{2.18}
$$

holds for every sequence $(a_n)_{n\in\mathbb{Z}} \in \ell_2$.

Proof. Let $(a_n)_{n\in\mathbb{Z}} \in \ell_2$ be given. We denote

$$b_n := \begin{cases} a_{\frac{n}{2}}, & \text{if } n \text{ is even,} \\ a_{-\frac{n-1}{2}}, & \text{if } n \text{ is odd,} \end{cases}, \quad \mu_n := \begin{cases} \lambda_{\frac{n}{2}}, & \text{if } n \text{ is even,} \\ \lambda_{-\frac{n-1}{2}}, & \text{if } n \text{ is odd,} \end{cases}, \quad \text{and } \delta_n := \begin{cases} \epsilon_{\frac{n}{2}}, & \text{if } n \text{ is even,} \\ \epsilon_{-\frac{n-1}{2}}, & \text{if } n \text{ is odd,} \end{cases}$$

for $n \in \mathbb{N}$. Then we obtain

$$\int_a^b \left| \sum_{n\in\mathbb{Z}} a_n e^{(i\lambda_n+\epsilon_n)t} \right|^2 dt = \int_a^b \left| \sum_{n\in\mathbb{N}} b_n e^{(i\mu_n+\delta_n)t} \right|^2 dt.$$

Applying Theorem 2.1.17, there exist $C_1, C_2 > 0$ such that

$$C_1 \sum_{n\in\mathbb{N}} |b_n|^2 \le \int_a^b \left| \sum_{n\in\mathbb{Z}} a_n e^{(i\lambda_n+\epsilon_n)t} \right|^2 dt \le C_2 \sum_{n\in\mathbb{N}} |b_n|^2 .$$

Since $\sum_{n\in\mathbb{N}} |b_n|^2 = \sum_{n\in\mathbb{Z}} |a_n|^2$, the proof follows. □

We now write the following result which is relevant to our work. In fact, this inequality helps us deal with the hyperbolic (transport) equation. We give a proof of this inequality by assuming the previous result. In this context, we refer to the article [CMRR14, Proposition 3.1] where this version is used to prove observability inequality of the linearized compressible Navier-Stokes system.

Theorem 2.1.18. *Let $(\lambda_n)_{n\in\mathbb{Z}}$ be a sequence of complex numbers with the following properties:*

(H1) $\lambda_k \neq \lambda_n$ for all $k, n \in \mathbb{Z}$ with $k \neq n$.

(H2) There exists $N \in \mathbb{N}$ large enough such that $\lambda_n = \beta + \gamma n i + e_n$ for all $|n| \ge N$, where $\beta \in \mathbb{C}$ and $(e_n)_{|n|\ge N} \in \ell_2$.

Then, for any $T > \frac{2\pi}{\gamma}$, there exist constants $C_1, C_2 > 0$ such that

$$C_1 \sum_{n\in\mathbb{Z}} |a_n|^2 \le \int_0^T \left| \sum_{n\in\mathbb{Z}} a_n e^{\lambda_n t} \right|^2 dt \le C_2 \sum_{n\in\mathbb{Z}} |a_n|^2 \tag{2.19}$$

holds for any sequence $(a_n)_{n\in\mathbb{Z}} \in \ell_2$.

Proof. Note that

$$\int_0^T \left| \sum_{n\in\mathbb{Z}} a_n e^{\lambda_n t} \right|^2 dt = \int_0^T e^{2\mathrm{Re}(\beta)t} \left| \sum_{n\in\mathbb{Z}} a_n e^{(i\gamma n + e_n)t} \right|^2 dt.$$

Therefore, applying Theorem 2.1.17-Corollary 2.1.3 together with the fact that $e^{2\mathrm{Re}(\beta)t}$ is bounded and has positive lower bound, the proof follows. □

We conclude this section with another version of Ingham-type inequality, often referred as the Müntz-Szász theorem of the parabolic Ingham's inequality. We give a proof of this inequality under some general assumptions on the sequence $(\lambda_n)_{n\in\mathbb{N}}$. The proof is given by Lopez and Zuazua in [LZ02, Proposition 3.2] for the case when the sequence $(\lambda_n)_{n\in\mathbb{N}}$ consists of real numbers.

Theorem 2.1.19. *Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers such that $\sum_{n\in\mathbb{N}} \frac{1}{|\lambda_n|} < \infty$ and let $(q_k)_{k\in\mathbb{N}}$ be biorthogonal to $(e^{-\lambda_n t})_{n\in\mathbb{N}}$ in $L^2(0,T)$ with the following estimate: for given $\epsilon > 0$ there exists a constant $C_\epsilon > 0$ such that*

$$\|q_k\|_{L^2(0,T)} \le C_\epsilon e^{\epsilon \mathrm{Re}(\lambda_n)} \quad \textit{for all } n \in \mathbb{N}.$$

Then, for any given $T > 0$, there exists $C > 0$ depending only on T such that

$$\int_0^T \left| \sum_{n\in\mathbb{N}} a_n e^{-\lambda_n t} \right|^2 dt \ge C \sum_{n\in\mathbb{N}} |a_n|^2 e^{-2\mathrm{Re}(\lambda_n)T} \tag{2.20}$$

for any sequence $(a_n)_{n\in\mathbb{N}} \in \ell_2$.

Proof. Let $T > 0$ be given. We have

$$a_k = \sum_{n\in\mathbb{N}} a_n \int_0^T e^{-\lambda_n t} q_k(t)dt = \int_0^T \sum_{n\in\mathbb{N}} a_n e^{-\lambda_n t} q_k(t)dt.$$

Applying Cauchy-Schwarz inequality, we get

$$|a_k|^2 \le \|q_k\|^2_{L^2(0,T)} \int_0^T \left|\sum_{n\in\mathbb{N}} a_n e^{-\lambda_n t}\right|^2 dt.$$

Multiplying both sides by $\frac{1}{|\lambda_k|}$ and summing over k, we get

$$\sum_{k\in\mathbb{N}} \frac{1}{|\lambda_k|} \frac{|a_k|^2}{\|q_k\|^2_{L^2(0,T)}} \le \sum_{k\in\mathbb{N}} \frac{1}{|\lambda_k|} \int_0^T \left|\sum_{n\in\mathbb{N}} a_n e^{-\lambda_n t}\right|^2 dt.$$

Using the biorthogonal estimate for $\epsilon = T$, we can write

$$\frac{1}{|\lambda_k|\,\|q_k\|^2_{L^2(0,T)}} \ge \frac{Ce^{-T\mathrm{Re}(\lambda_k)}}{|\lambda_k|} \ge Ce^{-2\mathrm{Re}(\lambda_k)T} \frac{e^{T\mathrm{Re}(\lambda_k)}}{|\lambda_k|} \ge Ce^{-2\mathrm{Re}(\lambda_k)T}$$

for some $C > 0$ depending only on T. With this estimate and the fact that $\sum_{n\in\mathbb{N}} \frac{1}{|\lambda_n|} < \infty$, the proof follows. □

Remark 2.1.4. *The above result shows that the existence of a biorthogonal family for the sequence of exponentials $(e^{-\lambda_n t})_{n\in\mathbb{N}}$ with suitable bounds is enough to deduce the parabolic Ingham's inequality* (2.20)*. Thus, we can obtain this result by assuming conditions on the sequence $(\lambda_n)_{n\in\mathbb{N}}$ mentioned in each of the Theorems 2.1.12–2.1.15.*

In a similar fashion, we can obtain the inequality (2.20) when the indices runs over $\mathbb{Z}$, under the same hypothesis of Theorem 2.1.19 but with $n \in \mathbb{Z}$, as explained in Corollary 2.1.3, see [CMRR14] for instance.

Corollary 2.1.4. *Let $(\lambda_n)_{n\in\mathbb{Z}}$ be a sequence of complex numbers such that $\sum_{n\in\mathbb{Z}} \frac{1}{|\lambda_n|} < \infty$ and let $(q_k)_{k\in\mathbb{Z}}$ be biorthogonal to $(e^{-\lambda_n t})_{n\in\mathbb{Z}}$ in $L^2(0,T)$ with the following estimate: For given $\epsilon > 0$ there exists a constant $C_\epsilon > 0$ such that*

$$\|q_k\|_{L^2(0,T)} \le C_\epsilon e^{\epsilon \mathrm{Re}(\lambda_n)} \quad \textit{for all } n \in \mathbb{Z}.$$

Then, for any given $T > 0$, there exists $C > 0$ depending only on T such that

$$\int_0^T \left|\sum_{n\in\mathbb{Z}} a_n e^{-\lambda_n t}\right|^2 dt \ge C \sum_{n\in\mathbb{Z}} |a_n|^2 e^{-2\mathrm{Re}(\lambda_n)T} \tag{2.21}$$

for any sequence $(a_n)_{n\in\mathbb{Z}} \in \ell_2$.

Remark 2.1.5. *Apart from this technique (finding biorthogonal sequences), there are many different methods available in the literature for proving the parabolic Ingham's inequality* (2.20)*. We refer to the works [AI95, JTZ97, You01, FCGBdT10, Edw06, LZ02, Lóp99, KL05, MZ04] for variations of proofs of the parabolic Ingham's inequality* (2.20)*.*

Remark 2.1.6. *Like the hyperbolic Ingham inequality, we do not have the reverse inequality, that is there is no constant $D > 0$ such that the inequality*

$$\int_0^T \left|\sum_{n\in\mathbb{N}} a_n e^{-\lambda_n t}\right|^2 dt \le D \sum_{n\in\mathbb{N}} |a_n|^2 e^{-2\mathrm{Re}(\lambda_n)T} \tag{2.22}$$

holds for any $(a_n)_{n\in\mathbb{N}} \in \ell_2$*. Indeed, for fixed* $N \in \mathbb{N}$*, we take* $\lambda_n = n^2$ *for* $n \in \mathbb{N}$ *and a sequence* $(a_n)_{n\in\mathbb{N}} \in \ell_2$ *as*

$$a_n := \begin{cases} \frac{1}{N}, & \text{if } n = N, \\ 0, & \text{if } n \neq N. \end{cases}$$

If the inequality (2.22) *is true, we obtain*

$$\frac{1}{N^2}\int_0^T e^{-2N^2 t} dt \leq \frac{D}{N^2} e^{-2N^2 T} \implies \frac{e^{2N^2 T} - 1}{2N^2} \leq D, \text{ for all } N \in \mathbb{N},$$

which is a contradiction. Consequently, the inequality (2.22) *cannot hold.*

However, using Cauchy-Schwarz inequality, we see that there exists a constant $D > 0$ *such that*

$$\int_0^T \left| \sum_{n\in\mathbb{N}} a_n e^{-\lambda_n t} \right|^2 dt \leq D \sum_{n\in\mathbb{N}} |a_n|^2$$

holds for all sequence $(a_n)_{n\in\mathbb{N}} \in \ell_2$*.*

2.2 Controllability and Observability

The aim of this section is to present an overview of the controllability and observability notions for both finite and infinite dimensional linear systems. We recall some of the important results that are relevant to this thesis and give proofs for the sake of completeness. All of these contents presented in this section can be found in any control theory book, for instance in [Cor07, Liu10, Zab20, TW09, CZ95]; see also [MZ04, Zua07, Ros07, Erv14, Boy23, Tré23]. Moreover, we give some comments about nonlinear systems at the end of this section.

Before proceeding, we first mention that there are essentially two types of methods to study the controllability of a linear system, namely the direct methods and the duality methods. The direct method refers to proving controllability by explicitly constructing the control(s), whereas the duality method is based on proving certain observability inequalities of the associated adjoint systems which then gives controllability of the linear system. We mention below some of the direct and duality methods to prove the controllability of linear control systems in both finite and infinite dimensions.

- **Direct methods.**
  - (i) The extension method: see for instance [Rus74, Rus78, Lit78, Cor07],
  - (ii) The method of moments: see for instance [FR71, AI95, KL05, Cor07],
  - (iii) The flatness approach: see for instance [FLMR95, MRFR98, LMR00, PR01, MRR18],
- **Duality methods.**
  - (i) Ingham's inequalities and harmonic analysis: see for instance [Rus67, You01, KL05],
  - (ii) Multipliers method: see for instance [Lio88, Kom94, Zua07],
  - (iii) Carleman's inequalities: See for instance [DE22, FI96, Yam09].

In this thesis, we will see some applications of both the direct and duality methods to prove controllability of the linear systems, in particular, the method of moments, Ingham-type inequalities and some multiplier approach.

2.2.1 Finite dimensional linear systems

Let $T > 0$. We consider the following control system posed in finite dimensional space:

$$\begin{cases} u'(t) = Au(t) + Bf(t), & t \in (0,T), \\ u(0) = u_0, \end{cases} \tag{2.23}$$

where $A \in M_n(\mathbb{R}), B \in M_{n,m}(\mathbb{R})$. The function $u : [0,T] \to \mathbb{R}^n$ represents the state vector, $f : [0,T] \to \mathbb{R}^m$ the control vector and $u_0 \in \mathbb{R}^n$ is the initial state. In practical situations, we always want the number of control components to be less than the state, that is, $m \leq n$.

In this setup, we first write the following result which guarantees the existence of a unique weak solution of the system (2.23).

Lemma 2.2.1. *Let $u_0 \in \mathbb{R}^n$ be given. If $f \in L^2(0,T;\mathbb{R}^m)$, then the system (2.23) admits a unique weak solution $u \in C^0([0,T];\mathbb{R}^n)$ and is given by*

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}Bf(s)ds, \quad \textit{for } t \in [0,T]. \tag{2.24}$$

If $f \in C^0([0,T];\mathbb{R}^m)$, this result is a consequence of the Picard-Lindeloff's theorem of existence and uniqueness. The proof will be similar for the case when $f \in L^2(0,T;\mathbb{R}^m)$, see for instance [Per01, Chapter 1, Section 1.10] and [Zab20, Chapter 1, Theorem 1.1]. To find the expression of the solution, we will apply the Duhamel's formula as follows:

We first consider the system

$$\begin{cases} u_1'(t) = Au_1(t), & t \in (0,T), \\ u_1(0) = u_0. \end{cases} \tag{2.25}$$

The solution of (2.25) is given by $u_1(t) = e^{tA}u_0$ for $t \in [0,T]$. We next consider the system

$$\begin{cases} u_2'(t) = Au_2(t), & t \in (s,T), \\ u_2(s) = Bf(s), \end{cases} \tag{2.26}$$

where $s \in [0,T]$. Then, the solution of (2.26) is $u_2(t,s) = e^{(t-s)A}Bf(s)$ for $t \in [s,T]$. Finally, we consider the system

$$\begin{cases} u_3'(t) = Au_3(t) + Bf(t), & t \in (0,T), \\ u_3(0) = 0. \end{cases} \tag{2.27}$$

By Duhamel's principle, the solution of this system (2.27) is given by

$$u_3(t) = \int_0^t u_2(t,s)ds, \quad t \in [0,T].$$

Therefore, the solution of the system (2.23) is

$$u(t) = u_1(t) + u_3(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}Bf(s)ds, \quad t \in [0,T].$$

Once we have the existence of a unique solution u in the space $C^0([0,T];\mathbb{R}^n)$, we can define the controllability notions for the system (2.23) (see the figures below).

Definition 2.2.1. *We say the system (2.23) (or the pair (A,B)) is*

(i) ***exactly controllable*** *at time $T > 0$ if, for any given initial state $u_0 \in \mathbb{R}^n$ and final state $u_T \in \mathbb{R}^n$ there exists a control $f \in L^2(0,T;\mathbb{R}^m)$ such that the associated solution of (2.23) satisfies*

$$u(T) = u_T \ \textit{in } \mathbb{R}^n.$$

(ii) ***null controllable*** *at time $T > 0$ if, for any given initial state $u_0 \in \mathbb{R}^n$ there exists $f \in L^2(0,T;\mathbb{R}^m)$ such that the associated solution of (2.23) satisfies*

$$u(T) = 0.$$

(iii) ***approximately controllable*** *at time $T > 0$ if, for any given initial state $u_0 \in \mathbb{R}^n$, final state $u_T \in \mathbb{R}^n$ and given $\epsilon > 0$, there exists $f_\epsilon \in L^2(0,T;\mathbb{R}^m)$ such that the associated solution of (2.23) satisfies*

$$\|u_\epsilon(T) - u_T\|_{\mathbb{R}^n} \le \epsilon.$$

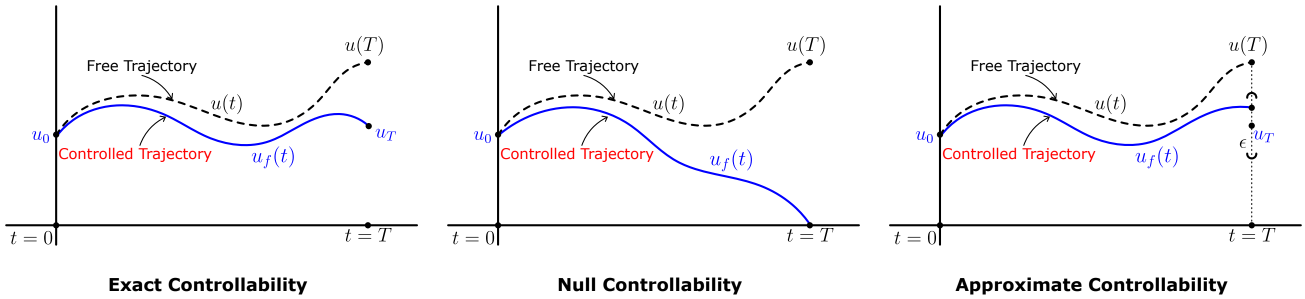

Figure 2.1: The dotted trajectory represents the solution of (2.23) with $f = 0$.

From the definition, it is clear that exact controllability of the system (2.23) always imply null and approximate controllability. Moreover, we explained below that for the finite dimensional linear system (2.23) all these controllability notions are equivalent.

- (**Null $\implies$ Exact**): Let the system (2.23) be null controllable at time $T > 0$. Let $u_0, u_T \in \mathbb{R}^n$ be given. Then, we can find a control $f \in L^2(0,T;\mathbb{R}^m)$ such that the solution of

$$\begin{cases} w'(t) = Aw(t) + Bf(t), & t \in (0,T), \\ w(0) = u_0 - v(0) \end{cases} \tag{2.28}$$

  satisfies $w(T) = 0$, where v is a solution of the following homogeneous system:

$$\begin{cases} v'(t) = Av(t), & t \in (0,T), \\ v(T) = u_T. \end{cases} \tag{2.29}$$

  Thus, the function $u = v+w$ satisfies the equation $u'(t) = Au(t)+Bf(t)$ for $t \in (0,T)$ with $u(0) = u_0$ and $u(T) = u_T$. This proves that the system (2.23) is exactly controllable at time T.

- (**Approximate $\implies$ Exact**): We now assume that the system (2.23) is approximately controllable at time $T > 0$. Let $u_0 \in \mathbb{R}^n$ be given. Then, the set defined by

$$\mathcal{R}(T,u_0) := \{u(T) : \ u \text{ solves (2.23) with } f \in L^2(0,T;\mathbb{R}^m)\} \tag{2.30}$$

  is a dense subspace of $\mathbb{R}^n$. Since any subspace of $\mathbb{R}^n$ which is dense is the $\mathbb{R}^n$ itself, we obtain $\mathcal{R}(T,u_0) = \mathbb{R}^n$. This proves that the system (2.23) is exactly controllable at time T.

Before going any further, we first give the following examples of finite dimensional linear systems and study the controllability properties at time $T > 0$.

Example 2.2.1. *Let us consider the case*

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then the system (2.23) can be written as

$$\begin{cases} u_1'(t) = u_2(t), & t \in (0,T), \\ u_2'(t) = -u_1(t) + f(t), & t \in (0,T), \\ u_1(0) = u_{0,1}, \quad u_2(0) = u_{0,2}, \end{cases} \tag{2.31}$$

where $u = (u_1, u_2)$ *and* $u_0 = (u_{0,1}, u_{0,2})$. *We now prove that this system is exactly controllable at any time* $T > 0$. *Let* $(u_{0,1}, u_{0,2}), (u_{T,1}, u_{T,2}) \in \mathbb{R}^2$ *be the initial and final states respectively. Our goal is to find a control* $f \in L^2(0,T;\mathbb{R}^2)$ *such that the following identities hold:*

$$u_1(0) = u_{0,1}, \quad u_2(0) = u_{0,2}, \quad u_1(T) = u_{T,1}, \quad u_2(T) = u_{T,2}. \tag{2.32}$$

Since $u_2 = u_1'$, *we can rewrite the system (2.31) as*

$$u_1''(t) + u_1(t) = f(t), \quad t \in (0,T), u_1(0) = u_{0,1}, \quad u_1'(0) = u_{0,2}. \tag{2.33}$$

and therefore the conditions (2.32) reduces to

$$u_1(0) = u_{0,1}, \quad u_1'(0) = u_{0,2}, \quad u_1(T) = u_{T,1}, \quad u_1'(T) = u_{T,2}. \tag{2.34}$$

There are many ways to constructing such function u_1, *for instance, we consider the function* u_1 *of the form*

$$u_1(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3, \quad t \in [0,T]. \tag{2.35}$$

Then we get the a system of linear equations:

$$\begin{cases} a_0 = u_{0,1}, \quad a_1 = u_{0,2}, \\ a_0 + a_1 T + a_2 T^2 + a_3 T^3 = u_{T,1}, \\ a_1 + 2a_2 T + 3a_3 T^2 = u_{T,2}. \end{cases}$$

The solution to this system of equations is given by

$$\begin{cases} a_0 = u_{0,1}, \quad a_1 = u_{0,2}, \\ a_2 = -3u_{0,1} - 2Tu_{0,2} + 3u_{T,1} - Tu_{T,2} \\ a_3 = 2u_{0,1} + Tu_{0,2} - 2u_{T,1} + Tu_{T,2}. \end{cases} \tag{2.36}$$

With these values of a_j *for* $j = 0, 1, 2, 3$, *we now define the control* f *as*

$$f(t) = u_1''(t) + u_1(t), \quad t \in (0,T),$$

where u_1 *is given by (2.35)–(2.36). Clearly,* (u_1, u_2) *with* $u_2 = u_1'$ *solves the system (2.31) and the identities (2.32). This proves that the system (2.31) is exactly controllable at any time* $T > 0$.

Example 2.2.2. *We now consider the case*

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then the system (2.23) can be written as

$$\begin{cases} u_1'(t) = u_1(t), \quad t \in (0,T), \\ u_2'(t) = u_2(t) + f(t), \quad t \in (0,T), \\ u_1(0) = u_{0,1}, \quad u_2(0) = u_{0,2}. \end{cases} \tag{2.37}$$

Here $u = (u_1, u_2)$ *and* $u_0 = (u_{0,1}, u_{0,2})$. *We now prove that this system cannot be exactly controllable at any time* $T > 0$ *whatever we choose the control* f. *In fact, for* $u_{0,1} \neq 0$, *the solution component* u_1 *is given by*

$$u_1(t) = e^t u_{0,1}, \quad t \in [0,T],$$

which is independent of the control function f and also $u_1(t) \neq 0$ for all $t \in [0,T]$. Therefore, given the final state of the form $(0,a)$ with $a \in \mathbb{R}$, we cannot find any f such that the solution satisfies $(u_1(T), u_2(T)) = (0,a)$. However, it is easy to see that the system is exactly controllable at any time T by using two controls acting in each equations. So, the number of controls matters to achieve controllability of the linear systems.

Since exact, null and approximate controllability notions are equivalent for the system (2.23), we will concentrate only on the null controllability. To prove null controllability of the system (2.23) at time $T > 0$, we need to find a control $f \in L^2(0,T;\mathbb{R}^m)$ such that the identity

$$e^{TA}u_0 + \int_0^T e^{(T-s)A}Bf(s)ds = 0$$

holds for every initial state $u_0 \in \mathbb{R}^n$. However, in general, it might be difficult to construct a control f that satisfies the above identity. For this reason, we will study the adjoint of the system (2.23) and derive some equivalent criterion for null controllability in terms of the adjoint state. More precisely, we consider the adjoint system corresponding to (2.23) as

$$\begin{cases} -\varphi'(t) = A^*\varphi(t), & t \in (0,T), \\ \varphi(T) = \varphi_T, \end{cases} \tag{2.38}$$

where $\varphi_T \in \mathbb{R}^n$. Note that, we can write explicitly the solution to this system as

$$\varphi(t) = e^{(T-t)A^*}\varphi_T, \quad \text{for } t \in [0,T]. \tag{2.39}$$

With the help of this adjoint equation, we now state the following result which gives an equivalent criterion for null controllability of the system (2.23).

Lemma 2.2.2. *The system (2.23) is null controllable at time $T > 0$ if, and only if, for given $u_0 \in \mathbb{R}^n$ there exists a $f \in L^2(0,T;\mathbb{R}^m)$ such that the following identity*

$$\int_0^T \langle f(t), B^*\varphi(t)\rangle_{\mathbb{R}^m}\, dt + \langle u_0, \varphi(0)\rangle_{\mathbb{R}^n} = 0 \tag{2.40}$$

holds for every $\varphi_T \in \mathbb{R}^n$, where φ is the solution of the adjoint system (2.38).

Proof. Let $\varphi_T \in \mathbb{R}^n$ and let φ be the solution of (2.38). Taking inner product in (2.23) with φ and integrating over $(0,T)$, we get

$$\int_0^T \langle u'(t), \varphi(t)\rangle_{\mathbb{R}^n}\, dt = \int_0^T \langle Au(t), \varphi(t)\rangle_{\mathbb{R}^n}\, dt + \int_0^T \langle Bf(t), \varphi(t)\rangle_{\mathbb{R}^n}\, dt.$$

Integrating by parts, we obtain

$$\begin{aligned} -\int_0^T \langle u(t), \varphi'(t)\rangle_{\mathbb{R}^n}\, dt + \langle u(T), \varphi(T)\rangle_{\mathbb{R}^n} - \langle u_0, \varphi(0)\rangle_{\mathbb{R}^n} \\ = \int_0^T \langle u(t), A^*\varphi(t)\rangle_{\mathbb{R}^n}\, dt + \int_0^T \langle f(t), B^*\varphi(t)\rangle_{\mathbb{R}^m}\, dt. \end{aligned}$$

Since φ solves (2.38), we deduce the identity

$$\langle u(T), \varphi_T\rangle_{\mathbb{R}^n} = \langle u_0, \varphi(0)\rangle_{\mathbb{R}^n} + \int_0^T \langle f(t), B^*\varphi(t)\rangle_{\mathbb{R}^m}\, dt, \tag{2.41}$$

with $\varphi_T \in \mathbb{R}^n$. Thus, if the system (2.23) is null controllable at time $T > 0$, we have $u(T) = 0$ and therefore

$$\langle u_0, \varphi(0)\rangle_{\mathbb{R}^n} + \int_0^T \langle f(t), B^*\varphi(t)\rangle_{\mathbb{R}^n}\, dt = 0,$$

for all $\varphi_T \in \mathbb{R}^n$, giving the identity (2.40). Conversely, if for given $u_0 \in \mathbb{R}^n$ there exists a $f \in L^2(0,T;\mathbb{R}^m)$ such that the identity (2.40) holds for every $\varphi_T \in \mathbb{R}^n$, then from (2.41), we deduce that $\langle u(T), \varphi_T\rangle_{\mathbb{R}^n} = 0$ for all $\varphi_T \in \mathbb{R}^n$ and hence $u(T) = 0$. This completes the proof. □

We note here that the identity (2.40) gives an optimality condition for the critical points of certain quadratic (cost) functional. More precisely, we have the following result:

Lemma 2.2.3. *For given $u_0 \in \mathbb{R}^n$, let us define a quadratic functional $J : \mathbb{R}^n \to \mathbb{R}$ by*

$$J(\varphi_T) := \frac{1}{2} \int_0^T \|B^* \varphi(t)\|^2_{\mathbb{R}^m} \, dt + \langle u_0, \varphi(0) \rangle_{\mathbb{R}^n}, \quad \varphi_T \in \mathbb{R}^n, \tag{2.42}$$

where φ is the solution of (2.38). If J admits a minimizer $\hat{\varphi}_T \in \mathbb{R}^n$, then the function $f(t) = B^* \hat{\varphi}(t)$ for $t \in (0, T)$, where $\hat{\varphi}$ is the solution of (2.38) with $\hat{\varphi}(T) = \hat{\varphi}_T$, is a (null) control of the system (2.23).

Proof. Let $\hat{\varphi}_T \in \mathbb{R}^n$ be a minimizer of J and $\hat{\varphi}$ be the solution of (2.38) corresponding to this terminal state $\hat{\varphi}_T$. Then, we have

$$\lim_{h \to 0} \frac{J(\hat{\varphi}_T + h\varphi_T) - J(\hat{\varphi}_T)}{h} = 0 \tag{2.43}$$

for all $\varphi_T \in \mathbb{R}^n$. Let φ denotes the solution of (2.38) with the final state φ_T. Then, by linearity of the equation, we can say that $\hat{\varphi} + h\varphi$ is the solution of (2.38) corresponding to the final state $\hat{\varphi}_T + h\varphi_T$. We now compute

$$\begin{aligned} J(\hat{\varphi}_T + h\varphi_T) - J(\hat{\varphi}_T) &= \frac{1}{2} \int_0^T \|B^*(\hat{\varphi}(t) + h\varphi(t))\|^2_{\mathbb{R}^m} \, dt + \langle u_0, \hat{\varphi}(0) + h\varphi(0) \rangle_{\mathbb{R}^n} \\ &\quad - \frac{1}{2} \int_0^T \|B^* \hat{\varphi}(t)\|^2_{\mathbb{R}^m} \, dt - \langle u_0, \hat{\varphi}(0) \rangle_{\mathbb{R}^n} \\ &= h \int_0^T \langle B^* \hat{\varphi}(t), B^* \varphi(t) \rangle_{\mathbb{R}^m} \, dt + \frac{h^2}{2} \int_0^T \|B^* \varphi(t)\|^2_{\mathbb{R}^m} \, dt + h \langle u_0, \varphi(0) \rangle_{\mathbb{R}^n}. \end{aligned}$$

Therefore, the relation (2.43) yields the identity

$$\int_0^T \langle B^* \hat{\varphi}(t), B^* \varphi(t) \rangle_{\mathbb{R}^m} \, dt + \langle u_0, \varphi(0) \rangle_{\mathbb{R}^n} = 0$$

for all $\varphi_T \in \mathbb{R}^n$. Applying Lemma 2.2.2, it follows that the system (2.23) is null controllable at time T by using the control $f(t) = B^* \hat{\varphi}(t)$ for $t \in (0, T)$. This completes the proof. □

The above result shows that it is enough to find a minimizer of the functional J for proving the null controllability of the system (2.23). To prove the quadratic functional J admits a minimizer, we use the following well-known result of the calculus of variations:

Theorem 2.2.1. *Let $F : \mathbb{R}^n \to \mathbb{R}$ be a continuous function satisfying $\lim\limits_{|x| \to \infty} F(x) = \infty$. Then F admits a minimizer $\hat{x} \in \mathbb{R}^n$.*

This result can be generalized in reflexive Banach spaces, see for instance the book [Kes09, Proposition 5.6.1]. With the help of this result, we now find equivalent conditions for exact, null and approximate controllability in terms of the adjoint state.

Theorem 2.2.2. *The following statements hold:*

(i) The system (2.23) is null controllable at time $T > 0$ if, and only if, there exists $C > 0$ such that the inequality

$$\int_0^T \|B^* \varphi(t)\|^2_{\mathbb{R}^m} \, dt \geq C \, \|\varphi(0)\|^2_{\mathbb{R}^n} \tag{2.44}$$

holds for all $\varphi_T \in \mathbb{R}^n$, where φ is the solution of the adjoint system (2.38).

(ii) The system (2.23) is exactly controllable at time $T > 0$ if, and only if, there exists $C > 0$ such that the inequality

$$\int_0^T \|B^* \varphi(t)\|^2_{\mathbb{R}^m} \, dt \geq C \, \|\varphi_T\|^2_{\mathbb{R}^n} \tag{2.45}$$

holds for all $\varphi_T \in \mathbb{R}^n$, where φ is the solution of the adjoint system (2.38).

(iii) The system (2.23) is approximately controllable at time $T > 0$ if, and only if, the following property holds:

$$\text{If } \varphi \text{ solves (2.38) and } B^*\varphi(t) = 0 \text{ for all } t \in [0,T], \text{ then } \varphi_T = 0. \tag{2.46}$$

Proof. First we recall the expression of the solution $\varphi(t) = e^{(T-t)A^*}\varphi_T$ for all $t \in [0,T]$. It is easy to see that the map $F : \mathbb{R}^n \to \mathbb{R}^n$ defined by $F(\varphi_T) := \varphi(0) = e^{TA^*}\varphi_T$ is a bounded invertible linear operator on $\mathbb{R}^n$. Therefore, the inequalities (2.44) and (2.45) are equivalent. On the other hand, if the property (2.46) holds, then $\|\varphi_T\| := \left(\int_0^T \|B^*\varphi(t)\|^2_{\mathbb{R}^m}\,dt\right)^{\frac{1}{2}}$ defines a norm on $\mathbb{R}^n$. Since any two norms in a finite dimensional space are equivalent, we have for some $C > 0$ that

$$\int_0^T \|B^*\varphi(t)\|^2_{\mathbb{R}^m}\,dt \geq C\,\|\varphi_T\|^2_{\mathbb{R}^n}\,.$$

As a consequence, the inequalities (2.44), (2.45) and the property (2.46) are equivalent for finite dimensional linear systems. Thus, we only prove Part (i) of this result.

Let $u_0 \in \mathbb{R}^n$. If the inequality (2.45) holds for all $\varphi_T \in \mathbb{R}^n$, then the quadratic functional J defined by (2.42) is coercive. In fact

$$J(\varphi_T) \geq \frac{C}{2}\,\|\varphi(0)\|^2_{\mathbb{R}^n} - \|u_0\|_{\mathbb{R}^n}\,\|\varphi(0)\|_{\mathbb{R}^n} \geq \frac{C}{2}\,\|\varphi_T\|^2_{\mathbb{R}^n} - \|u_0\|_{\mathbb{R}^n}\,\|\varphi_T\|_{\mathbb{R}^n}\,,$$

thanks to the Cauchy-Schwarz inequality $\left|\langle u_0, \varphi(0)\rangle_{\mathbb{R}^n}\right| \leq \|u_0\|_{\mathbb{R}^n}\,\|\varphi(0)\|_{\mathbb{R}^n}$. Thus,

$$\lim_{\|\varphi_T\|_{\mathbb{R}^n}\to\infty} J(\varphi_T) = \infty.$$

Therefore, J admits a minimizer $\hat{\varphi}_T \in \mathbb{R}^n$, thanks to Theorem 2.2.1. Applying Lemma 2.2.3, we can say that the system (2.23) is null controllable at time T.

Conversely, we suppose that the system (2.23) is null controllable at time $T > 0$. We will prove the inequality (2.44) via contradiction argument. If the inequality (2.44) is not true, then there exists a sequence $(\varphi_T^k)_{k\in\mathbb{N}} \subset \mathbb{R}^n$ such that $\left\|\varphi^k(0)\right\|_{\mathbb{R}^n} = 1$ for all $k \in \mathbb{N}$ and $\int_0^T \left\|B^*\varphi^k(t)\right\|^2_{\mathbb{R}^m} dt \to 0$ as $k \to \infty$, where φ^k is the solution of (2.38) with final state φ_T^k for $k \in \mathbb{N}$. Since $\left\|\varphi^k(0)\right\|_{\mathbb{R}^n} = 1$ for all $k \in \mathbb{N}$, the sequence $(\varphi_T^k)_{k\in\mathbb{N}}$ is bounded in $\mathbb{R}^n$ and therefore, up to a subsequence, $\varphi_T^k \to \varphi_T$ as $k \to \infty$, for some $\varphi_T \in \mathbb{R}^n$. Let φ denote the solution of (2.38) corresponding to this φ_T. Then

$$\|\varphi(0)\|_{\mathbb{R}^n} = \lim_{k\to\infty}\left\|\varphi^k(0)\right\|_{\mathbb{R}^n} = 1, \text{ and } \int_0^T \|B^*\varphi(t)\|^2_{\mathbb{R}^m}\,dt = \lim_{k\to\infty}\int_0^T \left\|B^*\varphi^k(t)\right\|^2_{\mathbb{R}^m} dt = 0. \tag{2.47}$$

On the other hand, since the system (2.23) is null controllable at time T, by Lemma 2.2.2, for any given $u_0 \in \mathbb{R}^n$, there exists a $f \in L^2(0,T;\mathbb{R}^m)$ such that the identities

$$\int_0^T \left\langle f(t), B^*\varphi^k(t)\right\rangle_{\mathbb{R}^m} dt + \left\langle u_0, \varphi^k(0)\right\rangle_{\mathbb{R}^n} = 0$$

holds for all $k \in \mathbb{N}$. Taking limit as $k \to \infty$ in this identity, we deduce that

$$\int_0^T \langle f(t), B^*\varphi(t)\rangle_{\mathbb{R}^m}\,dt + \langle u_0, \varphi(0)\rangle_{\mathbb{R}^n} = 0.$$

But from (2.47), $B^*\varphi(t) = 0$ for all $t \in [0,T]$ and therefore $\langle u_0, \varphi(0)\rangle_{\mathbb{R}^n} = 0$ for every $u_0 \in \mathbb{R}^n$. This implies $\varphi(0) = 0$, which contradicts the fact $\|\varphi(0)\|_{\mathbb{R}^n} = 1$ (see eq. (2.47)), and therefore the inequality (2.44) holds for all $\varphi_T \in \mathbb{R}^n$.

This completes the proof. □

The inequalities (2.44) and (2.45) are called the observability inequalities associated to the adjoint system (2.38). The above result shows that proving controllability of the system (2.23) is equivalent to prove certain observability inequalities corresponding to the adjoint system. Moreover, the property (2.46) is called the unique continuation property, which gives approximate controllability of the system (2.23). In the next section, we see that all these notions can be generalized for the infinite dimensional linear systems. However, there are other several concepts available in the literature to prove controllability of the finite dimensional linear system (2.23) which may or may not have direct generalization into the infinite dimensional systems. We conclude this section by writing only the statement of some of the famous and well-known results; proof of which can be found in many books, for instance in [Zab20, TW09].

Theorem 2.2.3. *The following statements are equivalent:*

(i) The system (2.23) is null controllable at time $T > 0$.

(ii) The Kalman matrix defined by

$$[A \mid B] = [B\ AB\ A^2B\ \dots\ A^{n-1}B]$$

has rank n.

(iii) The controllability Gramian

$$Q_T = \int_0^T e^{tA}BB^*e^{tA^*}dt$$

is invertible.

Theorem 2.2.4 (Fattorini-Hautus test)**.** *The following statements are equivalent:*

(i) The system (2.23) is null controllable at time $T > 0$.

(ii) For every $\lambda \in \mathbb{C}$, the matrix $[\lambda I - A\ B]$ has rank n.

(iii) For every $\lambda \in \sigma(A)$, the matrix $[\lambda I - A\ B]$ has rank n.

The above results indicates that, if the system (2.23) is controllable at some time $T > 0$, then it is controllable at every time T and the matrix Q_T is invertible for every $T > 0$. In the next few sections, we will see that this phenomenon might not always possible in the case of infinite dimensional linear systems or even non-linear systems posed in finite dimension.

2.2.2 Infinite dimensional linear systems

Let H and U be Hilbert spaces. We consider the following control system posed in infinite dimensional space:

$$\begin{cases} u'(t) = Au(t) + Bf(t), & t \in (0,T), \\ u(0) = u_0, \end{cases} \tag{2.48}$$

where $T > 0$, $A : \mathcal{D}(A) \subset H \to H$ is a closed and densely defined linear operator that generates a C^0-semigroup $\{S(t)\}_{t\geq 0}$ on H and $B : U \to H$ is the control operator. The function $u : [0,T] \to H$ represents the state, $f : [0,T] \to U$ the control and $u_0 \in H$ is the initial state.

The operator B can be bounded or unbounded. In this thesis, we consider only the case when $B : U \to H$ is an unbounded linear operator and address the controllability properties of the system (2.48). In the case when B is bounded, similar controllability properties can be studied and in this context, we refer to the books [Zab20, TW09, CZ95] for more detail.

We note that the adjoint of A, denoted by $A^* : \mathcal{D}(A^*) \subset H \to H$, also generates a C^0-semigroup $\{S^*(t)\}_{t\geq 0}$ on H, where $S^*(t)$ is the adjoint of the operator $S(t)$ in H. Let us denote $\mathcal{D}(A^*)'$ as the dual of $\mathcal{D}(A^*)$ with respect to the pivot space H, that is

$$\mathcal{D}(A^*) \subset H \cong H' \subset \mathcal{D}(A^*)'.$$

We assume that $B : U \to \mathcal{D}(A^*)'$ is bounded. Then the adjoint $B^* : \mathcal{D}(A^*) \to U$ is also a bounded linear operator. We further assume the following condition

$$\int_0^T \|B^* S^*(t)\varphi\|_U^2 \, dt \le C \|\varphi\|_H^2, \quad \text{for all } \varphi \in \mathcal{D}(A^*), \tag{2.49}$$

where $C > 0$ is a constant depending only on T. This condition is known as the admissibility condition/inequality, and it shows that we can uniquely extend the operator $\mathcal{F} : \mathcal{D}(A^*) \to C^0([0,T];U)$ defined by

$$F(\varphi) := B^* S^*(\cdot)\varphi, \quad \varphi \in \mathcal{D}(A^*)$$

as a continuous linear map from H into $L^2(0,T;U)$. In this setup, we first define the notion of a solution for the system (2.48).

Definition 2.2.2 (Strong solution)**.** *We say a continuous function $u : [0,T] \to H$ is a strong solution of (2.48) if the following conditions are satisfied:*

(i) $u(t) \in \mathcal{D}(A)$ *for all* $t \in [0,T]$*,*

(ii) $u(0) = u_0$

(iii) u *is differentiable on* $(0,T)$ *and* $u'(t) = Au(t) + Bf(t)$ *for all* $t \in (0,T)$*.*

Then we have the following existence and uniqueness result for the system (2.48).

Lemma 2.2.4. *Let $u_0 \in \mathcal{D}(A)$ be given. If $f \in C^1([0,T];U)$, then the system (2.48) admits a unique strong solution $u \in C^0([0,T];H)$ and is given by*

$$u(t) = S(t)u_0 + \int_0^t S(t-s)Bf(s)ds, \quad \text{for } t \in [0,T]. \tag{2.50}$$

The proof follows from the properties of the C^0-semigroup $\{S(t)\}_{t\ge 0}$, see for instance Theorem 2.1.2. We note that, if f is not continuous, then we cannot get a strong solution of the system (2.48). In this case, we will define the notion of a weak solution for this system. First, we will write the adjoint system associated to (2.48) as follows:

$$\begin{cases} -\varphi'(t) = A^*\varphi(t), & t \in (0,T), \\ \varphi(T) = \varphi_T, \end{cases} \tag{2.51}$$

where $\varphi_T \in H$. The solution to this system is given by $\varphi(t) = S^*(T-t)\varphi_T$ for $t \in [0,T]$.

Definition 2.2.3 (Weak solution)**.** *For any given $u_0 \in H$ and $f \in L^2(0,T;U)$, we say a function $u \in C^0([0,T];H)$ is a weak solution of (2.48) if the following identity*

$$\langle u(t), \varphi(t)\rangle_H - \langle u(0), \varphi(0)\rangle_H = \int_0^t \langle f(s), B^*\varphi(s)\rangle_U \, ds, \quad \forall t \in [0,T], \tag{2.52}$$

holds for all $\varphi_T \in H$, where φ is the solution of (2.51).

We note here that the term in the right hand side of the above expression is well-defined, thanks to the admissibility condition (2.49). With this definition, we write the following result which gives existence of a unique weak solution to the system (2.48).

Theorem 2.2.5. *For any given $u_0 \in H$ and $f \in L^2(0,T;U)$, the system (2.48) admits a unique weak solution $u \in C^0([0,T];H)$. Moreover, we have the following estimate*

$$\|u\|_{C^0([0,T];H)} \le C\left(\|u_0\|_H + \|f\|_{L^2(0,T;U)}\right), \tag{2.53}$$

where $C > 0$ is a constant depending only on T.

We refer to the book of Coron [Cor07, Theorem 2.37] for a proof of this result. Once we have the above Theorem, we can define the controllability notions for the system (2.48).

Definition 2.2.4. *We say the system* (2.48) *is*

(i) ***exactly controllable*** *at time* $T > 0$ *in the space* H *if, for any given initial state* $u_0 \in H$ *and final state* $u_T \in H$*, there exists a control* $f \in L^2(0,T;U)$ *such that the associated solution of* (2.48) *satisfies*

$$u(T) = u_T \text{ in } H.$$

(ii) ***null controllable*** *at time* $T > 0$ *in the space* H *if, for any given initial state* $u_0 \in H$ *there exists* $f \in L^2(0,T;U)$ *such that the associated solution of* (2.48) *satisfies*

$$u(T) = 0.$$

(iii) ***approximately controllable*** *at time* $T > 0$ *in the space* H *if, for any given initial state* $u_0 \in H$*, final state* $u_T \in H$ *and given* $\epsilon > 0$*, there exists* $f_\epsilon \in L^2(0,T;U)$ *such that the associated solution of* (2.48) *satisfies*

$$\|u_\epsilon(T) - u_T\|_H \leq \epsilon.$$

Note that exact controllability always implies null and approximate controllability. However, the converse is not true, in general, for infinite dimensional linear systems; see Section 2.4 for more details.

Let us assume, for the time, that the system (2.48) has a unique solution $u \in C^0([0,T];H)$ given by

$$u(t) = S(t)u_0 + \int_0^t S(t-s)Bf(s)ds$$

for all $t \in [0,T]$ (see Lemma 2.2.4). Then, from the expression of the solution, we have

$$u(T) = S(T)u_0 + \int_0^T S(T-s)Bf(s)ds.$$

Thus, proving exact controllability of the system (2.48) at time T in H is equivalent to find a control $f \in L^2(0,T;U)$ such that the following relation

$$\int_0^T S(T-s)Bf(s)ds = u_T - S(T)u_0$$

holds for every $u_0, u_T \in H$. For null and approximate controllability of the system (2.48), we will get similar relations on the control f. This motivates us to define a linear map $f \mapsto \int_0^T S(T-s)Bf(s)ds$ in appropriate Hilbert spaces and study the properties of this map. Also, note from Lemma 2.2.4 that, the strong solution of (2.48) with $u_0 = 0$ and $f \in C^1([0,T];U)$ can be written as $u(T) = \int_0^T S(T-s)Bf(s)ds$. With this formula, we can now define the above map as follows:

Let $T > 0$ be given. We define a linear map $F_T : L^2(0,T;U) \to H$ by

$$F_T(f) := u(T), \quad f \in L^2(0,T;U), \tag{2.54}$$

where $u \in C^0([0,T];H)$ is the unique weak solution of (2.48) with $u_0 = 0$ and $f \in L^2(0,T;U)$. Then, we can find equivalent conditions for exact, null and approximate controllability for the system (2.48).

Theorem 2.2.6. *The following statements hold:*

(i) The system (2.48) *is exactly controllable at time* $T > 0$ *in* H *if and only if* $\mathrm{Range}(F_T) = H$.

(ii) The system (2.48) *is null controllable at time* $T > 0$ *in* H *if and only if* $\mathrm{Range}(S(T)) \subset \mathrm{Range}(F_T)$.

(iii) The system (2.48) *is approximately controllable at time* $T > 0$ *in* H *if and only if* $\mathrm{Range}(F_T)$ *is dense in* H.

Proof. We prove each parts separately.

(i) Let us first assume that the system (2.48) be exactly controllable at time T in the space H. This means, for any given $z \in H$ and initial state $u_0 = 0$, we can find a $f \in L^2(0,T;U)$ such that $u(T) = z$ in H, that is, $F_T(f) = z$, which proves that $\text{Range}(F_T) = H$.

Conversely, we assume $\text{Range}(F_T) = H$. Let $u_0, u_T \in H$ be given. Let u_1 denote the weak solution of (2.48) with initial state u_0 and $f = 0$. Since $\text{Range}(F_T) = H$, we can find a $f \in L^2(0,T;U)$ such that $F_T(f) = u_T - u_1(T)$, that is $u_2(T) = u_T - u_1(T)$, where u_2 is the solution of (2.48) with initial state 0 and above function f. Then, by linearity of the system (2.48), the function $u := u_1 + u_2$ is a solution of (2.48) with initial state $u_0 \in H$ and the above function $f \in L^2(0,T;U)$. Moreover, this solution satisfies $u(T) = u_1(T) + u_2(T) = u_T$, which proves that the system (2.48) is exactly controllable at time T in the space H.

(ii) Let the system (2.48) be null controllable at time T in the space H. Let $u_0 \in H$ be given. We will prove that $S(T)u_0 \in \text{Range}(F_T)$. Note that, $u_1(t) := S(t)u_0$ is the weak solution of (2.48) with this u_0 and $f = 0$. On the other hand, since the system (2.48) is null controllable at time T in H, there exists a control $f \in L^2(0,T;U)$ such that the solution u_2 of (2.48) starting from $-u_0 \in H$ satisfies $u_2(T) = 0$. Then the function $u := u_1 + u_2$ is the weak solution of (2.48) with initial state 0 and control $f \in L^2(0,T;U)$. Moreover, $u(T) = u_1(T) = S(T)u_0$, that is $F_T(f) = S(T)u_0$, which implies $S(T)u_0 \in \text{Range}(F_T)$. Since $u_0 \in H$ was arbitrary, the proof follows.

Conversely, we assume $\text{Range}(S(T)) \subset \text{Range}(F_T)$. Let $u_0 \in H$ be given. Since $-S(T)u_0 \in \text{Range}(F_T)$, we get a $f \in L^2(0,T;U)$ such that $F_T(f) = -S(T)u_0$, that is the solution u_1 of (2.48) with initial state 0 and this f satisfies $u_1(T) = -S(T)u_0$. Also, note that $u_2(t) := S(t)u_0$ is the weak solution of (2.48) with initial state u_0 and $f = 0$. Then the function $u := u_1 + u_2$ is the weak solution of (2.48) with the above initial state $u_0 \in H$ and control $f \in L^2(0,T;U)$. Moreover, $u(T) = u_1(T) + u_2(T) = 0$. This proves that the system (2.48) is null controllable at time T in H.

(iii) We finally assume that the system (2.48) is approximately controllable at time T in the space H. This implies for given $z \in H$, initial state 0 and given $\epsilon > 0$, there exists a control $f \in L^2(0,T;U)$ such that the solution u of (2.48) satisfies $\|u(T) - z\|_H \le \epsilon$, that is $\|F_T(f) - z\|_H \le \epsilon$, proving that $\text{Range}(F_T)$ is dense in H.

Conversely, if $\text{Range}(F_T)$ is dense in H then, for given $u_0, u_T \in H$ and $\epsilon > 0$, we can find a $f \in L^2(0,T;U)$ such that $\|F_T(f) + u_1(T) - u_T\|_H \le \epsilon$, where u_1 is the weak solution of (2.48) with the above $u_0 \in H$ and $f = 0$. This implies $\|u_2(T) + u_1(T) - u_T\|_H \le \epsilon$, where u_2 is the unique weak solution of (2.48) with initial state 0 and the above function f. Let us now define $u := u_1 + u_2$. Then u is the weak solution of (2.48) with the above initial state $u_0 \in H$ and control $f \in L^2(0,T;U)$ satisfying $\|u(T) - u_T\|_H = \|u_1(T) + u_2(T) - u_T\|_H \le \epsilon$. This proves that the system (2.48) is approximately controllable at time T in the space H.

This completes the proof. □

In general, finding the range of a linear operator is quite difficult, so we will further reduce the above conditions with the help of some useful results from functional analysis. First, let us recall the following result which is written in more general Banach spaces.

Theorem 2.2.7. *Let X, Y and Z be Banach spaces and let $F : X \to Y$ and $G : Y \to Z$ be linear operators. Then:*

(i) F is surjective if and only if there exists a $C > 0$ such that $\|F^*(z)\|_{X'} \ge C\|z\|_{Y'}$ holds for all $z \in Y'$.

(ii) $\text{Range}(F)$ *is dense in Y if and only if $F^*(z) = 0$ for all $z \in Y'$ implies $z = 0$.*

(iii) If Y is reflexive, then $\text{Range}(F) \subset \text{Range}(G)$ *if and only if there exists $C > 0$ such that $\|F^*(z)\|_{X'} \le C\|G^*(z)\|_{Y'}$ for all $z \in Z'$.*

We refer to the books of Brezis [Bre11, Theorem 2.20, Corollary 2.18] and Coron [Cor07, Lemma 2.46, Lemma 2.48] for a proof of Theorem 2.2.7; see also [Zab20, Tré23, TWX20]. In this thesis, we will assume this result and then, with the help of Theorem 2.2.6, we deduce some equivalent inequalities for the controllability of the system (2.48). The proof is straightforward from Theorem 2.2.7 and so we leave the details.

Theorem 2.2.8. *The following statements hold:*

(i) The system (2.48) is exactly controllable at time $T > 0$ in H if and only if there exists a $C > 0$ such that $\|F_T^*(z)\|_{L^2(0,T;U)} \geq C\|z\|_H$ holds for all $z \in H$.

(ii) The system (2.48) is null controllable at time $T > 0$ in H if and only if there exists a $C > 0$ such that $\|S^*(T)(z)\|_H \leq C\|F_T^*(z)\|_{L^2(0,T;U)}$ holds for all $z \in H$.

(iii) The system (2.48) is approximately controllable at time $T > 0$ in H if and only if $F_T^*(z) = 0$ for all $z \in H$ implies $z = 0$.

In view of this result, we now find the adjoint operator $F_T^* : H \to L^2(0,T;U)$. Let $\varphi_T \in \mathcal{D}(A^*)$ and $f \in L^2(0,T;U)$ be given. Let $u \in C^0([0,T];H)$ be the unique weak solution of (2.48) with this f and initial state $u_0 = 0$. Then, we have

$$\left\langle f, F_T^*(\varphi_T)\right\rangle_{L^2(0,T;U)} = \langle F_T(f), \varphi_T\rangle_H = \langle u(T), \varphi_T\rangle_H .$$

Since u is a weak solution of (2.48), we have the following relation (see eq. (2.52)):

$$\langle u(T), \varphi_T\rangle_H = \int_0^T \langle f(t), B^*\varphi(t)\rangle_U \, dt.$$

Thus, we obtain

$$\left\langle f, F_T^*(\varphi_T)\right\rangle_{L^2(0,T;U)} = \langle f, B^*\varphi\rangle_{L^2(0,T;U)}$$

and therefore $F_T^* : H \to L^2(0,T;U)$ is defined as

$$F_T^*(\varphi_T) = B^*\varphi, \quad \text{for all } \varphi_T \in \mathcal{D}(A^*).$$

Since $\mathcal{D}(A^*)$ is dense in H, the operator $F_T^* : \mathcal{D}(A^*) \to L^2(0,T;U)$ has a unique extension on H. Thus, denoting the same function, the adjoint operator $F_T^* : H \to L^2(0,T;U)$ is defined as

$$F_T^*(\varphi_T) := B^*\varphi, \quad \text{for all } \varphi_T \in H. \tag{2.55}$$

Then, the statements of Theorem 2.2.8 reduces to the following (see Theorem 2.2.2 for a comparison with the finite dimensional linear systems):

Theorem 2.2.9. *The following statements hold:*

(i) The system (2.48) is exactly controllable at time $T > 0$ in H if, and only if, there exists a $C > 0$ such that the following observability inequality

$$\int_0^T \|B^*\varphi(t)\|_U^2 \, dt \geq C\|\varphi_T\|_H^2 \tag{2.56}$$

holds for all $\varphi_T \in \mathcal{D}(A^*)$, where φ is the solution of the adjoint system (2.51).

(ii) The system (2.48) is null controllable at time $T > 0$ in H if, and only if, there exists a $C > 0$ such that the observability inequality

$$\int_0^T \|B^*\varphi(t)\|_U^2 \, dt \geq C\|\varphi(0)\|_H^2 \tag{2.57}$$

holds for all $\varphi_T \in \mathcal{D}(A^*)$, where φ is the solution of the adjoint system (2.51).

(iii) The system (2.48) *is approximately controllable at time* $T > 0$ *in* H *if, and only if, the following unique continuation principle holds:*

$$\text{If } \varphi \text{ solves (2.51) and } B^*\varphi = 0 \text{ in } L^2(0,T;U), \text{ then } \varphi_T = 0 \text{ in } H. \tag{2.58}$$

We note here that in finite dimensional setup, we have proved this result directly by constructing a quadratic functional $J : \mathbb{R}^n \to \mathbb{R}$. In comparison, for infinite dimensional linear systems, the function $J : H \to \mathbb{R}$ may not be well-defined due to the less regularity of the solution. However, one can modify the cost functional by choosing φ_T more regular (say from $\mathcal{D}(A^*)$) and prove that controllability is equivalent to these observability inequalities. In this context, we refer to the end of Section 2.4 for more details.

We now state a very useful result on approximate controllability of (2.48) in the presence of backward uniqueness property of the associated homogeneous system (that is, with no control input). First, let us consider the following homogeneous system

$$\begin{cases} u'(t) = Au(t), & t \in (0,T), \\ u(0) = u_0, \end{cases} \tag{2.59}$$

where u_0 is the initial state belong to some Hilbert space H. We assume that this system (2.59) has a unique solution $u \in C^0([0,T];H)$.

Definition 2.2.5. *We say the system* (2.59) *satisfy the backward uniqueness property if*

$$u(T) = 0 \text{ in } H \text{ implies } u \equiv 0.$$

The backward uniqueness property is very important in control theoretic perspective. Note that, the backward uniqueness of (2.59) will imply the same for adjoint system (2.51): "If $\varphi(0) = 0$ in H, then necessarily $\varphi \equiv 0$". Using this property, we will show that the approximate controllability can be achieved from null controllability of the infinite dimensional linear system (2.48).

Proposition 2.2.1. *Let the homogeneous system* (2.59) *satisfy the backward uniqueness property. Let us also assume that the control system* (2.48) *is null controllable at some time* $T > 0$ *in the space* H*. Then, the system* (2.48) *is approximately controllable at that time* T *in* H*.*

Proof. Since the system (2.48) is null controllable at time $T > 0$, applying Theorem 2.2.9-Part (ii), we have the following observability inequality

$$\int_0^T \|B^*\varphi(t)\|_U^2 \, dt \geq C \, \|\varphi(0)\|_H^2$$

for all $\varphi_T \in \mathcal{D}(A^*)$. To prove approximate controllability of the system (2.48) at time T in H, it is enough to prove the unique continuation principle (2.58). Let $B^*\varphi = 0$ in $L^2(0,T;U)$. The above observability inequality is then yield $\varphi(0) = 0$ in H. Thanks to the backward uniqueness property, we deduce that $\varphi \equiv 0$ and hence $\varphi_T = 0$ in H. This completes the proof. □

Apart from the above result, we now state two results, which shows that the approximate controllability can also achieved from null controllability in the absence of backward uniqueness. For the proof of first part, we refer to the book [Cor07, Theorem 2.45, Page 57], whereas the second part follows directly from Theorem 2.2.6.

Theorem 2.2.10. *Let* H *be a Hilbert space. Then:*

1. *If the system* (2.48) *is null controllable at every time* $T > 0$ *in* H*, then the system* (2.48) *is approximately controllable at every time* $T > 0$ *in* H*.*

2. *If the system* (2.48) *is null controllable at some time* $T > 0$ *in* H *and* $\text{Range}(S(T))$ *is dense in* H*, then the system* (2.48) *is approximately controllable at that time* T *in* H*.*

We finally conclude this section with some important results that are stated in finite dimensional setup at the end of the previous section (see Theorem 2.2.3 and Theorem 2.2.4). Note that, there is no natural generalization of the Kalman rank condition in the infinite dimensional case. However, the Fattorini-Hautus test can be generalized only for the approximate controllability under some general assumptions on the operator A. Also, recall from Theorem 2.2.3 that for finite dimensional linear systems, controllability at time T is equivalent to the invertibility of the Gramian matrix Q_T. We can generalize this result for the infinite dimensional linear systems when the operator $B : U \to H$ is bounded, by defining the controllability Gramian $Q_T : H \to H$ as

$$Q_T(z) := \int_0^T S(t)BB^*S^*(t)dt, \quad z \in H. \tag{2.60}$$

Note that, Q_T is a well-defined bounded linear operator on H which is self-adjoint and non-negative definite (since B is bounded). We only state these results and for the proof, we refer to the lecture note [Boy23, Theorem III.3.7, Page 40] and the book [Zab20, Sections 15.2–15.3]. We mention here that the Fattorini-Hautus test has been used in several places of this thesis.

Theorem 2.2.11 (Fattorini-Hautus test)**.** *Let H be Hilbert space and the resolvent operator $(\lambda I - A)^{-1}$ is compact on H for every $\lambda \in \rho(A)$. Let us also assume that the semigroup generated by $-A^*$ is analytic. Then the system (2.48) is approximately controllable at time $T > 0$ in H if and only if*

$$\ker(\lambda I - A^*) \cap \ker(B^*) = \{0\}.$$

Theorem 2.2.12. *The following statements hold:*

(i) The system (2.48) is exactly controllable at time T in H if and only if $\mathrm{Range}(Q_T^{\frac{1}{2}}) = H$.

(ii) The system (2.48) is null controllable at time T in H if and only if $\mathrm{Range}(S(T)) \subset \mathrm{Range}(Q_T^{\frac{1}{2}})$.

(iii) The system (2.48) is approximately controllable at time T in H if and only if $\mathrm{Range}(Q_T^{\frac{1}{2}})$ *is dense in H.*

2.2.3 Nonlinear systems

In this section, we give a brief introduction to the controllability of nonlinear systems in finite and infinite dimensions. We present one example of an infinite dimensional nonlinear system in Section 2.5 which is relevant to this thesis. First, we write a general nonlinear system in $\mathbb{R}^n$ as follows:

$$\begin{cases} u'(t) = F(u(t), f(t)), & t \in (0,T), \\ u(0) = u_0. \end{cases} \tag{2.61}$$

Here $T > 0$, $u : [0,T] \to \mathbb{R}^n$ represents the state vector, $f : [0,T] \to \mathbb{R}^m$ the control vector and $u_0 \in \mathbb{R}^n$ is the initial state. We assume that the nonlinear function $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is regular enough. In this setup, we first define the equilibrium point of this system.

Definition 2.2.6. *We say a point $(\bar{u}, \bar{f}) \in \mathbb{R}^n \times \mathbb{R}^m$ is an equilibrium (or steady state) of the system (2.61) if*

$$F(\bar{u}, \bar{f}) = 0.$$

We wish to study controllability properties of (2.61) around some equilibrium point. For this, we assume that this system (2.61) has a unique solution in the whole interval $[0,T]$. Then, we define the controllability notion for this system (see the figure below).

Definition 2.2.7. *We say the system (2.61) is* ***small-time locally controllable*** *around the equilibrium $(\bar{u}, \bar{f})$ if, for given $T > 0$, there exists a $\epsilon > 0$ such that for chosen $u_0, u_T \in \mathbb{R}^n$ with $\|u_0 - \bar{u}\|_{\mathbb{R}^n} < \epsilon$ and $\|u_T - \bar{u}\| < \epsilon$, there exists a measurable function $f : [0,T] \to \mathbb{R}^m$ such that $u(T) = u_T$ in $\mathbb{R}^n$.*

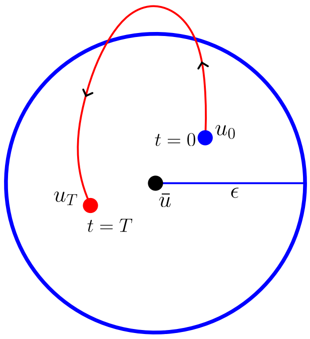

Figure 2.2: Small-time local controllability

If the above property holds at a given time $T > 0$, then we say the system (2.61) is locally controllable at time T around $(\bar{u}, \bar{f})$. In addition, if we don't have smallness condition on the initial and final states, we say the nonlinear system (2.61) is **globally controllable**.

Studying local controllability of the nonlinear system is very difficult compared to the finite dimensional linear systems. There are a few techniques available in the literature for studying the small-time local controllability of the general nonlinear system (2.61) including the linear test, the return method and Lie algebra technique (which relies on iterated Lie brackets). We refer to the books [Cor07, Chapter 3] and [Zab20, Chapter 6] for more insights in this matter. In this section, we only state some of these results and provide some examples of nonlinear systems in finite dimension.

From the definition of small-time local controllability, it is easy to see that, for $F(u(t), f(t)) = Au(t) + Bf(t)$ with $A \in M_n(\mathbb{R})$ and $B \in M_{n,m}(\mathbb{R})$, controllability of the pair (A, B) implies local controllability of the linear system

$$\begin{cases} u'(t) = Au(t) + Bf(t), \quad t \in (0, T). \\ u(0) = u_0 \end{cases}$$

around $(\bar{u}, \bar{f}) = (0, 0) \in \mathbb{R}^n \times \mathbb{R}^m$ at every $T > 0$. Moreover, the following result shows that we can achieve local controllability of the nonlinear system (2.61) from the controllability of the corresponding linear system.

Theorem 2.2.13. *Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be continuously differentiable in a neighborhood of the equilibrium point $(\bar{u}, \bar{f})$. If the linearized system around $(\bar{u}, \bar{f})$*

$$\begin{cases} u'(t) = \dfrac{\partial F}{\partial u}(\bar{u}, \bar{f})u(t) + \dfrac{\partial F}{\partial f}(\bar{u}, \bar{f})f(t), \quad t \in (0, T), \\ u(0) = u_0 \end{cases}$$

is controllable, then the nonlinear system (2.61) is locally controllable around $(\bar{u}, \bar{f})$ at every $T > 0$.

We refer to [Cor07, Theorem 3.8] and [Zab20, Theorem 6.6] for a proof of this result. Note that, converse of the above Theorem is not true, in general. More precisely, there are locally controllable nonlinear systems such that the linearized systems are not controllable, see Example 2.2.3 below.

Example 2.2.3. *Let us consider $u = (u_1, u_2, u_3) \in \mathbb{R}^3, f = (f_1, f_2) \in \mathbb{R}^2$ and*

$$F(u, f) = (f_1, f_2, u_1 f_2 - u_2 f_1).$$

Note that $\{(\bar{u}, 0) \ : \ \bar{u} = (\bar{u}_1, \bar{u}_2, \bar{u}_3) \in \mathbb{R}^3\} \subset \mathbb{R}^3 \times \mathbb{R}^2$ is the set of all equilibrium points of this system. The linearized system around these equilibrium points is given by

$$\begin{cases} u'(t) = Au(t) + Bf(t), \quad t \in (0, T), \\ u(0) = u_0, \end{cases}$$

where

$$A := \frac{\partial F}{\partial u}(\bar{u}, 0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B := \frac{\partial F}{\partial f}(\bar{u}, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -u_2 & u_1 \end{pmatrix}.$$

Since Rank of the Kalman matrix is 2, the linear system is not controllable. However, one can prove using Lie algebra technique that the nonlinear system is small-time locally controllable at any equilibrium point $(\bar{u}, 0)$, see [Cor07, Example 3.20, Page 135] or [Zab20, Example 6.3, Page 93] for more details.

We conclude this section with an example of a nonlinear system that is not small-time locally controllable.

Example 2.2.4. *Let us consider $u = (u_1, u_2, u_3) \in \mathbb{R}^3, f(t) \in \mathbb{R}$ and*

$$F(u, f) = (f, u_3, -u_2 + 2u_1 f).$$

Note that $(0,0) \in \mathbb{R}^3 \times \mathbb{R}$ is an equilibrium point of this system. The linearized system around this equilibrium point is given by

$$\begin{cases} u'(t) = Au(t) + Bf(t), & t \in (0, T), \\ u(0) = u_0, \end{cases}$$

where

$$A := \frac{\partial F}{\partial u}(0, 0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad B := \frac{\partial F}{\partial f}(0, 0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Since rank of the Kalman matrix is 2, the linear system is not controllable and we show below that the nonlinear system is also not small-time locally controllable around $(0, 0)$.
We first write the nonlinear system as

$$u_1'(t) = f(t), \quad u_2'(t) = u_3(t), \quad u_3'(t) = -u_2(t) + 2u_1(t)f(t),$$

for $t \in (0, T)$. Thus, we obtain

$$u_2''(t) + u_2(t) = 2u_1(t)u_1'(t), \quad t \in (0, T).$$

Assuming the initial state $u(0) = (0, 0, 0)$, we find that

$$u_2(T) = \int_0^T \cos(T - t) u_1^2(t) dt.$$

This shows that $u_2(T) \geq 0$ if $0 < T \leq \frac{\pi}{2}$. As a consequence, the nonlinear system cannot be small-time locally controllable around $(0, 0)$.

Like the finite dimensional case, there are no general results available in the literature to prove small-time local controllability of the infinite dimensional nonlinear systems. In this context, we refer to the book [Cor07, Chapter 4], where local controllability of several nonlinear equations (in infinite dimension) has been proved by using variations of fixed-point methods; see also the book [Zab20]. In Chapter 5, we will show another variation of fixed point, known as "the source term method" to prove small-time local controllability of a coupled 2-parabolic system. Moreover, in Section 2.5, we give some overview on the local controllability of a 1-d nonlinear heat equation (with square nonlinearity), where we have utilized the controllability of the linearized system.

2.3 The transport equation

This section is devoted to the controllability properties of the transport equation posed in one dimension. This equation plays a crucial role in this thesis because of its presence in the linearized compressible Navier-Stokes system, as mentioned in the introduction. For this reason, we will present a detailed study of controllability of this equation by using one boundary control. The results and proofs addressed here are taken from the book [Cor07, Chapter 2]. In this context, we refer to the book [?] for a study of general hyperbolic systems.

Let $T, L > 0$. The transport equation in the interval $(0, L)$ is given by

$$\rho_t + c\rho_x = 0,$$

where $c > 0$ and $\rho := \rho(t,x)$ is the state. We take the initial condition as

$$\rho(0,x) = \rho_0(x), \quad x \in (0,L).$$

We consider one of the following boundary conditions on ρ:

- ⋄ **(Dirichlet):** $\rho(t,0) = 0, \quad \text{for } t \in (0,T),$
- ⋄ **(Periodic):** $\rho(t,0) = \rho(t,L), \quad \text{for } t \in (0,T).$

We first consider the Dirichlet case and provide a detailed study of controllability of this equation using only one boundary control. The periodic case will be similar to the Dirichlet setup, so we will give some comments at the end of this section.

2.3.1 Dirichlet setup

Let $T, L > 0$ be given. We consider the following system:

$$\begin{cases} \rho_t + c\rho_x = 0, & \text{in } (0,T) \times (0,L), \\ \rho(t,0) = p(t), & \text{for } t \in (0,T), \\ \rho(0,x) = \rho_0(x), & \text{in } (0,L). \end{cases} \tag{2.62}$$

Here $c > 0$, $\rho = \rho(t,x)$ is the state, ρ_0 is the initial state and p is a boundary control. We consider the state space as $L^2(0,L)$, the control space as $L^2(0,T)$ and define the unbounded linear operator $A : \mathcal{D}(A) \subset L^2(0,L) \to L^2(0,L)$ as follows.

$$\begin{cases} \mathcal{D}(A) = \left\{ f \in H^1(0,L) \ : \ f(0) = 0 \right\}, \\ Af := -cf_x, \quad f \in \mathcal{D}(A). \end{cases} \tag{2.63}$$

We note here that the adjoint of the operator A is given by

$$\begin{cases} \mathcal{D}(A^*) = \left\{ g \in H^1(0,L) \ : \ g(L) = 0 \right\}, \\ A^* g := cg_x, \quad g \in \mathcal{D}(A^*). \end{cases} \tag{2.64}$$

In this setup, we first write the following result, which shows that the operator $(A, \mathcal{D}(A))$ generates a C^0-semigroup $\{S(t)\}_{t\geq 0}$ of contractions in $L^2(0,L)$. As a consequence, the adjoint operator $(A^*, \mathcal{D}(A^*))$ also generates a C^0-semigroup $\{S^*(t)\}_{t\geq 0}$ of contractions in $L^2(0,L)$.

Lemma 2.3.1. *The operator $(A, \mathcal{D}(A))$ generates a C^0-semigroup of contractions $\{S(t)\}_{t\geq 0}$ in $L^2(0,L)$.*

Proof. We will apply the Lumer-Philips theorem (Corollary 2.1.2) to prove this result. More precisely, it is enough to prove that A is a densely defined closed linear operator in $L^2(0,L)$ and both A, A^* are dissipative in $L^2(0,L)$.

- Since $C_c^\infty(0,L) \subset \mathcal{D}(A)$ is dense in $L^2(0,L)$, therefore $\mathcal{D}(A)$ is dense in $L^2(0,L)$. Thus, A is densely defined.

- Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}(A)$ such that $f_n \to f$ in $L^2(0,L)$ and $Af_n \to g$ in $L^2(0,L)$ for some $f,g \in L^2(0,L)$. This implies

$$\lim_{n\to\infty}\int_0^L (-cf_n)_x\varphi dx = \int_0^L g\varphi dx, \quad \forall\varphi \in C_c^\infty(0,L).$$

  An integration by parts yields

$$\lim_{n\to\infty}\int_0^L cf_n\varphi_x dx = \int_0^L g\varphi dx, \quad \forall\varphi \in C_c^\infty(0,L).$$

  Since $f_n \to f$ in $L^2(0,L)$, we readily have

$$\int_0^L cf\varphi_x dx = \int_0^L g\varphi dx, \quad \forall\varphi \in C_c^\infty(0,L).$$

  This proves that $f \in H^1(0,L)$ and $-cf_x = g$. It remains to prove that $f(0) = 0$. Since $f_n \in \mathcal{D}(A)$, therefore $f_n(0) = 0$ for all $n \in \mathbb{N}$. Let $\varphi \in C^\infty[0,L]$ be such that $\varphi(L) = 0$ and $\varphi(0) \neq 0$. Then we have after an integration by parts

$$\int_0^L g\varphi dx = \int_0^L (-cf)_x\varphi dx = \int_0^L (cf)\varphi_x dx - cf(0)\varphi(0).$$

  On the other hand

$$\int_0^L (cf)\varphi_x dx = \lim_{n\to\infty}\int_0^L (cf_n)\varphi_x dx = \lim_{n\to\infty}\int_0^L (-cf_n)_x\varphi dx = \int_0^L g\varphi dx.$$

  Comparing these above two identities, we deduce that $cf(0)\varphi(0) = 0$, which implies $f(0) = 0$ as $\varphi(0) \neq 0$. Thus, $f \in \mathcal{D}(A)$ and therefore A is closed.

- Let $f \in \mathcal{D}(A)$. Then

$$\langle Af, f\rangle_{L^2(0,L)} = -c\int_0^L f_x(x)f(x)dx = -\frac{c}{2}f^2(L) \leq 0,$$

  and therefore A is dissipative in $L^2(0,L)$.

- Let $g \in \mathcal{D}(A^*)$. Then

$$\langle A^*g, g\rangle_{L^2(0,L)} = c\int_0^L g_x(x)g(x)dx = -\frac{c}{2}g^2(0) \leq 0,$$

  and hence A^* is also dissipative in $L^2(0,L)$.

The proof completes. □

Thanks to this result, we can guarantee the existence and uniqueness of a strong solution to the system (2.62) when the initial state ρ_0 and control p are more regular.

Lemma 2.3.2. *Let us assume that $\rho_0 \in \mathcal{D}(A)$ and $p \in C^2([0,T])$ satisfies the compatibility condition $p(0) = 0$. Then the system (2.62) admits a unique strong solution*

$$\rho \in C^1([0,T]; L^2(0,L)) \cap C^0([0,T]; H^1(0,L)).$$

Proof. Let $\rho_0 \in \mathcal{D}(A)$ and $p \in C^2[0,T]$ with $p(0) = 0$. We define the function $\tilde{\rho}(t,x) = \rho(t,x) - p(t)$ for $(t,x) \in [0,T]\times[0,L]$. Then $\tilde{\rho}$ satisfies

$$\begin{cases} \tilde{\rho}_t = A\tilde{\rho} + f, & \text{in } (0,T)\times(0,L), \\ \tilde{\rho}(t,0) = 0, & \text{for } t \in (0,T), \\ \tilde{\rho}(0,x) = \rho_0(x), & \text{in } (0,L), \end{cases} \tag{2.65}$$

with $f(t,x) := -p'(t)$ for $(t,x) \in (0,T)\times(0,L)$. Since $f \in C^1([0,T]\times[0,L])$, by semigroup property (see Corollary 2.1.1), this system (2.65) has a unique strong solution $\tilde{\rho}$ in the space

$$C^1([0,T];L^2(0,L)) \cap C^0([0,T];\mathcal{D}(A)).$$

Consequently, the system (2.62) has a unique strong solution ρ in the space $C^1([0,T];L^2(0,L)) \cap C^0([0,T];H^1(0,L))$. This completes the proof. □

To guarantee the existence of a unique solution when $\rho_0 \in L^2(0,L)$ and $p \in L^2(0,T)$, we need to define the notion of a weak solution for the system (2.62) (see Definition 2.2.3). For this, we consider the adjoint system corresponding to (2.62) as follows:

$$\begin{cases} -\sigma_t - c\sigma_x = 0, & \text{in } (0,T)\times(0,L),\\ \sigma(t,L) = 0, & \text{for } t \in (0,T),\\ \sigma(T,x) = \sigma_T(x), & \text{in } (0,L), \end{cases} \tag{2.66}$$

where $\sigma_T \in L^2(0,L)$. Then, using the adjoint semigroup $\{S^*(t)\}_{t\geq 0}$, we have the following result:

Lemma 2.3.3. *For any given $\sigma_T \in \mathcal{D}(A^*)$, the adjoint system (2.66) admits a unique strong solution*

$$\sigma \in C^1([0,T];L^2(0,L)) \cap C^0([0,T];\mathcal{D}(A^*)).$$

We now define the notion of a weak solution of the system (2.62) when $\rho_0 \in L^2(0,L)$ and $p \in L^2(0,T)$.

Definition 2.3.1 (Weak solution)**.** *Let $\rho_0 \in L^2(0,L)$ and $p \in L^2(0,T)$ be given. We say a function $\rho \in C^0([0,T];L^2(0,L))$ is a weak solution of (2.62) if, for every $\sigma_T \in \mathcal{D}(A^*)$ the following identity holds true:*

$$\int_0^L \rho(t,x)\sigma(t,x)dx - \int_0^L \rho_0(x)\sigma(0,x)dx - c\int_0^t p(s)\sigma(s,0)ds = 0, \ \forall t \in [0,T]. \tag{2.67}$$

We note here that this formula is well-defined because of the fact that $\sigma(\cdot,0) \in L^2(0,T)$, thanks to Lemma 2.3.3. Using this definition, we have the following well-posedness result for the system (2.62).

Theorem 2.3.1. *For any given $\rho_0 \in L^2(0,L)$ and $p \in L^2(0,T)$, the system (2.62) admits a unique weak solution*

$$\rho \in C^0([0,T];L^2(0,L)).$$

Moreover, this solution ρ satisfies

$$\|\rho\|_{C^0([0,T];L^2(0,L))} \leq C\left(\|\rho_0\|_{L^2(0,L)} + \|p\|_{L^2(0,T)}\right), \tag{2.68}$$

for some $C > 0$ depending only on T, c.

Proof. We first prove uniqueness of the solution. Let us suppose that $\rho_1, \rho_2 \in C^0([0,T];L^2(0,L))$ be two weak solutions of the system (2.62) and denote $\rho := \rho_1 - \rho_2$. Then $\rho \in C^0([0,T];L^2(0,L))$ satisfies

$$\begin{cases} \rho_t + c\rho_x = 0, & \text{in } (0,T)\times(0,L),\\ \rho(t,0) = 0, & \text{for } t \in (0,T),\\ \rho(0,x) = 0, & \text{in } (0,L). \end{cases}$$

Thus, we have from the definition of weak solution (see (2.67)):

$$\int_0^L \rho(t,x)\sigma(t,x)dx = 0, \quad \forall t \in [0,T],$$

for all $\sigma_T \in \mathcal{D}(A^*)$. Thus we have $\rho(t,\cdot) = 0$ for all $t \in [0,T]$.
On the other hand, to prove the existence of a weak solution, we consider the following cases:

Case 1. Let us first assume that $\rho_0 \in \mathcal{D}(A)$ and $p \in C^2[0,T]$ with $p(0) = 0$. Then, applying Lemma 2.3.2, there is a unique strong solution $\rho \in C^1([0,T]; L^2(0,L)) \cap C^0([0,T]; \mathcal{D}(A))$. We now prove the estimate (2.68). Taking L^2-inner product in (2.62) with ρ and integrating over $[0,t]$, we have

$$\int_0^t \int_0^L \rho_t \rho dx ds + c \int_0^t \int_0^L \rho_x \rho dx ds = 0.$$

Integrating by parts and using the boundary-initial conditions, we deduce that

$$\int_0^L \left[\rho^2(t,x) - \rho_0^2(x)\right] dx + c \int_0^t \left[\rho^2(s,L) - p^2(s)\right] ds = 0.$$

This gives

$$\begin{aligned}\int_0^L \rho^2(t,x)dx &= \int_0^L \rho_0^2(x)dx - c\int_0^t \rho^2(s,L)ds + c\int_0^t p^2(s)ds \\ &\leq \int_0^L \rho_0^2(x)dx + c\int_0^T p^2(t)dt.\end{aligned}$$

Taking supremum over $t \in [0,T]$, we obtain the inequality (2.68) when $\rho_0 \in \mathcal{D}(A)$ and $p \in C^2[0,T]$ with $p(0) = 0$.

Case 2. We now consider the case when $\rho_0 \in L^2(0,L)$ and $p \in L^2(0,T)$. Since $C_c^\infty(0,T)$ is dense in $L^2(0,T)$, there exist sequences $(\rho_0^n)_{n\in\mathbb{N}} \subset \mathcal{D}(A)$ and $(p^n)_{n\in\mathbb{N}} \subset C^2[0,T]$ with $p^n(0) = 0$ for all $n \in \mathbb{N}$ such that

$$\rho_0^n \to \rho_0 \text{ in } L^2(0,L), \text{ and } p^n \to p \text{ in } L^2(0,T). \tag{2.69}$$

Then, applying Case 1, for each $n \in \mathbb{N}$, we find a unique strong solution $\rho^n \in C^0([0,T]; L^2(0,L))$ of the system

$$\begin{cases}\rho_t^n + c\rho_x^n = 0, & \text{in } (0,T)\times(0,L),\\ \rho^n(t,0) = p^n(t), & \text{for } t \in (0,T),\\ \rho^n(0,x) = \rho_0^n(x), & \text{in } (0,1).\end{cases} \tag{2.70}$$

Moreover, we have the following estimate

$$\|\rho^n\|_{C^0([0,T];L^2(0,L))} \leq C\left(\|\rho_0^n\|_{L^2(0,L)} + \|p^n\|_{L^2(0,T)}\right), \quad \text{for all } n \in \mathbb{N}. \tag{2.71}$$

Let $\sigma \in C^1([0,T]; L^2(0,L)) \cap C^0([0,T]; \mathcal{D}(A^*))$ be the strong solution of the adjoint system (2.66) with $\sigma_T \in \mathcal{D}(A^*)$. Taking $L^2(0,L)$-inner product in (2.70) with σ and integrating over $[0,t]$, we get

$$\int_0^t \int_0^L \rho_t^n \sigma dx ds + c\int_0^t \int_0^L \rho_x^n \sigma dx ds = 0, \quad \forall t \in [0,T].$$

Integrating by parts and using the boundary-initial conditions, we deduce that

$$\int_0^L \rho^n(t,x)\sigma(t,x)dx - \int_0^L \rho_0^n(x)\sigma(0,x)dx - c\int_0^t p^n(s)\sigma(s,0)ds = 0, \; \forall t \in [0,T]. \tag{2.72}$$

Let $m,n \in \mathbb{N}$. By linearity of the equation (2.62), the solution corresponding to the initial state $\rho_0^n - \rho_0^m$ and control $p^n - p^m$ is $\rho^n - \rho^m$. From (2.71), we can say that this solution $\rho^n - \rho^m$ satisfies the following estimate

$$\|\rho^n - \rho^m\|_{C^0([0,T];L^2(0,L))} \leq C\left(\|\rho_0^n - \rho_0^m\|_{L^2(0,L)} + \|p^n - p^m\|_{L^2(0,T)}\right), \quad \text{for all } m,n \in \mathbb{N}.$$

Thanks to the convergence property (2.69), it follows that the sequence $(\rho^n)_{n\in\mathbb{N}}$ is Cauchy in the space $C^0([0,T]; L^2(0,L))$. Let $\rho^n \to \rho$ in $C^0([0,T]; L^2(0,L))$ for some $\rho \in C^0([0,T]; L^2(0,L))$. Then, $\rho^n(T) \to \rho(T)$ in $L^2(0,L)$ and therefore passing limit as $n \to \infty$ in the equation (2.72), we deduce the identity (2.67). This shows that ρ is a weak solution of (2.62). To obtain the desired estimate (2.68), we pass the limit as $n \to \infty$ in the inequality (2.71).

This completes the proof. □

Remark 2.3.1. *We want to mention here that the solution to the transport equation (2.62) can be written explicitly using the method of characteristics (see the figure below) and is given by*

$$\rho(t,x) := \begin{cases} \rho_0(x-ct), & \text{if } t < \frac{x}{c}, \\ p\left(t-\frac{x}{c}\right), & \text{if } t > \frac{x}{c}, \end{cases}$$

for all $(t,x) \in (0,T)\times(0,L)$. However, from this expression, one cannot conclude that ρ belong to $C^0([0,T];L^2(0,L))$, because we do not have any information of the solution on the line $x = ct$. Therefore, the concept of weak solution is very useful for this system and once we have existence of a weak solution ρ in $C^0([0,T];L^2(0,L))$, we can also obtain the following hidden regularity property of this system.

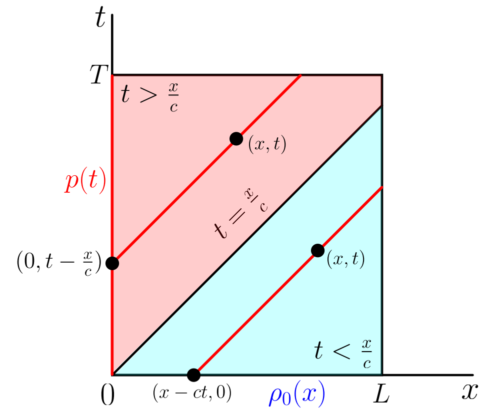

Figure 2.3: The characteristics curves are straight lines parallel to $t = \frac{x}{c}$

Lemma 2.3.4 (A hidden regularity property)**.** *Let $\rho_0 \in L^2(0,L)$ and $p \in L^2(0,T)$ be given. Then the solution $\rho \in C^0([0,T];L^2(0,L))$ of the system*

$$\begin{cases} \rho_t + c\rho_x = 0, & \text{in } (0,T)\times(0,L), \\ \rho(t,0) = p(t), & \text{for } t \in (0,T), \\ \rho(0,x) = \rho_0(x), & \text{in } (0,L) \end{cases} \tag{2.73}$$

satisfies the hidden regularity property

$$\rho(\cdot,L) \in L^2(0,T).$$

Proof. We consider the following cases:

Case 1. Let us first assume that $\rho_0 \in \mathcal{D}(A)$ and $p \in C^2[0,T]$ with $p(0) = 0$. Then there is a unique strong solution $\rho \in C^1([0,T];L^2(0,L)) \cap C^0([0,T];\mathcal{D}(A))$, thanks to Lemma 2.3.2. Taking $L^2(0,L)$-inner product in (2.62) with ρ and integrating over $[0,T]$, we have

$$\int_0^T\int_0^L \rho_t\rho dxdt + c\int_0^T\int_0^L \rho_x\rho dxdt = 0.$$

Integrating by parts and using the boundary-initial conditions, we deduce that

$$\int_0^L \left[\rho^2(T,x) - \rho_0^2(x)\right] dx + c\int_0^T [\rho^2(t,L) - p^2(t)]dt = 0.$$

Thus, we can write

$$c\int_0^T \rho^2(t,L)dt = \int_0^L \rho_0^2(x)dx - \int_0^L \rho^2(T,x)dx + c\int_0^T p^2(t)dt$$
$$\leq \int_0^L \rho_0^2(x)dx + c\int_0^T p^2(t)dt.$$

This proves the inequality

$$\int_0^T \rho^2(t,L)dt \leq \frac{1}{c}\int_0^L \rho_0^2(x)dx + \int_0^T p^2(t)dt$$

when $\rho_0 \in \mathcal{D}(A)$ and $p \in C^2[0,T]$ with $p(0) = 0$.

Case 2. We now consider the case when $\rho_0 \in L^2(0,L)$ and $p \in L^2(0,T)$. Then there exist sequences $(\rho_0^n)_{n\in\mathbb{N}} \subset \mathcal{D}(A)$ and $(p^n)_{n\in\mathbb{N}} \subset C^2[0,T]$ with $p^n(0) = 0$ for all $n \in \mathbb{N}$ such that

$$\rho_0^n \to \rho_0 \text{ in } L^2(0,L), \text{ and } p^n \to p \text{ in } L^2(0,T).$$

For each $n \in \mathbb{N}$, let ρ^n denotes the strong solution of (2.62) with initial state ρ_0^n and control p^n. Also, let ρ denotes the solution of (2.62) with the above $\rho_0 \in L^2(0,L)$ and $p \in L^2(0,T)$. Then, $\rho^n, \rho \in C^0([0,T];L^2(0,L))$ and by uniqueness of solutions, we have that

$$\rho^n \to \rho \quad \text{in } C^0([0,T];L^2(0,L)).$$

On the other hand, applying Case 1, the solution ρ^n satisfies the following estimate

$$\|\rho^n(\cdot,L)\|_{L^2(0,T)} \leq C\left(\|\rho_0^n\|_{L^2(0,L)} + \|p^n\|_{L^2(0,T)}\right) \tag{2.74}$$

for all $n \in \mathbb{N}$ and some constant $C > 0$. Then, by linearity of the system, we have

$$\|\rho^n(\cdot,L) - \rho^m(\cdot,L)\|_{L^2(0,T)} \leq C\left(\|\rho_0^n - \rho_0^m\|_{L^2(0,L)} + \|p^n - p^m\|_{L^2(0,T)}\right), \quad \text{for all } m,n \in \mathbb{N}. \tag{2.75}$$

Since $\rho_0^n \to \rho_0$ in $L^2(0,L)$ and $p^n \to p$ in $L^2(0,T)$, it follows that the sequence $(\rho^n(\cdot,L))_{n\in\mathbb{N}}$ is Cauchy in the space $L^2(0,T)$. Let us define

$$\rho(\cdot,L) := \lim_{n\to\infty} \rho^n(\cdot,L) \quad \text{in } L^2(0,T).$$

Since the constant C appearing in (2.75) does not depend on the choice of the sequences $(\rho_0^n)_{n\in\mathbb{N}}$ and $(p^n)_{n\in\mathbb{N}}$, the above function is well-defined. Now, passing limit as $n \to \infty$ in the inequality (2.74), we deduce that

$$\|\rho(\cdot,L)\|_{L^2(0,T)} \leq C\left(\|\rho_0\|_{L^2(0,L)} + \|p\|_{L^2(0,T)}\right)$$

when $\rho_0 \in L^2(0,L)$ and $p \in L^2(0,T)$.

This completes the proof. □

In a similar way, we can also obtain the hidden regularity property for the adjoint system (2.66):

Lemma 2.3.5. *Let $\sigma_T \in L^2(0,T)$ be given. Then the solution $\sigma \in C^0([0,T];L^2(0,L))$ to the adjoint system (2.66) satisfies*

$$\sigma(\cdot,0) \in L^2(0,T).$$

More precisely, the following estimate

$$\int_0^T \sigma^2(t,0)dt \leq C\|\sigma_T\|^2_{L^2(0,L)}$$

holds for some constant $C > 0$.

These hidden regularity properties are very useful in the context of controllability of the infinite dimensional linear systems (that contains a transport equation) using a boundary control; see Chapters 3–4 for instance. In this section also, we will see the use of these hidden regularity properties to achieve controllability of the system (2.62). Note that, the existence result (Theorem 2.3.1) shows that the value of the solution at time T is well-defined in $L^2(0,L)$ and therefore we can study the controllability properties for the system (2.62) in the space $L^2(0,L)$. We first write the following result which shows that exact and null controllability are equivalent for this system.

Theorem 2.3.2. *Let $T > 0$ be given. Then the system (2.62) is exactly controllable at time T in $L^2(0,L)$ if and only if it is null controllable at time T in $L^2(0,L)$.*

Proof. Let us assume that the system (2.62) be null controllable at time T in the space $L^2(0,L)$. Let $\rho_0, \rho_T \in L^2(0,L)$ be given. We consider the following system

$$\begin{cases} \bar{\rho}_t + c\bar{\rho}_x = 0, & \text{in } (0,T)\times(0,L), \\ \bar{\rho}(t,0) = 0, & \text{for } t \in (0,T), \\ \bar{\rho}(0,x) = \rho_T(L-x), & \text{in } (0,L). \end{cases} \tag{2.76}$$

Thanks to Lemma 2.3.4, we have $\bar{\rho}(\cdot,L) \in L^2(0,T)$. Denote $\tilde{\rho}(t,x) := \bar{\rho}(T-t, L-x)$ for $(t,x) \in (0,T)\times(0,L)$. Then $\tilde{\rho} \in C^0([0,T];L^2(0,L))$ and is a solution of the system

$$\begin{cases} \tilde{\rho}_t + c\tilde{\rho}_x = 0, & \text{in } (0,T)\times(0,L), \\ \tilde{\rho}(t,L) = 0, & \text{for } t \in (0,T), \\ \tilde{\rho}(T,x) = \rho_T(x), & \text{in } (0,L). \end{cases} \tag{2.77}$$

With the help of this solution and due to our assumption, we find the existence of a control $\hat{p} \in L^2(0,T)$ such that the solution $\hat{\rho} \in C^0([0,T];L^2(0,L))$ to the system

$$\begin{cases} \hat{\rho}_t + c\hat{\rho}_x = 0, & \text{in } (0,T)\times(0,L), \\ \hat{\rho}(t,0) = \hat{p}(t), & \text{for } t \in (0,T), \\ \hat{\rho}(0,x) = \rho_0(x) - \tilde{\rho}(0,x), & \text{in } (0,L) \end{cases} \tag{2.78}$$

satisfies $\hat{\rho}(T,x) = 0$ in $(0,L)$. Denote $\rho := \tilde{\rho} + \hat{\rho}$. Then ρ satisfies the system (2.62) with $p = \bar{\rho}(T-\cdot, L) + \hat{p} \in L^2(0,T)$. Moreover, we have

$$\rho(T,x) = \bar{\rho}(0, L-x) + \hat{\rho}(T,x) = \rho_T(x) \text{ in } (0,L).$$

This proves that the system (2.62) is exactly controllable at time T in $L^2(0,L)$.

Converse part is obvious. This completes the proof. □

We therefore study the exact controllability of this system (2.62). The following result shows that a minimal time is required to achieve exact controllability of the transport equation (2.62). This is one of the main difference between finite and infinite dimensional linear systems. Recall that in finite dimensional setup, no restriction on T is required to achieve controllability (Theorem 2.2.3).

Theorem 2.3.3. *The system (2.62) is exactly controllable at time T in $L^2(0,L)$ if and only if $T \geq \frac{L}{c}$.*

Proof. We will use the explicit expression of the solution

$$\rho(t,x) = \begin{cases} \rho_0(x-ct), & \text{if } x > ct, \\ p(ct-x), & \text{if } x < ct, \end{cases} \tag{2.79}$$

to prove this result. Let us first assume that $0 < T < \frac{L}{c}$. We choose the initial state $\rho_0(x) = 1$ and final state $\rho_T(x) = 0$ for all $x \in (0,L)$. Since $0 < T < \frac{L}{c}$, the solution of (2.62) with this initial state satisfies $\rho(T,x) = \rho_0(x-cT) = 1$ for all $x \in (cT, L)$ (see the figure below). This implies there cannot exists any $p \in L^2(0,T)$ such that $\rho(T,x) = \rho_T(x)$ for all $x \in (0,L)$. As a consequence, the system (2.62) cannot be exactly controllable at time T in $L^2(0,L)$.

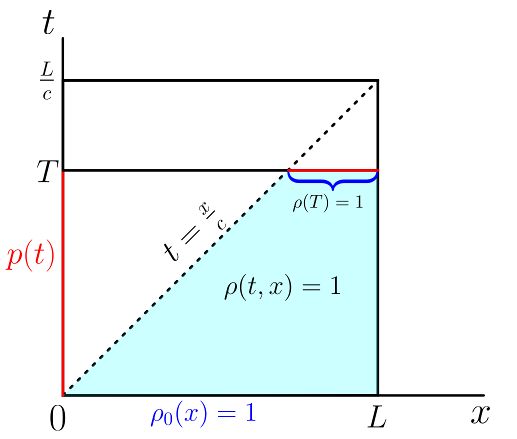

Figure 2.4: The control p do not have any effect in the lower region.

On the other hand, let us assume $T \geq \frac{L}{c}$. Let $\rho_0, \rho_T \in L^2(0, L)$ be given. We define a function $p \in L^2(0, T)$ as

$$p(t) := \begin{cases} 0, & \text{if } t \in \left(0, T - \dfrac{L}{c}\right), \\ \rho_T(c(T - t)), & \text{if } t \in \left(T - \dfrac{L}{c}, T\right). \end{cases}$$

Since $T \geq \frac{L}{c}$, the solution of (2.62) with the initial state ρ_0 and the above control p satisfies

$$\rho(T, x) = p\left(T - \frac{x}{c}\right) = \rho_T(x), \quad x \in (0, L),$$

see the figure below. Hence, the system is exactly controllable at time T in $L^2(0, L)$.

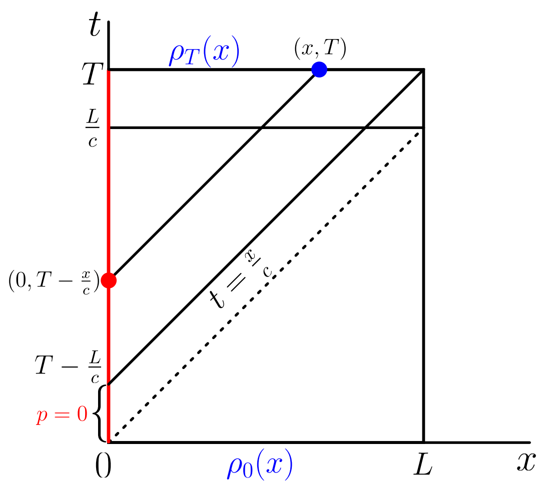

Figure 2.5: The solution at time T do not depend on the choice of the control p in $(0, T - \frac{L}{c})$.

This completes the proof. □

Recall that, in Section 2.2.2, we have derived some equivalent conditions for exact, null and approximate controllability in terms of the operator F_T and its adjoint. In this case, we define the map $F_T : L^2(0, T) \to L^2(0, L)$ by

$$F_T(p) := \rho(T, \cdot),$$

where ρ is the unique weak solution of (2.62) with $\rho_0 = 0$ and $p \in L^2(0, T)$. Using Theorem 2.3.1, we know that $\rho \in C^0([0, T]; L^2(0, L))$, and therefore F_T is a well-defined linear map. From the definition of weak solution (eq. (2.67)), the adjoint of this map $F_T^* : L^2(0, L) \to L^2(0, T)$ can be computed as

$$F_T^*(\sigma_T) := \sigma(\cdot, 0), \quad \text{for } \sigma_T \in L^2(0, L).$$

Note that this map is well-defined, thanks to the hidden regularity property of the adjoint system (2.66) (see Lemma 2.3.5). Consequently, the operator $B^* : \mathcal{D}(A^*) \to \mathbb{R}$ is defined as

$$B^*(\varphi) := \varphi(0), \text{ for all } \varphi \in \mathcal{D}(A^*).$$

With these maps, we can now state the following result, thanks to Theorem 2.2.8 and Theorem 2.2.9.

Theorem 2.3.4. *The system* (2.62) *is exactly controllable at time T in $L^2(0,L)$ if and only if there exists a $C > 0$ such that the following observability inequality*

$$\int_0^T |\sigma(t,0)|^2\, dt \geq C \, \|\sigma_T\|^2_{L^2(0,L)} \tag{2.80}$$

holds for all $\sigma_T \in \mathcal{D}(A^*)$.

Thus, to prove exact controllability of the system (2.62), it is enough to prove the above observability inequality (2.80). We use two different methods, one is using explicit expression of the solution and the other is via multiplier method, to prove this observability inequality.

Theorem 2.3.5. *The system* (2.62) *is exactly controllable at time T in the space $L^2(0,L)$ if and only if $T \geq \frac{L}{c}$.*

Proof. **Method 1: Explicit Solution.** Let $0 < T < \frac{L}{c}$. We choose a function $\sigma_T \in L^2(0,L)$ as

$$\sigma_T(x) := \begin{cases} \dfrac{1}{\sqrt{cT}}, & \text{if } 0 < x < cT, \\ 0, & \text{if } cT < x < L. \end{cases}$$

Then the solution of (2.66) with this σ_T satisfies $\sigma(t,0) = 0$ for all $t \in (0,T)$. This contradicts the inequality (2.80) as $\|\sigma_T\|_{L^2(0,L)} = 1$. Therefore, the system (2.62) cannot be exactly controllable at time T in $L^2(0,L)$.

We now assume that $T \geq \frac{L}{c}$. Thanks to Theorem 2.3.4, it is enough to prove the observability inequality (2.80). Note that, we have from the characteristics

$$\sigma(t,0) = \begin{cases} 0, & \text{if } 0 < t < T - \dfrac{L}{c}, \\ \sigma_T(c(T-t)), & \text{if } T - \dfrac{L}{c} < t < T. \end{cases}$$

This yields

$$\int_0^T |\sigma(t,0)|^2\, dt = \int_{T-\frac{L}{c}}^T |\sigma_T(c(T-t))|^2\, dt = \frac{1}{c}\int_0^L |\sigma_T(x)|^2\, dx = \frac{1}{c}\, \|\sigma_T\|^2_{L^2(0,L)},$$

proving the observability inequality (2.80).

Method 2: Multiplier Method. In this case, we assume that $T > \frac{L}{c}$. We cannot conclude exact controllability of (2.62) at the optimal time $T = \frac{L}{c}$ using this method. However, this can be done by using the explicit expression of the solution, as mentioned above. Here, we present this method because of its various importance in several places. To prove the observability inequality (2.80) with $T > \frac{L}{c}$, let us assume that $\sigma_T \in \mathcal{D}(A^*)$. Then, using Lemma 2.3.3, the solution σ of (2.66) belongs to the space

$$C^1([0,T]; L^2(0,L)) \cap C^0([0,T]; \mathcal{D}(A^*)).$$

Taking $L^2(0,L)$-inner product in (2.66) with σ and integrating over $[t,T]$, we deduce that

$$-\int_t^T \int_0^L \sigma_t \sigma dx ds - c\int_t^T \int_0^L \sigma_x \sigma dx ds = 0, \quad t \in (0,T).$$

Integrating by parts and using the boundary-initial conditions, we get

$$-\int_0^L [\sigma_T^2(x) - \sigma^2(t,x)]dx + c\int_t^T \sigma^2(s,0)ds = 0, \quad t \in (0,T).$$

This yields

$$\begin{aligned}\int_0^L \sigma^2(t,x)dx &= \int_0^L \sigma_T^2(x)dx - c\int_t^T \sigma^2(s,0)ds \\ &\geq \int_0^L \sigma_T^2(x)dx - c\int_0^T \sigma^2(s,0)ds, \quad t \in (0,T).\end{aligned}$$

An integration over the interval $(0,T)$ gives

$$\int_0^T\int_0^L \sigma^2(t,x)dxdt \geq T\int_0^L \sigma_T^2(x)dx - cT\int_0^T \sigma^2(s,0)ds. \tag{2.81}$$

On the other hand, taking $L^2(0,L)$-inner product in (2.66) with $x\sigma$ and integrating over the time interval $[0,T]$, we get

$$-\int_0^T\int_0^L x\sigma_t\sigma dxdt - c\int_0^T\int_0^L x\sigma_x\sigma dxdt = 0.$$

Integrating by parts and using the boundary-initial conditions, we deduce that

$$-\int_0^L [x\sigma_T^2(x) - x\sigma^2(0,x)]dx + c\int_0^T\int_0^L \sigma^2 dxdt = 0,$$

This gives

$$\int_0^T\int_0^L \sigma^2 dxdt = \frac{1}{c}\int_0^L [x\sigma_T^2(x) - x\sigma^2(0,x)]dx \leq \frac{L}{c}\int_0^L \sigma_T^2(x)dx. \tag{2.82}$$

Combining the inequalities (2.81) and (2.82), we obtain

$$T\int_0^L \sigma_T^2(x)dx - cT\int_0^T \sigma^2(s,0)ds \leq \frac{L}{c}\int_0^L \sigma_T^2(x)dx.$$

Thus, we finally have

$$\left(T - \frac{L}{c}\right)\int_0^L \sigma_T^2(x)dx \leq cT\int_0^T \sigma^2(s,0)ds.$$

Since $T > \frac{L}{c}$, the observability inequality (2.80) follows for all $\sigma_T \in \mathcal{D}(A^*)$. This completes the proof. □

2.3.2 Periodic setup

Let $T, L > 0$ be given. We now consider the following system:

$$\begin{cases}\rho_t + c\rho_x = 0, & \text{in } (0,T)\times(0,L), \\ \rho(t,0) = \rho(t,1) + p(t), & \text{for } t \in (0,T), \\ \rho(0,x) = \rho_0(x), & \text{in } (0,L).\end{cases} \tag{2.83}$$

Here $c > 0$, $\rho = \rho(t,x)$ is the state, $\rho_0 \in L^2(0,L)$ is the initial state and $p \in L^2(0,T)$ is a boundary control. We define the unbounded linear operator $A : \mathcal{D}(A) \subset L^2(0,L) \to L^2(0,L)$ as follows:

$$\begin{cases}\mathcal{D}(A) = \left\{f \in H^1(0,L) \ : \ f(0) = f(L)\right\}, \\ Af := -cf_x, \quad f \in \mathcal{D}(A).\end{cases} \tag{2.84}$$

The adjoint of the operator A is given by

$$\begin{cases}\mathcal{D}(A^*) = \left\{g \in H^1(0,L) \ : \ g(0) = g(L)\right\}, \\ A^*g := cg_x, \quad g \in \mathcal{D}(A^*).\end{cases} \tag{2.85}$$

Then, we have the following well-posedness result; the proof of which is similar to the Dirichlet case and so we omit the details.

Lemma 2.3.6. *The operator* $(A, \mathcal{D}(A))$ *(resp.* $(A^*, \mathcal{D}(A^*))$*) generates a* C^0*-semigroup of contractions* $\{S(t)\}_{t\geq 0}$ *in* $L^2(0,L)$*. Moreover, for given any* $\rho_0 \in L^2(0,L)$ *and* $p \in L^2(0,T)$*, the system* (2.83) *has a unique weak solution* $\rho \in C^0([0,T]; L^2(0,L))$ *satisfying the following estimate:*

$$\|\rho\|_{C^0([0,T];L^2(0,L))} \leq C\left(\|\rho_0\|_{L^2(0,L)} + \|p\|_{L^2(0,T)}\right),$$

for some constant $C > 0$ *depending only on* T, c*.*

Further, we have the hidden regularity property $\rho(\cdot, 0) \in L^2(0,T)$*.*

We then compute the adjoint map $F_T^* : L^2(0,L) \to L^2(0,T)$ as

$$F_T^*(\sigma_T) := \sigma(\cdot, 0), \text{ for } \sigma_T \in L^2(0,L),$$

where σ is the solution of the adjoint system

$$\begin{cases} -\sigma_t - c\sigma_x = 0, & \text{in } (0,T)\times(0,L), \\ \sigma(t,0) = \sigma(t,L), & \text{for } t \in (0,T), \\ \sigma(T,x) = \sigma_T(x), & \text{in } (0,L). \end{cases} \tag{2.86}$$

As a consequence, the operator $B^* : \mathcal{D}(A^*) \to \mathbb{R}$ is defined as

$$B^*(\varphi) := \varphi(0), \text{ for all } \varphi \in \mathcal{D}(A^*).$$

We finally prove the following controllability result for the system (2.83). We present a different method to prove the corresponding observability inequality by writing the solution of the adjoint system in terms of a basis consisting of the eigenfunctions of the adjoint operator A^*.

Theorem 2.3.6. *The system* (2.83) *is exactly controllable at time* T *in* $L^2(0,L)$ *if and only if* $T \geq \frac{L}{c}$*.*

Proof. Note that, thanks to Theorem 2.2.6, exact controllability of the system (2.83) is equivalent to proving the observability inequality

$$\int_0^T |\sigma(t,0)|^2\, dt \geq C\, \|\sigma_T\|^2_{L^2(0,L)} \tag{2.87}$$

for all $\sigma_T \in \mathcal{D}(A^*)$, where σ is the solution of (2.86). Let us first assume that $0 < T < \frac{L}{c}$. We choose a non-trivial $\sigma_T \in \mathcal{D}(A^*)$ such that $\operatorname{supp}(\sigma_T) \subset (cT, L)$. Then, by the method of characteristics, the solution σ of the adjoint system (2.86) satisfies $\sigma(t,0) = \sigma(t,L) = 0$ for all $t \in (0,T)$ but $\sigma \neq 0$ in $(0,T)\times(0,L)$. This contradicts the observability inequality (2.87) and, as a consequence, the system (2.83) cannot be exactly controllable at time T in $L^2(0,L)$.

We now assume that $T \geq \frac{L}{c}$. Let $\sigma_T \in \mathcal{D}(A^*)$ be given. It is easy to see that the eigenvalues of the operator $(A^*, \mathcal{D}(A^*))$ are $\lambda_n = \frac{2icn\pi}{L}$ and the corresponding eigenfunctions are $\varphi_n(x) := e^{\frac{2in\pi x}{L}}$ for all $n \in \mathbb{Z}$. We therefore write σ_T as

$$\sigma_T(x) = \sum_{n\in\mathbb{Z}} a_n e^{\frac{2in\pi x}{L}}, \quad \text{for } x \in (0,L),$$

and the corresponding solution as

$$\sigma(t,x) = \sum_{n\in\mathbb{Z}} a_n e^{\frac{2icn\pi}{L}(T-t)} e^{\frac{2in\pi x}{L}}, \quad \text{for } (t,x) \in (0,T)\times(0,L),$$

for some $(a_n)_{n\in\mathbb{Z}} \in \ell_2$. Since $T \geq \frac{L}{c}$, we have

$$\int_0^T |\sigma(t,0)|^2\, dt = \int_0^T \left|\sum_{n\in\mathbb{Z}} a_n e^{\frac{2icn\pi}{L}(T-t)}\right|^2 dt = \int_0^T \left|\sum_{n\in\mathbb{Z}} a_n e^{\frac{2icn\pi}{L}t}\right|^2 dt \geq \int_0^{\frac{L}{c}} \left|\sum_{n\in\mathbb{Z}} a_n e^{\frac{2icn\pi}{L}t}\right|^2 dt.$$

Changing the variable $t \mapsto \frac{c}{L}t$, we get

$$\int_0^T |\sigma(t,0)|^2 \, dt \geq \int_0^1 \left| \sum_{n\in\mathbb{Z}} a_n e^{2in\pi t} \right|^2 dt = \sum_{n\in\mathbb{Z}} |a_n|^2, \tag{2.88}$$

thanks to the Parseval's identity. We similarly have

$$\|\sigma_T\|^2_{L^2(0,L)} = \int_0^L \left| \sum_{n\in\mathbb{Z}} a_n e^{\frac{2in\pi x}{L}} \right|^2 dx = \int_0^1 \left| \sum_{n\in\mathbb{Z}} a_n e^{2in\pi x} \right|^2 dx = \sum_{n\in\mathbb{Z}} |a_n|^2. \tag{2.89}$$

Combining the above two estimates (2.88) and (2.89), the observability inequality (2.87) follows. This completes the proof. □

2.4 The heat equation

In this section, we will consider the one dimensional heat equation and study the controllability properties using only a boundary control. In the next few chapters, all these properties will be very useful in the context of controllability of the linearized compressible Navier-Stokes system (see Chapters (3)–4) or in the case of nonlinear systems considered in Chapter 5. The contents of this section can be found in any control theory books/ lecture notes, for instance in [MZ04, Section 2.5], [Boy23, Chapter 4], [Cor07, Section 2.5]. In addition, controllability of the heat equation using a localized distributed control is also studied in the above-mentioned references. In this thesis, we will concentrate only on the boundary controllability of the heat equation.

Let $T, L > 0$. The heat equation in the interval $(0, L)$ is given by

$$u_t - \nu u_{xx} = 0,$$

where $\nu > 0$ is called the diffusion coefficient and $u = u(t, x)$ is the state. We take the initial condition as

$$u(0, x) = u_0(x), \quad x \in (0, L).$$

In this case, we consider one of the following three boundary conditions on u:

- ⋄ **(Dirichlet):** $u(t, 0) = 0, \quad u(t, L) = 0, \quad \text{for } t \in (0, T),$
- ⋄ **(Neumann):** $u_x(t, 0) = 0, \quad u_x(t, L) = 0, \quad \text{for } t \in (0, T),$
- ⋄ **(Periodic):** $u(t, 0) = u(t, L), \quad u_x(t, 0) = u_x(t, L), \quad \text{for } t \in (0, T).$

We will present a detailed study of the controllability properties of the heat equation in the Dirichlet case using only one boundary control. The Neumann and periodic case will be similar to the Dirichlet setup, so we will omit the details here. In fact, the Neumann case is studied with detail in Chapter 5 for both linear and nonlinear heat equations, see also Section 2.5. Moreover, similar controllability studies for the heat equation with periodic boundary conditions is included in Chapter 3.

2.4.1 Dirichlet setup

Let $T, L > 0$ be given. We consider the following system:

$$\begin{cases} u_t - \nu u_{xx} = 0, & \text{in } (0,T) \times (0,L), \\ u(t,0) = 0, \quad u(t,L) = q(t), & \text{for } t \in (0,T), \\ u(0,x) = u_0(x), & \text{in } (0,L). \end{cases} \tag{2.90}$$

Here $\nu > 0$ is called the diffusion coefficient, $u = u(t, x)$ is the state, u_0 is the initial state and q is the boundary control. In this section, we will study the controllability properties for this system at any time $T > 0$ in the space $H^{-1}(0, L)$ using the boundary control $q \in L^2(0, T)$ acting at $x = L$.

We first define the unbounded linear operator $A : \mathcal{D}(A) \subset L^2(0,L) \to L^2(0,L)$ as follows.

$$\begin{cases} \mathcal{D}(A) = \left\{ f \in H^2(0,L) \ : \ f(0) = f(L) = 0 \right\}, \\ Af := -\nu f_{xx}, \quad f \in \mathcal{D}(A). \end{cases} \tag{2.91}$$

Note that the operator A is self-adjoint, that is

$$\begin{cases} \mathcal{D}(A^*) = \mathcal{D}(A), \\ A^* g := -\nu g_{xx}, \quad g \in \mathcal{D}(A^*). \end{cases} \tag{2.92}$$

We now write following result which shows that the operator $(-A, \mathcal{D}(A))$ generates a C^0-semigroup of contractions on $L^2(0,L)$.

Lemma 2.4.1. *The operator $(-A, \mathcal{D}(A))$ generates a C^0-semigroup of contractions $\{S(t)\}_{t \geq 0}$ on $L^2(0,L)$.*

Proof. We will apply the Lumer-Philips theorem (see Corollary 2.1.2) to prove this result. Since $A = A^*$, it is enough to prove that A is a densely defined closed linear operator in $L^2(0,L)$ and that $-A$ is dissipative in $L^2(0,L)$.

- Since $C_c^\infty(0,L) \subset \mathcal{D}(A)$ is dense in $L^2(0,L)$, therefore $\mathcal{D}(A)$ is dense in $L^2(0,L)$. Thus, A is densely defined.

- Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}(A)$ such that $f_n \to f$ in $L^2(0,L)$ and $Af_n \to g$ in $L^2(0,L)$ for some $f, g \in L^2(0,L)$. This implies

  $$\lim_{n\to\infty} \int_0^L (-\nu f_n)_{xx} \varphi dx = \int_0^L g\varphi dx, \quad \forall \varphi \in C_c^\infty(0,L).$$

  Integrating by parts twice yields

  $$\lim_{n\to\infty} \int_0^L (-\nu f_n) \varphi_{xx} dx = \int_0^L g\varphi dx, \quad \forall \varphi \in C_c^\infty(0,L).$$

  Since $f_n \to f$ in $L^2(0,L)$, we readily have

  $$\int_0^L (-\nu f) \varphi_{xx} dx = \int_0^L g\varphi dx, \quad \forall \varphi \in C_c^\infty(0,L). \tag{2.93}$$

  On the other hand, since the sequence $((f_n)_{xx})_{n\in\mathbb{N}}$ is bounded in $L^2(0,L)$, it follows that (f_n) is bounded in $H^1(0,L)$ (thanks to the Poincaré Inequality). Therefore, up to a subsequence, the sequence $(f_n)_{n\in\mathbb{N}}$ converges weakly in $H^1(0,L)$ to some function $\tilde{f} \in H^1(0,L)$. By uniqueness of the limit, we see that $\tilde{f} = f$ and consequently $f \in H^1(0,L)$. Then, from (2.93), we deduce that $f \in H^2(0,L)$ and $-\nu f_{xx} = g$.

  It remains to prove that $f(0) = f(L) = 0$. Since $f_n \in \mathcal{D}(A)$, therefore $f_n(0) = f_n(L) = 0$ for all $n \in \mathbb{N}$. Let $\varphi \in C^\infty[0,L]$ be such that $\varphi(0) = \varphi(L) = \varphi_x(L) = 0$ and $\varphi_x(0) \neq 0$. For example, one can take $\varphi(x) = x(x-L)^2$ for all $x \in [0,L]$. Then we have after twice integration by parts

  $$\int_0^L g\varphi dx = \int_0^L (-\nu f)_{xx} \varphi dx = \int_0^L (-\nu f) \varphi_{xx} dx - \nu f(0)\varphi_x(0).$$

  On the other hand

  $$\int_0^L (-\nu f)\varphi_{xx} dx = \lim_{n\to\infty} \int_0^L (-\nu f_n)\varphi_{xx} dx = \lim_{n\to\infty} \int_0^L (-\nu f_n)_{xx} \varphi dx = \int_0^L g\varphi dx.$$

  Comparing these above two identities, we deduce that $\nu f(0)\varphi_x(0) = 0$, which implies $f(0) = 0$ as $\varphi_x(0) \neq 0$. Similarly, one can prove that $f(L) = 0$. Thus, $f \in \mathcal{D}(A)$ and therefore A is closed.

- Let $f \in \mathcal{D}(A)$. Then
$$\langle Af, f\rangle_{L^2(0,L)} = -\nu \int_0^L f_{xx} f dx = \nu \int_0^L f_x^2 dx \geq 0,$$
and therefore $-A$ is dissipative in $L^2(0,L)$.

The proof completes. □

Thanks to this result, we can guarantee the existence and uniqueness of a strong solution of the system (2.90) when the initial state u_0 and control q are regular enough.

Theorem 2.4.1. *Let us assume that $u_0 \in \mathcal{D}(A)$ and $q \in C^2([0,T])$ satisfies the compatibility condition $q(0) = 0$. Then the system (2.90) admits a unique strong solution*
$$u \in C^1([0,T]; L^2(0,L)) \cap C^0([0,T]; H^2(0,L)).$$

Proof. Let $u_0 \in \mathcal{D}(A)$ and $q \in C^2[0,T]$ with $q(0) = 0$. We define the function $\tilde{u}(t,x) = u(t,x) - \frac{x}{L}q(t)$ for $(t,x) \in [0,T] \times [0,L]$. Then $\tilde{u}$ satisfies
$$\begin{cases} \tilde{u}_t = A\tilde{u} + f, & \text{in } (0,T) \times (0,L), \\ \tilde{u}(t,0) = 0, \quad \tilde{u}(t,L) = 0, & \text{for } t \in (0,T), \\ \tilde{u}(0,x) = u_0(x), & \text{in } (0,L), \end{cases} \tag{2.94}$$
with $f(t,x) := -\frac{x}{L}q'(t)$ for $t \in (0,T)$ and $x \in (0,L)$. Since $f \in C^1([0,T] \times [0,L])$, by semigroup property (see Corollary 2.1.1), this system has a unique strong solution $\tilde{u}$ in the space
$$C^1([0,T]; L^2(0,L)) \cap C^0([0,T]; \mathcal{D}(A)).$$
Consequently, the system (2.62) has a unique strong solution u in the space $C^1([0,T]; L^2(0,L)) \cap C^0([0,T]; H^2(0,L))$. This completes the proof. □

To guarantee the existence of a unique solution when $u_0 \in L^2(0,L)$ and $q \in L^2(0,T)$, we need to define the notion of a weak solution to the system (2.90). For this, we consider the adjoint system corresponding to (2.90) as follows:
$$\begin{cases} -v_t - \nu v_{xx} = 0, & \text{in } (0,T) \times (0,L), \\ v(t,0) = 0, \quad v(t,L) = 0, & \text{for } t \in (0,T), \\ v(T,x) = v_T(x), & \text{in } (0,L). \end{cases} \tag{2.95}$$
Here $v_T \in L^2(0,L)$. Then, using the adjoint semigroup $\{S^*(t)\}_{t \geq 0}$, we have the following result:

Lemma 2.4.2. *For any given $v_T \in \mathcal{D}(A^*)$, the adjoint system (2.95) admits a unique strong solution*
$$v \in C^1([0,T]; L^2(0,L)) \cap C^0([0,T]; \mathcal{D}(A^*)).$$

We then consider the following homogeneous system with a source term:
$$\begin{cases} u_t - \nu u_{xx} = f, & \text{in } (0,T) \times (0,L), \\ u(t,0) = 0, \quad u(t,L) = 0, & \text{for } t \in (0,T), \\ u(0,x) = u_0(x), & \text{in } (0,L). \end{cases} \tag{2.96}$$
For this system, we study some well-posedness results which will be very useful in the later chapters of this thesis. First, we define the notion of a weak solution for this system (2.96) when $u_0 \in L^2(0,L)$ and $f \in L^2(0,T; L^2(0,L))$.

Definition 2.4.1 (Weak solution)**.** *Let $u_0 \in L^2(0,L)$ and $f \in L^2(0,T;L^2(0,L))$ be given. We say a function $u \in C^0([0,T];L^2(0,L))$ is a weak solution of* (2.96) *if for every $v_T \in \mathcal{D}(A^*)$ the following identity holds true:*

$$\int_0^L u(t,x)v(t,x)dx - \int_0^L u_0(x)v(0,x)dx = \int_0^t \int_0^L f(s,x)v(s,x)dxds, \ \forall t \in [0,T]. \tag{2.97}$$

Then, with this definition of a weak solution, one can have the following result:

Theorem 2.4.2. *Let $u_0 \in L^2(0,L)$ and $f \in L^2(0,T;L^2(0,L))$ be given. Then the system* (2.96) *has a unique weak solution u in the space*

$$C^0([0,T];L^2(0,L)) \cap L^2(0,T;H_0^1(0,L)).$$

Moreover, there exists a $C > 0$ depending only on ν, T such that

$$\|u\|_{C^0([0,T];L^2(0,L))} + \|u\|_{L^2(0,T;H_0^1(0,L))} \le C\left(\|u_0\|_{L^2(0,L)} + \|f\|_{L^2(0,T;L^2(0,L))}\right). \tag{2.98}$$

Furthermore, if $u_0 \in H_0^1(0,L)$, this solution u satisfies the following estimate:

$$\|u\|_{C^0([0,T];H_0^1(0,L))} + \|u\|_{L^2(0,T;H^2(0,L))} \le C\left(\|u_0\|_{H_0^1(0,L)} + \|f\|_{L^2(0,T;L^2(0,L))}\right) \tag{2.99}$$

for some $C > 0$ depending only on ν, T.

Proof. We will prove each part separately.

- Let us first prove uniqueness of the solutions. If u_1 and u_2 are two solutions of (2.96), then the function $u := u_1 - u_2$ satisfies the system

$$\begin{cases} u_t - \nu u_{xx} = 0, & \text{in } (0,T)\times(0,L), \\ u(t,0) = 0, \quad u(t,L) = 0, & \text{for } t \in (0,T), \\ u(0,x) = 0, & \text{in } (0,L). \end{cases}$$

By the definition of weak solution (eq. (2.97)), we have

$$\int_0^L u(t,x)v(t,x)dx = 0, \quad \text{for all } v_T \in \mathcal{D}(A^*),$$

which implies $u(t) = 0$ in $L^2(0,L)$ for all $t \in (0,T)$.

We now prove the existence of a solution to the system (2.96). Let us first consider the case when $u_0 \in \mathcal{D}(A)$ and $f \in C^1([0,T]\times[0,L])$. Then the system (2.96) has a unique strong solution $u \in C^1([0,T];L^2(0,L)) \cap C^0([0,T];\mathcal{D}(A))$, thanks to the semigroup property (see Lemma 2.4.2). We now prove the estimate (2.98) in this case.

Taking $L^2(0,L)$-inner product in (2.96) with u, we get

$$\frac{1}{2}\frac{d}{dt}\int_0^L u^2(t,x)dx - \nu \int_0^L u_{xx}u dx = \int_0^L fu dx, \quad \forall t \in [0,T].$$

Integrating by parts and using the Young's inequality, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_0^L u^2(t,x)dx + \nu \int_0^L u_x^2(t,x)dx &= \int_0^L fu dx \\ &\le \frac{1}{2}\int_0^L u^2(t,x)dx + \frac{1}{2}\int_0^L f^2(t,x)dx, \quad \forall t \in [0,T]. \end{aligned} \tag{2.100}$$

Ignoring the term $\nu \int_0^L u_x^2(t,x)dx$, we have

$$\frac{1}{2}\frac{d}{dt}\int_0^L u^2(t,x)dx \le \frac{1}{2}\int_0^L u^2(t,x)dx + \frac{1}{2}\int_0^L f^2(t,x)dx, \quad \forall t \in [0,T].$$

Applying Gronwall's inequality (see [Eva10, Appendix B]), we deduce that

$$\frac{1}{2}\int_0^L u^2(t,x)dx \leq e^t\left(\int_0^L u_0^2(x)dx + \frac{1}{2}\int_0^t\int_0^L f^2(t,x)dx\right), \quad \forall t \in [0,T].$$

Taking supremum over $t \in [0,T]$, we obtain

$$\|u\|^2_{C^0([0,T];L^2(0,L))} \leq 2e^T\left(\|u_0\|^2_{L^2(0,L)} + \|f\|^2_{L^2(0,T;L^2(0,L))}\right).$$

On the other hand, we have from (2.100) that

$$\frac{1}{2}\frac{d}{dt}\int_0^L u^2(t,x)dx + \nu\int_0^L u_x^2(t,x)dx \leq \frac{1}{2}\|u\|^2_{C^0([0,T];L^2(0,L))} + \frac{1}{2}\int_0^L f^2(t,x)dx, \quad \forall t \in [0,T].$$

Integrating over the interval $[0,T]$, we obtain

$$\frac{1}{2}\int_0^L [u^2(T,x) - u_0^2(x)]dx + \nu\int_0^T\int_0^L u_x^2(t,x)dxdt \leq \frac{T}{2}\|u\|^2_{C^0([0,T];L^2(0,L))} + \frac{1}{2}\int_0^T\int_0^L f^2(t,x)dxdt.$$

Ignoring the term $\frac{1}{2}\int_0^L u^2(T,x)dx$ and using the previous $C^0([0,T];L^2(0,L))$-estimate, we deduce that

$$\|u\|^2_{L^2(0,T;H_0^1(0,L))} \leq Ce^T\left(\|u_0\|^2_{L^2(0,L)} + \|f\|^2_{L^2(0,T;L^2(0,L))}\right).$$

Here $C > 0$ is a constant depending only on ν.

Let us now consider the case when $u_0 \in L^2(0,L)$ and $f \in L^2(0,T;L^2(0,L))$. Let $(u_0^n)_{n\in\mathbb{N}} \subset \mathcal{D}(A)$ be a sequence such that $u_0^n \to u_0$ in $L^2(0,L)$ and let $(f^n)_{n\in\mathbb{N}} \subset C^1([0,T]\times[0,L])$ be such that $f^n \to f$ in $L^2(0,T;L^2(0,L))$. For each $n \in \mathbb{N}$, let u^n denotes the strong solution of (2.96) with initial state u_0^n and source term f^n. Since $u_0^n \in \mathcal{D}(A)$ and $f^n \in C^1([0,T]\times[0,L])$ for all $n \in \mathbb{N}$, we have from the previous case that

$$\|u^n\|_{C^0([0,T];L^2(0,L))} + \|u^n\|_{L^2(0,T;H_0^1(0,L))} \leq Ce^T\left(\left\|u_0^n\right\|_{L^2(0,L)} + \|f^n\|_{L^2(0,T;L^2(0,L))}\right), \quad \text{for all } n \in \mathbb{N}. \tag{2.101}$$

Let $v_T \in \mathcal{D}(A^*)$ and let $v \in C^1([0,T];L^2(0,L)) \cap C^0([0,T];\mathcal{D}(A^*))$ be the strong solution of (2.95) (see Lemma 2.4.2). Since u^n is the unique solution of (2.96), we have from (2.97) that

$$\int_0^L u^n(t,x)v(t,x)dx - \int_0^L u_0^n(x)v(0,x)dx = \int_0^t\int_0^L f^n(s,x)v(s,x)dxdt, \tag{2.102}$$

for all $n \in \mathbb{N}$ and $t \in [0,T]$. Let $m,n \in \mathbb{N}$. Note that $u^n - u^m$ is the strong solution of (2.96) corresponding to the initial state $u_0^n - u_0^m$ and source term $f^n - f^m$. Thus, we see from (2.101) that this solution satisfies the estimate

$$\|u^n - u^m\|_{C^0([0,T];L^2(0,L))\cap L^2(0,T;H_0^1(0,L))} \leq C\left(\left\|u_0^n - u_0^m\right\|_{L^2(0,L)} + \|f^n - f^m\|_{L^2(0,T;L^2(0,L))}\right)$$

for all $m,n \in \mathbb{N}$. This implies the sequence $(u^n)_{n\in\mathbb{N}}$ is Cauchy in the space $C^0([0,T];L^2(0,L)) \cap L^2(0,T;H_0^1(0,L))$. Let $u^n \to u$ in $C^0([0,T];L^2(0,L)) \cap L^2(0,T;H_0^1(0,L))$ for some function $u \in C^0([0,T];L^2(0,L)) \cap L^2(0,T;H_0^1(0,L))$. Then, passing limit as $n \to \infty$ in the identity (2.102), we see that u is a weak solution of the system (2.96) with the above initial state $u_0 \in L^2(0,L)$ and source term $f \in L^2(0,T;L^2(0,L))$. To prove the desired estimate (2.98), we pass limit as $n \to \infty$ in the inequality (2.101). This completes the proof of first part.

- To prove the estimate (2.99), we first assume that $u_0 \in \mathcal{D}(A)$ and $f \in C^1([0,T]\times[0,L])$. Then, the system (2.96) has a unique strong solution $u \in C^1([0,T];L^2(0,L)) \cap C^0([0,T];\mathcal{D}(A))$. Taking $L^2(0,L)$-inner product in (2.96) with u_t, we get

$$\int_0^L u_t^2dx - \nu\int_0^L u_tu_{xx}dx = \int_0^L fu_tdx, \quad t \in [0,T].$$

An integration by parts gives

$$\int_0^L u_t^2 dx + \nu \int_0^L u_{tx} u_x dx = \int_0^L f u_t dx, \quad t \in [0,T].$$

The boundary term vanishes because $u_t(t,0) = u_t(t,L) = 0$ for $t \in [0,T]$. Thus we can write

$$\frac{\nu}{2}\frac{d}{dt}\int_0^L u_x^2 dx + \int_0^L u_t^2 dx = \int_0^L f u_t dx \le \frac{1}{2}\int_0^L \left(f^2 + u_t^2\right) dx, \quad t \in [0,T].$$

This yields

$$\frac{1}{2}\int_0^L u_t^2 dx + \frac{\nu}{2}\frac{d}{dt}\int_0^L u_x^2 dx \le \frac{1}{2}\int_0^L f^2 dx, \quad t \in [0,T].$$

Ignoring the first term, we have

$$\nu \frac{d}{dt}\int_0^L u_x^2 dx \le \int_0^L f^2 dx, \quad t \in [0,T].$$

Integrating over $[0,t]$, we get that

$$\int_0^L u_x^2(t,x) dx - \int_0^L u_x^2(0,x) dx \le \int_0^t \int_0^L f^2 dx ds, \quad t \in [0,T].$$

Taking supremum over the interval $[0,T]$, we deduce that

$$\|u\|^2_{C^0([0,T];H_0^1(0,L))} \le T\left(\|u_0\|^2_{H_0^1(0,L)} + \|f\|_{L^2(0,T;L^2(0,L))}\right),$$

thanks to the Poincaré inequality. To prove the required $L^2(H^2)$-estimate, we take inner product in (2.96) with u_{xx} and integrate over $[0,T]$. We get

$$\int_0^T \int_0^L u_t u_{xx} dx dt - \nu \int_0^T \int_0^L u_{xx}^2 dx dt = \int_0^T \int_0^L f u_{xx} dx.$$

Integrating by parts, we can get

$$\nu \int_0^T \int_0^L u_{xx}^2 dx dt = -\int_0^T \int_0^L u_{tx} u_x dx dt - \int_0^T \int_0^L f u_{xx} dx.$$

An integration by parts again yields

$$\begin{aligned}
\nu \int_0^T \int_0^L u_{xx}^2 dx dt &= -\int_0^L [u_x^2(T,x) - u_x^2(0,x)] dx - \int_0^T \int_0^L f u_{xx} dx \\
&\le \int_0^L u_x^2(0,x) dx + \int_0^T \int_0^L \left(\frac{1}{4\epsilon} f^2 + \epsilon u_{xx}^2\right) dx dt,
\end{aligned}$$

where we have used the Young's inequality $ab \le \epsilon a^2 + \frac{b^2}{4\epsilon}$ for some $\epsilon > 0$. Thus, we obtain

$$(\nu - \epsilon)\int_0^T \int_0^L u_{xx}^2 dx dt \le \int_0^L u_x^2(0,x) dx + \frac{1}{4\epsilon}\int_0^T \int_0^L f^2 dx dt.$$

Choosing $\epsilon > 0$ small enough so that $\nu - \epsilon > 0$, we deduce that

$$\|u\|^2_{L^2(0,T;H^2(0,L))} \le C\left(\|u_0\|^2_{H_0^1(0,L)} + \|f\|^2_{L^2(0,T;L^2(0,L))}\right),$$

thanks to the previous $C^0([0,T];H_0^1(0,L))$-estimate of u. Let us now consider the case when $u_0 \in H_0^1(0,L)$ and $f \in L^2(0,T;L^2(0,L))$. Let $(u_0^n)_{n\in\mathbb{N}} \subset \mathcal{D}(A)$ and $(f^n)_{n\in\mathbb{N}} \subset C^1([0,T] \times [0,L])$

be sequences such that $u_0^n \to u_0$ in $H_0^1(0,L)$ and $f^n \to f$ in $L^2(0,T;L^2(0,L))$. Then, we have the following estimate:

$$\|u^n\|_{C^0([0,T];H_0^1(0,L))} + \|u^n\|_{L^2(0,T;H^2(0,L))} \le C\left(\|u_0^n\|_{H_0^1(0,L)} + \|f^n\|_{L^2(0,T;L^2(0,L))}\right), \quad \text{for all } n \in \mathbb{N}.$$

This implies $(u^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the space $C^0([0,T];H_0^1(0,L)) \cap L^2(0,T;H^2(0,L))$ and by uniqueness of the solutions, the sequence $(u^n)_{n\in\mathbb{N}}$ converges to the solution u with $u \in C^0([0,T];H_0^1(0,L)) \cap L^2(0,T;H^2(0,L))$ corresponding to the above $u_0 \in H_0^1(0,L)$ and $f \in L^2(0,T;L^2(0,L))$. Then the desired estimate (2.99) follows by passing limit as $n \to \infty$ in the above inequality.

This completes the proof. □

We now consider the adjoint system with a source term:

$$\begin{cases} -v_t - \nu v_{xx} = g, & \text{in } (0,T)\times(0,L), \\ v(t,0) = 0, \quad v(t,L) = 0, & \text{for } t \in (0,T), \\ v(T,x) = v_T(x), & \text{in } (0,L), \end{cases} \tag{2.103}$$

where $v_T \in L^2(0,L)$ and $g \in L^2(0,T;L^2(0,L))$. Then, we have the following result; proof of which is similar to the above Theorem and so we omit the details.

Lemma 2.4.3. *Let $v_T \in L^2(0,L)$ and $g \in L^2(0,T;L^2(0,L))$ be given. Then there exists a unique weak solution v of (2.103) in the space*

$$C^0([0,T];L^2(0,L)) \cap L^2(0,T;H_0^1(0,L)).$$

Moreover, this solution v satisfies the estimate

$$\|v\|_{C^0([0,T];L^2(0,L))} + \|v\|_{L^2(0,T;H_0^1(0,L))} \le C\left(\|v_T\|_{L^2(0,L)} + \|g\|_{L^2(0,T;L^2(0,L))}\right) \tag{2.104}$$

for some $C > 0$ depending only on ν, T.

Further, if $v_T \in H_0^1(0,L)$, this solution v satisfies the following estimate:

$$\|v\|_{C^0([0,T];H_0^1(0,L))} + \|v\|_{L^2(0,T;H^2(0,L))} \le C\left(\|v_T\|_{H_0^1(0,L)} + \|g\|_{L^2(0,T;L^2(0,L))}\right) \tag{2.105}$$

for some $C > 0$ depending only on ν, T.

We are now ready to define the notion of a weak solution for the main control system (2.90) when $u_0 \in H^{-1}(0,L)$ and $q \in L^2(0,T)$. In the literature, this type of solution (defined below) is often referred as the “solution in the sense of transposition”.

Definition 2.4.2. *Let the initial state $u_0 \in H^{-1}(0,L)$ and control $q \in L^2(0,T)$ be given. We say a function $u \in C^0([0,T];H^{-1}(0,L))$ is a weak solution (or a solution in the sense of transposition) of (2.90) if, for every $v_T \in \mathcal{D}(A^*)$ the following identity holds true:*

$$\langle u(t), v(t)\rangle_{H^{-1},H_0^1} - \langle u_0, v(0)\rangle_{H^{-1},H^1} + \nu \int_0^t q(s) v_x(s,L)\,ds = 0, \ \textit{for all } t \in [0,T], \tag{2.106}$$

where v is the strong solution of the adjoint system (2.95).

We note here that the above identity is well-defined because $v(0,\cdot) \in \mathcal{D}(A^*)$ and $v_x(\cdot,L) \in L^2(0,T)$, thanks to Lemma 2.4.2. Using this definition of a weak solution, we can now guarantee the existence of a unique weak solution to the system (2.90). The statement is written below:

Theorem 2.4.3. *For any given $u_0 \in H^{-1}(0,L)$ and $q \in L^2(0,T)$, the system (2.90) admits a unique weak solution*

$$u \in C^0([0,T]; H^{-1}(0,L)).$$

Moreover, the solution u satisfies

$$\|u\|_{C^0([0,T];H^{-1}(0,L))} \le C\left(\|u_0\|_{H^{-1}(0,L)} + \|q\|_{L^2(0,T)}\right), \tag{2.107}$$

for some $C > 0$ depending only on ν, T.

Proof. We first prove uniqueness of the solutions. Let us suppose that $u_1, u_2 \in C^0([0,T]; H^{-1}(0,L))$ be two weak solutions of the system (2.90) and denote $u := u_1 - u_2$. Then $u \in C^0([0,T]; H^{-1}(0,L))$ satisfies the system

$$\begin{cases} u_t - u_{xx} = 0, & \text{in } (0,T)\times(0,L),\\ u(t,0) = 0, \quad u(t,L) = 0, & \text{for } t \in (0,T),\\ u(0,x) = 0, & \text{in } (0,L). \end{cases}$$

From the definition of weak solution (see (2.106)), we readily have

$$\langle u(t), v(t)\rangle_{H^{-1},H_0^1} = 0, \quad \forall t \in [0,T],$$

for all $v_T \in \mathcal{D}(A^*)$. This implies $u(t) = 0$ in $L^2(0,L)$ for all $t \in [0,T]$.

We now prove the existence of a weak solution to the system (2.90). We will consider $u_0 \in \mathcal{D}(A)$ and $q \in C^2[0,T]$ with $q(0) = 0$ and prove the result. Then, using a similar density argument as we did in the proof of Theorem 2.4.2, the same will be true when $u_0 \in H^{-1}(0,L)$ and $q \in L^2(0,T)$. Since $u_0 \in \mathcal{D}(A)$ and $q \in C^2[0,T]$ satisfies $q(0) = 0$, applying Theorem 2.4.1, there is a unique strong solution $u \in C^1([0,T]; L^2(0,L)) \cap C^0([0,T]; H^2(0,L))$ of the system (2.90). We now prove the estimate (2.107). Let $\tau \in [0,T]$ be fixed and $\xi \in H_0^1(0,L)$ be given. We consider the following system

$$\begin{cases} -v_t - \nu v_{xx} = 0, & \text{in } (0,T)\times(0,L),\\ v(t,0) = 0, \quad v(t,L) = 0, & \text{for } t \in (0,T),\\ v(\tau,x) = \xi(x), & \text{in } (0,1). \end{cases} \tag{2.108}$$

Then $v \in C^0([0,\tau]; H_0^1(0,L)) \cap L^2(0,\tau; H^2(0,L))$, thanks to Lemma 2.4.3. Moreover, we have the following estimate:

$$\|v\|_{C^0([0,\tau];H_0^1(0,L))} + \|v\|_{L^2(0,\tau;H^2(0,L))} \le C\,\|\xi\|_{H_0^1(0,L)}. \tag{2.109}$$

Taking duality product in (2.90) with this v and integrating over the interval $(0,\tau)$, we get

$$\int_0^\tau \langle u_t(s), v(s)\rangle_{H^{-1},H_0^1}\,ds - \nu\int_0^\tau \langle u_{xx}(s), v(s)\rangle_{H^{-1},H_0^1}\,ds = 0.$$

Integrating by parts, we readily have

$$\langle u(\tau), \xi\rangle_{H^{-1},H_0^1} - \langle u_0, v(0)\rangle_{H^{-1},H_0^1} + \nu\int_0^\tau q(s) v_x(s,L)ds = 0. \tag{2.110}$$

Since $v \in L^2(0,\tau; H^2(0,L))$, the map $v \in L^2(0,\tau; H^2(0,L)) \mapsto v_x(\cdot,L) \in L^2(0,T)$ is bounded. Therefore

$$\begin{aligned} \left|\langle u(\tau), \xi\rangle_{H^{-1},H_0^1}\right|_{H^{-1},H_0^1} &\le \left|\langle u_0, v(0)\rangle_{H^{-1},H_0^1}\right| + \nu\int_0^\tau |q(s)|\,|v_x(s,L)|\,ds\\ &\le \|u_0\|_{H^{-1}(0,L)}\,\|v(0)\|_{H_0^1(0,L)} + \nu\,\|q\|_{L^2(0,T)}\,\|v_x(\cdot,L)\|_{L^2(0,T)}\\ &\le \|u_0\|_{H^{-1}(0,L)}\,\|v\|_{C^0([0,T];H_0^1(0,L))} + \nu\,\|q\|_{L^2(0,T)}\,\|v\|_{(L^2(0,T;H^2(0,L))}\\ &\le C\,\|\xi\|_{H_0^1(0,L)}\left(\|u_0\|_{H^{-1}(0,L)} + \|q\|_{L^2(0,T)}\right), \end{aligned}$$

thanks to the estimate (2.109). Thus we obtain

$$\|u(\tau)\|_{H^{-1}(0,L)} = \sup_{\|\xi\|_{H_0^1(0,L)}=1} \left|\langle u(\tau), \xi\rangle_{H^{-1},H_0^1}\right| \le C\left(\|u_0\|_{H^{-1}(0,L)} + \|q\|_{L^2(0,T)}\right).$$

Since $\tau \in [0,T]$ is arbitrary, we deduce that

$$\|u\|_{L^\infty(0,T;H^{-1}(0,L))} \le C\left(\|u_0\|_{H^{-1}(0,L)} + \|q\|_{L^2(0,T)}\right).$$

Applying the usual density argument, we can obtain the $C^0([0,T];H^{-1}(0,L))$-estimate on u. This completes the proof. □

The above result guarantees the existence of a unique solution $u \in C^0([0,T];H^{-1}(0,L))$ to the system (2.90) when $u_0 \in H^{-1}(0,L)$ and $q \in L^2(0,T)$. In addition to this, we can also obtain a regularity result for the system (2.90), which says that this solution u also belongs to the space $L^2(0,T;L^2(0,L))$. To prove this result, we require another definition of a weak solution to the system (2.90), which is written below.

Definition 2.4.3. *Let $u_0 \in H^{-1}(0,L)$ and $q \in L^2(0,T)$ be given. We say a function $u \in L^2(0,T;L^2(0,L))$ is a weak solution (or a solution in the sense of transposition) of the system (2.90) if, for every $g \in L^2(0,T;L^2(0,L))$ the following identity holds true:*

$$-\langle u_0, v(0)\rangle_{H^{-1},H^1} + \nu\int_0^T q(t)v_x(t,L)dt + \int_0^T\int_0^L ugdxdt = 0, \tag{2.111}$$

where v is the weak solution of the adjoint system (2.103) with $v_T = 0$.

It can be proved that this notion of defining a solution is equivalent to that considered in Definition 2.4.2, see for instance [Cor07, Section 2.5, Page 76]. With this definition, we now prove the regularity result (L^2-estimate) of the solution u to the system (2.90).

Proposition 2.4.1. *For given $u_0 \in H^{-1}(0,L)$ and $q \in L^2(0,T)$, the solution $u \in C^0([0,T];H^{-1}(0,L))$ of the system (2.90) belong to the space $L^2(0,T;L^2(0,L))$ and we have the following estimate:*

$$\|u\|_{L^2(0,T;L^2(0,L))} \le C\left(\|u_0\|_{H^{-1}(0,L)} + \|q\|_{L^2(0,T)}\right), \tag{2.112}$$

where $C > 0$ is a constant depending only on ν, T.

Proof. We will prove the required estimate by assuming $u_0 \in \mathcal{D}(A)$ and $q \in C^2[0,T]$ with $q(0) = 0$. Then, applying a density argument, we can get the estimate when $u_0 \in H^{-1}(0,L)$ and $q \in L^2(0,T)$. For $u_0 \in \mathcal{D}(A)$ and $q \in C^2[0,T]$ with $q(0) = 0$, the system (2.90) has a unique strong solution $u \in C^1([0,T];L^2(0,L)) \cap C^0([0,T];H^2(0,L))$, thanks to Theorem 2.4.1. Let v be the solution of the adjoint system (2.103) with terminal data $v_T = 0$ and source term $g \in L^2(0,T;L^2(0,L))$. Then, applying Lemma 2.4.3, this solution v satisfies $v \in C^0([0,T];H_0^1(0,L)) \cap L^2(0,T;H^2(0,L))$ with the following estimate

$$\|v\|_{C^0([0,T];H_0^1(0,L))} + \|v\|_{L^2(0,T;H^2(0,L))} \le C\|g\|_{L^2(0,T;L^2(0,L))}$$

for some constant $C > 0$. Taking duality product in (2.90) with v and integrating over $(0,T)$, we have

$$\int_0^T \langle u_t(t), v(t)\rangle_{H^{-1},H_0^1}\, ds - \nu\int_0^T \langle u_{xx}(t), v(t)\rangle_{H^{-1},H_0^1}\, ds = 0.$$

Integrating by parts and using the boundary-initial conditions, we deduce that

$$-\langle u(0), v(0)\rangle_{H^{-1},H_0^1} + \nu\int_0^T q(t)v_x(t,L)dt + \int_0^T\int_0^L ugdxdt = 0,$$

since v satisfies (2.103). Thus, we have

$$\begin{aligned}\left|\int_0^T\int_0^L ugdxdt\right| &\leq \left|\langle u(0), v(0)\rangle_{H^{-1},H_0^1}\right| + \nu\int_0^T |q(t)|\,|v_x(t,L)|\,dt \\ &\leq \|u(0)\|_{H^{-1}(0,L)}\,\|v(0)\|_{H_0^1} + \nu\,\|q\|_{L^2(0,T)}\,\|v_x(\cdot,L)\|_{L^2(0,T)} \\ &\leq C\,\|g\|_{L^2(0,T;L^2(0,L))}\left(\|u_0\|_{H^{-1}(0,L)} + \|q\|_{L^2(0,T)}\right).\end{aligned}$$

Therefore

$$\|u\|_{L^2(0,T;L^2(0,L))} = \sup_{\|g\|_{L^2(L^2)}=1}\left|\int_0^T\int_0^L ugdxdt\right| \leq C\left(\|u_0\|_{H^{-1}(0,L)} + \|q\|_{L^2(0,T)}\right),$$

proving the required estimate when $u_0 \in \mathcal{D}(A)$ and $q \in C^2[0,T]$ with $q(0) = 0$.

We now consider the case when $u_0 \in H^{-1}(0,L)$ and $q \in L^2(0,T)$. Then there exist sequences $(u_0^n)_{n\in\mathbb{N}} \subset \mathcal{D}(A)$ and $(q^n)_{n\in\mathbb{N}} \subset C^2[0,T]$ with $q^n(0) = 0$ for all $n \in \mathbb{N}$ such that $u_0^n \to u_0$ in $L^2(0,L)$ and $q^n \to q$ in $L^2(0,T)$. For each $n \in \mathbb{N}$, let u^n denote the strong solution of (2.90) with the above u_0^n and q^n. Then, applying Theorem 2.4.1, this solution u^n belongs to $C^1([0,T];L^2(0,L)) \cap C^0([0,T];H^2(0,L))$ for all $n \in \mathbb{N}$. Since $(u_0^n)_{n\in\mathbb{N}} \subset \mathcal{D}(A)$ and $(q^n)_{n\in\mathbb{N}} \subset C^2[0,T]$ with $q^n(0) = 0$ for all $n \in \mathbb{N}$, we have the following estimate

$$\|u^n\|_{L^2(0,T;L^2(0,L))} \leq C\left(\|u_0^n\|_{H^{-1}(0,L)} + \|q^n\|_{L^2(0,T)}\right), \quad \text{for all } n \in \mathbb{N}. \tag{2.113}$$

Let $g \in L^2(0,T;L^2(0,L))$ be given and let $v \in C^0([0,T];H_0^1(0,L)) \cap L^2(0,T;H^2(0,L))$ be the unique strong solution of (2.103) (see Lemma 2.4.3). Taking $L^2(0,L)$-inner product in (2.90) with v and integrating over $[0,T]$, we get

$$\int_0^T\int_0^L u_t^n v dxdt - \nu\int_0^T\int_0^L u_{xx}^n v dxdt = 0, \quad \text{for all } n \in \mathbb{N}.$$

Integrating by parts twice, we obtain

$$-\int_0^L u_0^n(x)v(0,x)dx + \nu\int_0^T q^n(t)v_x(t,L)dt + \int_0^T\int_0^L u^n g dxdt = 0, \quad \text{for all } n \in \mathbb{N}. \tag{2.114}$$

Let $m,n \in \mathbb{N}$. Note that $u^n - u^m$ is the strong solution of (2.96) corresponding to the initial state $u_0^n - u_0^m$ and control $q^n - q^m$. Thus, we see from (2.113) that this solution satisfies the estimate

$$\|u^n - u^m\|_{L^2(0,T;L^2(0,L))} \leq C\left(\|u_0^n - u_0^m\|_{L^2(0,L)} + \|q^n - q^m\|_{L^2(0,T)}\right)$$

for all $m,n \in \mathbb{N}$. This implies the sequence $(u^n)_{n\in\mathbb{N}}$ is Cauchy in the space $L^2(0,T;L^2(0,L))$. Let $u^n \to u$ in $L^2(0,T;L^2(0,L))$ for some function $u \in L^2(0,T;L^2(0,L))$. Then, passing limit as $n \to \infty$ in the identity (2.114), we see from Definition 2.4.3 that u is a weak solution of the system (2.96) with the above initial state $u_0 \in L^2(0,L)$ and control $q \in L^2(0,T)$. To prove the desired estimate (2.112), we pass limit as $n \to \infty$ in the inequality (2.113). This completes the proof.

□

Since we have the well-posedness result for the heat equation (2.90) in $H^{-1}(0,L)$, we can study the controllability properties of (2.90) in the space $H^{-1}(0,L)$. We first prove that the heat equation (2.90) cannot be exactly controllable at any time $T > 0$ in the space $H^{-1}(0,L)$ by using a boundary control $q \in L^2(0,T)$. For this, we consider the following homogeneous system

$$\begin{cases} u_t - \nu u_{xx} = 0, & \text{in } (0,T)\times(0,L), \\ u(t,0) = 0, \quad u(t,L) = 0, & \text{for } t \in (0,T), \\ u(0,x) = u_0(x), & \text{in } (0,L), \end{cases} \tag{2.115}$$

with $u_0 \in H^{-1}(0,L)$ and recall the unbounded operator $A : \mathcal{D}(A) \subset L^2(0,L) \to L^2(0,L)$ where

$$\begin{cases} \mathcal{D}(A) = \left\{ f \in H^2(0,L) \ : \ f(0) = f(L) = 0 \right\}, \\ Af := -\nu f_{xx}, \quad f \in \mathcal{D}(A). \end{cases} \tag{2.116}$$

The eigenvalues of this operator A are $\lambda_n := \frac{n^2\pi^2}{L^2}$ and the corresponding eigenfunctions are $\varphi_n(x) := \sin\left(\frac{n\pi x}{L}\right)$ for all $n \in \mathbb{N}$. Since these eigenfunctions $\{\varphi_n \ : \ n \in \mathbb{N}\}$ forms an orthogonal basis of $L^2(0,L)$ and hence a dense family in $H^{-1}(0,L)$, we can write $u_0 \in H^{-1}(0,L)$ as

$$u_0(x) = \sum_{n\in\mathbb{N}} a_n \sin\left(\frac{n\pi x}{L}\right), \quad x \in (0,L),$$

for some $(a_n)_{n\in\mathbb{N}}$ such that $\left(\frac{a_n}{n}\right)_{n\in\mathbb{N}} \in \ell_2$. The solution of (2.115) is then given by the separation of variable formula

$$u(t,x) = \sum_{n\in\mathbb{N}} a_n e^{-\frac{n^2\pi^2}{L^2}t} \sin\left(\frac{n\pi x}{L}\right), \text{ for } (t,x) \in (0,T) \times (0,L).$$

Note that $u(T) = 0$ in $H^{-1}(0,L)$ implies the coefficients $a_n = 0$ for all $n \in \mathbb{N}$, which immediately gives $u \equiv 0$. This shows that the system (2.115) satisfies the backward uniqueness property (see Definition 2.2.5). On the other hand, from this expression of the solution, we deduce that

$$u_t(T,x) = -\sum_{n\in\mathbb{N}} a_n \frac{n^2\pi^2}{L^2} e^{-\frac{n^2\pi^2}{L^2}T} \sin\left(\frac{n\pi x}{L}\right), \text{ for } x \in (0,L)$$

and therefore $u_{xx}(T) = u_t(T) \in L^2(0,L)$, which implies $u(T) \in H^2(0,L)$. On the other hand, we have

$$u_{txx}(T,x) = -\sum_{n\in\mathbb{N}} a_n \frac{n^4\pi^4}{L^4} e^{-\frac{n^2\pi^2}{L^2}T} \sin\left(\frac{n\pi x}{L}\right), \text{ for } x \in (0,L).$$

This gives $u_{xxxx}(T) = u_{txx}(T) \in L^2(0,T)$ and therefore $u(T) \in H^4(0,L)$. A repeated argument shows that $u(T) \in H^{2k}(0,L)$ for all $k \in \mathbb{N}$. As a consequence, we have $u(T) \in C^\infty(0,L]$. Using this argument in the main control system (2.90), we see that the solution is smooth far away from the right end $x = L$, that is $u(T) \in C^\infty(0,L)$. Therefore, the heat equation (2.90) cannot be exactly controllable at time T in the space $H^{-1}(0,L)$ (and in particular, in $L^2(0,L)$) by using a boundary control $q \in L^2(0,T)$. Thus, we only concentrate on the null controllability of this system at time T in $H^{-1}(0,L)$, since approximate controllability will follow due to the backward uniqueness property of the equation (2.115) (thanks to Proposition 2.2.1).

Recall that, in Sections 2.2.2–2.3, we have derived some equivalent conditions for null controllability in terms of the operator F_T and its adjoint. In this case, we define the map $F_T : L^2(0,T) \to H^{-1}(0,L)$ by

$$F_T(q) := u(T,\cdot),$$

where u is the unique weak solution of (2.90) with $u_0 = 0$ and $q \in L^2(0,T)$. Using Theorem 2.4.3, we know that $u \in C^0([0,T]; H^{-1}(0,L))$ and therefore F_T is a well-defined linear map. From the definition of weak solution (eq. (2.106)), the adjoint of this map $F_T^* : H_0^1(0,L) \to L^2(0,T)$ is given by

$$F_T^*(v_T) := v_x(\cdot, L), \quad \text{for } v_T \in H_0^1(0,L).$$

Thanks to Lemma 2.4.3, this map is well-defined and with the help of this map, we can now state the following result; the proof of which follows from Theorem 2.2.8.

Theorem 2.4.4. *The system (2.90) is null controllable at time T in $H^{-1}(0,L)$ if and only if there exists a $C > 0$ such that the following inequality*

$$\int_0^T |v_x(t,L)|^2 \, dt \geq C \, \|v(0)\|^2_{H_0^1(0,L)} \tag{2.117}$$

holds for all $v_T \in H_0^1(0,L)$.

Thus, to prove null controllability of the system (2.90), it is enough to prove the observability inequality (2.117). We state the result below.

Theorem 2.4.5. *The system* (2.90) *is null controllable at any time $T > 0$ in the space $H^{-1}(0, L)$.*

Proof. Let $T > 0$ be given and let $v_T \in H_0^1(0, L)$. Since the eigenfunctions $\left\{\sin\left(\frac{n\pi x}{L}\right) : n \in \mathbb{N}\right\}$ of A forms a complete (dense) family in $H_0^1(0, L)$, we can write

$$v_T(x) = \sum_{n\in\mathbb{N}} a_n \sin\left(\frac{n\pi x}{L}\right), \quad x \in (0, L),$$

for some $(a_n)_{n\in\mathbb{N}}$ such that $(na_n)_{n\in\mathbb{N}} \in \ell_2$. Then the solution to the adjoint system (2.95) is given by

$$v(t, x) = \sum_{n\in\mathbb{N}} a_n e^{-\frac{n^2\pi^2}{L^2}(T-t)} \sin\left(\frac{n\pi x}{L}\right), \quad (t, x) \in (0, T) \times (0, L).$$

Thus, we have

$$\int_0^T |v_x(t, L)|^2\, dt = \int_0^T \left|\sum_{n\in\mathbb{N}} a_n(-1)^n \frac{n\pi}{L} e^{-\frac{n^2\pi^2}{L^2}(T-t)}\right|^2 dt = \int_0^T \left|\sum_{n\in\mathbb{N}} a_n(-1)^n \frac{n\pi}{L} e^{-\frac{n^2\pi^2}{L^2}t}\right|^2 dt.$$

Note that, thanks to Theorem 2.1.13 (or Theorem 2.1.14), the family $\left(e^{-\frac{n^2\pi^2}{L^2}t}\right)_{n\in\mathbb{N}}$ has a biorthogonal sequence $(q_k)_{k\in\mathbb{N}}$ in $L^2(0, T)$ with the estimate $\|q_k\|_{L^2(0,T)} \le Ce^{\epsilon \mathrm{Re}(\lambda_k)}$ for all $k \in \mathbb{N}$ and $\epsilon > 0$. Thus, we can apply the parabolic Ingham's inequality (2.20) in Theorem 2.1.19 to deduce that

$$\int_0^T |v_x(t, L)|^2\, dt \ge \sum_{n\in\mathbb{N}} |a_n|^2 \frac{n^2\pi^2}{L^2} e^{-2\frac{n^2\pi^2}{L^2}T}. \tag{2.118}$$

On the other hand, we have

$$\|v(0)\|^2_{H_0^1(0,L)} = \int_0^L |v_x(0, x)|^2\, dx = \int_0^L \left|\sum_{n\in\mathbb{N}} a_n \frac{n\pi}{L} e^{-\frac{n^2\pi^2}{L^2}T} \cos\left(\frac{n\pi x}{L}\right)\right|^2 dx$$

Since the set $\left\{\cos\left(\frac{n\pi x}{L}\right) : n \in \mathbb{N} \cup \{0\}\right\}$ is an orthogonal basis in $L^2(0, L)$, we deduce that

$$\|v(0)\|^2_{H_0^1(0,L)} \le C \sum_{n\in\mathbb{N}} |a_n|^2 \frac{n^2\pi^2}{L^2} e^{-2\frac{n^2\pi^2}{L^2}T}. \tag{2.119}$$

Combining the estimates (2.118) and (2.119), we obtain the required observability inequality (2.117). This completes the proof. □

This method for proving null controllability of (2.90) is a crucial part in this thesis, in particular in Chapters 3 and 4. Apart from this method, we now give a different approach, the so called moments method (introduced in Section 2.1.3), to prove null controllability of the heat equation (2.90). This technique will be very useful in the later chapters of this thesis (see Chapters 4–5). Before proceeding, we first write the following result which gives an equivalent condition for null controllability of the system (2.90).

Lemma 2.4.4 (Equivalent criterion for null controllability)**.** *The heat equation* (2.90) *is null controllable at time $T > 0$ in $H^{-1}(0, L)$ if, and only if, for every $u_0 \in H^{-1}(0, L)$ there exists $q \in L^2(0, T)$ such that the following identity*

$$\int_0^T q(t)v_x(t, L)dt = \langle u_0, v(0)\rangle_{H^{-1},H_0^1} \tag{2.120}$$

holds for every $v_T \in H_0^1(0, L)$.

Proof. Let us first consider the case when $u_0 \in \mathcal{D}(A)$ and $q \in C^2[0,T]$ with $q(0) = 0$. Then the strong solution u to (2.90) satisfy $u \in C^1([0,T]; L^2(0,L)) \cap C^0([0,T]; \mathcal{D}(A))$. Let $v_T \in H_0^1(0,L)$. Then the solution v of the adjoint system (2.95) belong to the space $C^0([0,T]; H_0^1(0,L)) \cap L^2(0,T; H^2(0,L))$, thanks to Lemma 2.4.3. Taking inner product in (2.90) with this v, we get

$$\int_0^T \langle u_t, v\rangle_{H^{-1},H_0^1}\, dt - \nu \int_0^T \langle u_{xx}, v\rangle_{H^{-1},H_0^1}\, dt = 0.$$

Integrating by parts and using the boundary-initial conditions, we obtain

$$\langle u(T), v_T\rangle_{H^{-1},H_0^1} - \langle u_0, v(0)\rangle_{H^{-1},H_0^1} + \nu \int_0^T q(t) v_x(t,L) dt = 0,$$

for all $v_T \in H_0^1(0,L)$. Using a density argument, this identity is also true when $u_0 \in H^{-1}(0,L)$ and $q \in L^2(0,T)$. We now assume that the system (2.90) is null controllable at time T in $H^{-1}(0,L)$. Then, for every $u_0 \in H^{-1}(0,L)$ there exists a $q \in L^2(0,T)$ such that the associated solution satisfies $u(T) = 0$. Consequently, we have the required identity (2.120). On the other hand, if for every $u_0 \in H^{-1}(0,L)$ there exists a $q \in L^2(0,T)$ such that the identity (2.120) holds, then we deduce from the above relation that

$$\langle u(T), v_T\rangle_{H^{-1},H_0^1} = 0, \text{ for all } v_T \in H_0^1(0,L).$$

As a consequence, we obtain $u(T) = 0$ and therefore the system (2.90) is null controllable at time T in $H^{-1}(0,L)$. This completes the proof. □

We can further reduce the above equivalent identity into a set of moments problem by using the eigen-elements of the operator A. This moment problem will be similar to the one considered in Section 2.1.3 (see Example 2.1.6).

Lemma 2.4.5 (Reduction to the moments problem)**.** *The system (2.90) is null controllable at time $T > 0$ in $H^{-1}(0,L)$ if, and only if, there exists $q \in L^2(0,T)$ such that the following identities holds:*

$$\int_0^T q(T-t) e^{-\frac{n^2\pi^2}{L^2}t} dt = \omega_n, \quad \textit{for all } n \in \mathbb{N}, \tag{2.121}$$

where

$$\omega_n = \frac{(-1)^n L e^{-\frac{n^2\pi^2}{L^2}T}}{n\pi} \left\langle u_0, \sin\left(\frac{n\pi\,\cdot}{L}\right)\right\rangle_{H^{-1},H_0^1}, \quad n \in \mathbb{N}. \tag{2.122}$$

Proof. Recall that, the set of eigenfunctions $\{\varphi_n : n \in \mathbb{N}\}$ of A, where $\varphi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$ for $n \in \mathbb{N}$, forms a dense family in $H_0^1(0,L)$. With this φ_n, the solution of the adjoint system (2.95) is

$$v^n(t,x) = e^{-\frac{n^2\pi^2}{L^2}(T-t)} \sin\left(\frac{n\pi x}{L}\right), \quad t \in (0,T), \ x \in (0,L),$$

for all $n \in \mathbb{N}$. Plugging this value of v^n in (2.120), we readily have

$$\int_0^T q(t) e^{-\frac{n^2\pi^2}{L^2}(T-t)} (-1)^n \frac{n\pi}{L} dt = \left\langle u_0, e^{-\frac{n^2\pi^2}{L^2}T} \sin\left(\frac{n\pi\,\cdot}{L}\right)\right\rangle_{H^{-1},H_0^1}$$

for all $n \geq 1$. Changing the variable $t \mapsto T - t$ in the above integral, the proof follows. □

From the above Lemma, it is enough to solve the moments problem (2.121) for proving null controllability of the system (2.90), and with the help of a suitable biorthogonal family of the exponentials, we now solve this moments problem. The statement is given below:

Theorem 2.4.6. *Let $T > 0$ be given. Then the system (2.90) is null controllable at time T in the space $H^{-1}(0,L)$.*

Proof. We first apply Theorem 2.1.12 to obtain a biorthogonal sequence $(q_k)_{k\in\mathbb{N}}$ to the exponential family $(e^{-\frac{n^2\pi^2}{L^2}t})_{n\in\mathbb{N}}$ in $L^2(0,T)$ with the estimate

$$\|q_k\|_{L^2(0,T)} \le Me^{\epsilon\frac{k^2\pi^2}{L^2}}, \quad \text{for all } k \ge 1, \tag{2.123}$$

for some constant $M > 0$ and all $\epsilon > 0$. We define $q(T-t) := \sum_{k\in\mathbb{N}} \omega_k q_k(t)$, for $t \in (0,T)$, where ω_k is defined by (2.122). Then, it is easy to see that q satisfies (2.121). It remains to prove that $q \in L^2(0,T)$. In fact,

$$\begin{aligned}
\|q\|_{L^2(0,T)} &\le \sum_{k\in\mathbb{N}} |\omega_k| \, \|q_k\|_{L^2(0,T)}, \\
&\le M \sum_{k\in\mathbb{N}} \frac{Le^{-\frac{k^2\pi^2}{L^2}T}}{k\pi} \|u_0\|_{H^{-1}(0,L)} \left\| \sin\left(\frac{k\pi\,\cdot}{L}\right)\right\|_{H_0^1(0,L)} e^{\epsilon\frac{k^2\pi^2}{L^2}} \\
&\le M \|u_0\|_{H^{-1}(0,L)} \sum_{k\in\mathbb{N}} e^{-\frac{k^2\pi^2}{L^2}(T-\epsilon)}.
\end{aligned}$$

Choosing $\epsilon > 0$ small enough such that $T - \epsilon > 0$, we deduce that

$$\|q\|_{L^2(0,T)} \le M \|u_0\|_{H^{-1}(0,1)},$$

for some $M > 0$. Then, applying Lemma 2.4.5 the proof follows. □

Remark 2.4.1 (Control cost). *We can estimate the constant M appearing in the above inequality. The role of this constant (called the "cost of the control") appears when we deal with the nonlinear systems and prove local controllability using the controllability properties of the associated linearized system (see the next section and Chapter 5 for more details). To estimate this constant, we use the general biorthogonal result (Theorem 2.1.15) to obtain the bound of the biorthogonal sequence $(q_k)_{k\in\mathbb{N}}$ as*

$$\|q_k\|_{L^2(0,T)} \le Me^{M\sqrt{\lambda_k}+\frac{M}{T}}, \text{ for all } k \in \mathbb{N},$$

where recall that $\lambda_k = \frac{k^2\pi^2}{L^2}$ for $k \in \mathbb{N}$. Thus, we have

$$\begin{aligned}
\|q\|_{L^2(0,T)} &\le \sum_{k\in\mathbb{N}} \frac{e^{-\lambda_k T}}{\sqrt{\lambda_k}} \|u_0\|_{H^{-1}(0,L)} \left\|\sin(\sqrt{\lambda_k}\ \cdot)\right\|_{H_0^1(0,L)} \|q_k\|_{L^2(0,T)} \\
&\le M \|u_0\|_{H^{-1}(0,L)} \sum_{k\in\mathbb{N}} e^{-\lambda_k T + M\sqrt{\lambda_k}+\frac{M}{T}} \\
&\le M \|u_0\|_{H^{-1}(0,L)} \sum_{k\in\mathbb{N}} e^{\frac{M}{T}+\frac{M^2}{2T}} e^{-\frac{1}{2}\lambda_k T} \\
&\le Me^{\frac{M}{T}+\frac{M^2}{2T}} \|u_0\|_{H^{-1}(0,L)} \sum_{k\in\mathbb{N}} e^{-\frac{1}{2}\lambda_k T},
\end{aligned}$$

where we have used the Young's inequality $M\sqrt{\lambda_k} = \frac{M}{\sqrt{T}}\sqrt{\lambda_k T} \le \frac{M^2}{2T} + \frac{\lambda_k T}{2}$. On the other hand, we have that

$$\sum_{k\in\mathbb{N}} e^{-\frac{1}{2}\lambda_k T} \le \sum_{k\in\mathbb{N}} \frac{2}{\lambda_k T} \le \frac{C}{T} \tag{2.124}$$

for some $C > 0$ independent of T. Thus, we have

$$\|q\|_{L^2(0,T)} \le Ce^{\frac{C}{T}} \|u_0\|_{H^{-1}(0,L)} \tag{2.125}$$

for some $C > 0$ independent of T.

We conclude this section with the comment that, like the finite dimensional linear systems, one can prove the equivalence between controllability and observability by introducing a quadratic functional in appropriate function spaces, see for instance [MZ04, Zua07]; see also the proof of Theorem 2.2.2. More precisely, to prove null controllability of the heat equation (2.90) at any time $T > 0$ in the space $L^2(0, L)$, it is enough to prove the following observability inequality:

$$\int_0^T |v_x(t, L)|^2 \, dt \geq C \, \|v(0)\|^2_{L^2(0,L)} \tag{2.126}$$

for all $v_T \in \mathcal{D}(A^*)$, thanks to Theorem 2.2.8 (see Theorem 2.4.4 for comparison). For this, we define the following subspace of $L^2(0, L)$

$$\mathcal{H} := \left\{ \varphi_T \in L^2(0, L) \ : \ \text{the solution } \varphi \text{ of (2.95) satisfies } \int_0^T |\varphi_x(t, L)|^2 \, dt < \infty \right\}$$

and define a quadratic functional $J : \mathcal{H} \to \mathbb{R}$ by

$$J(\varphi_T) := \frac{1}{2} \int_0^T |\varphi_x(t, L)|^2 \, dt + \int_0^L u_0(x)\varphi(0, x)dx, \quad \varphi_T \in \mathcal{H}. \tag{2.127}$$

Note that, thanks to Lemma 2.4.2, we find that $\mathcal{D}(A^*) \subset \mathcal{H}$. Also, J is not coercive with respect to the usual L^2-norm. Thus, we define a new norm on H as follows:

$$\|\varphi_T\|_{\mathcal{H}} := \left(\int_0^T |\varphi_x(t, L)|^2 \, dt \right)^{\frac{1}{2}}. \tag{2.128}$$

To prove this a norm, we only need to verify the following property:

$$\|\varphi_T\|_{\mathcal{H}} = 0 \ \text{ implies } \varphi_T = 0.$$

Indeed, $\|\varphi_T\|_{\mathcal{H}} = 0$ implies $\varphi_x(t, L) = 0$ for a.e. $t \in (0, T)$. The observability inequality (2.126) is then gives $\varphi(0, x) = 0$ for a.e. $x \in (0, L)$. Since the heat equation has backward uniqueness property, we readily have $\varphi_T(x) = 0$ for a.e. $x \in (0, L)$.

With this new norm, the functional J is continuous and coercive on $\mathcal{H}$. Therefore, J has a minimizer (say $\hat{\varphi}_T$) in $\mathcal{H}$. Let $\hat{\varphi}$ denotes the solution of (2.95) with respect to this terminal data $\hat{\varphi}_T$. Then, we have

$$\int_0^T \varphi_x(t, L)\hat{\varphi}_x(t, L)dt + \int_0^L u_0(x)\varphi(0, x)dx = 0 \tag{2.129}$$

for all $\varphi_T \in \mathcal{H}$. On the other hand, we have for $q(t) = -\hat{\varphi}_x(t, L)$,

$$\int_0^T u(T, x)\varphi_T(x)dx - \int_0^L u_0(x)\varphi(0, x)dx - \int_0^T \varphi_x(t, L)\hat{\varphi}_x(t, L)dt = 0 \tag{2.130}$$

for all $\varphi_T \in \mathcal{H}$, thanks to eq. (2.106). Comparing the above two equations, we obtain

$$\int_0^T u(T, x)\varphi_T(x)dx = 0$$

for all $\varphi_T \in \mathcal{H}$. Since the space $\mathcal{H}$ is dense in $L^2(0, L)$, we deduce that $u(T, x) = 0$ in $(0, L)$. This proves that the system (2.90) is null controllable at time T in the space $L^2(0, L)$.

Remark 2.4.2. *We note here that the observation term $v_x(\cdot, L)$ does not necessarily belong to $L^2(0, T)$ if we take $v_T \in L^2(0, L)$. Also, note that the space $H^1_0(0, L) \subset \mathcal{H}$, thanks to the regularity result (Lemma 2.4.3). Further, one can prove that $H^s(0, L) \subset \mathcal{H}$ for any $s > \frac{1}{2}$.*

2.5 A nonlinear heat equation

In this section, our main focus is to give a brief introduction to the "source term method", introduced by Liu, Takahashi and Tucsnak in [LTT13], to prove small-time local null controllability of a nonlinear heat equation. This technique has been explained in detail in Chapter 5 for a nonlinear coupled parabolic system, so we leave the technical details here.

For given finite time $T > 0$, we consider the following system

$$\begin{cases} y_t - y_{xx} = f(y), & \text{in } (0,T) \times (0,L), \\ y_x(t,0) = q(t), \quad y_x(t,L) = 0, & \text{for } t \in (0,T), \\ y(0,x) = y_0(x), & \text{in } (0,L). \end{cases} \tag{2.131}$$

Here $y_0 \in L^2(0,L)$ is the initial state, $q \in L^2(0,T)$ is the (Neumann) boundary control and f is a nonlinear function with $f(0) = 0$. Before writing any results, let us first define the notion of controllability for this system.

Definition 2.5.1. *We say the system (2.131) is* ***small-time locally null controllable*** *around the equilibrium 0 in the space $L^2(0,L)$ if, for any given $T > 0$, there exists a $\delta > 0$ such that for given $y_0 \in L^2(0,L)$ with $\|y_0\|_{L^2(0,L)} \leq \delta$, there exists a control $q \in L^2(0,T)$ such that the associated solution y of (2.131) satisfies*

$$y(T) = 0.$$

If the above holds for any $y_0 \in L^2(0,L)$, we say the system is ***globally null controllable*** *in $L^2(0,L)$.*

There are a significant amount of local and global controllability results known for the nonlinear heat equation using a distributed or boundary control; see for instance the works [Bar00, DFCGBZ02, È95, HSLBP23, FPZ95, FC97, FI96, LB20a] and the references therein. In this section, we consider the simplest case when $f(y) = y^2$ in (2.131) and provide a brief idea of proving small-time local null controllability of (2.131), as mentioned in [LTT13]. We refer to Chapter 5 for more details.

Step 1. We first linearize the system (2.131) around the equilibrium point 0

$$\begin{cases} y_t - y_{xx} = 0, & \text{in } (0,T) \times (0,L), \\ y_x(t,0) = q(t), \quad y_x(t,L) = 0, & \text{for } t \in (0,T), \\ y(0,x) = y_0(x), & \text{in } (0,L), \end{cases} \tag{2.132}$$

and prove null controllability of this system at any time $T > 0$ in the space $L^2(0,L)$ by using a Neumann control q with the cost estimate

$$\|q\|_{L^2(0,T)} \leq Ce^{\frac{C}{T}} \|y_0\|_{L^2(0,L)},$$

for some constant $C > 0$ independent of T. To prove this result one can use, for instance, the method of moments, which we have described in Section 2.4 for the heat equation with Dirichlet boundary conditions; the same can be done for Neumann case also. We note here that, due to the Neumann boundary conditions, the solution y of the linearized system (2.132) belongs to $C^0([0,T]; L^2(0,L)) \cap L^2(0,T; H^1(0,L))$; the proof of which will be similar to Proposition 5.2.3. This is the main reason for considering the Neumann conditions instead of Dirichlet.

Step 2. We then consider the linearized system with a source term $\xi \in L^2(0,T; L^2(0,L))$

$$\begin{cases} y_t - y_{xx} = \xi, & \text{in } (0,T) \times (0,L), \\ y_x(t,0) = q(t), \quad y_x(t,L) = 0, & \text{for } t \in (0,T), \\ y(0,x) = y_0(x), & \text{in } (0,L) \end{cases} \tag{2.133}$$

and prove null controllability of this system at any time T in some weighted L^2 space. More precisely, we prove the following inequality

$$\left\|\frac{y}{w_0}\right\|_{C^0([0,T];L^2(0,L))\cap L^2(0,T;H^1(0,L))} + \left\|\frac{q}{w_0}\right\|_{L^2(0,T)} \leq Ce^{CT+\frac{C}{T}}\left(\|y_0\|_{L^2(0,L)} + \left\|\frac{\xi}{w_s}\right\|_{L^2(0,T;L^2(0,L))}\right),$$

for appropriate weight functions $w_0, w_s \in C^0[0,T]$ with $w_0(T) = w_s(T) = 0$, where $C > 0$ is a constant independent of T. Note that, the above inequality gives $y(T) = \frac{y(T)}{w_0(T)}w_0(T) = 0$ (as $\frac{y(T)}{w_0(T)}$ is bounded in $L^2(0,L)$), proving null controllability of the system (2.133). We refer to Proposition 5.4.1 for detailed explanations.

Step 3. Finally, for suitable $\delta > 0$, we define the mapping $F : S_\delta \to L^2(0,T;L^2(0,L))$ as $F(\xi) := y^2$ for $\xi \in S_\delta$, where S_δ is a δ-neighborhood around 0 of the w_s-weighted $L^2(0,T;L^2(0,L))$ space. Then, applying Banach fixed point theorem, we prove that there exists a $\delta > 0$ such that for $y_0 \in L^2(0,L)$ with $\|y_0\|_{L^2(0,L)} \leq \delta$, the map $F : S_\delta \to S_\delta$ has a unique fixed point $\hat{\xi} \in S_\delta$. This will imply that the solution y of (2.131) satisfies $y(T) = 0$, proving small-time local null controllability of the nonlinear system (2.131) in $L^2(0,L)$. We refer to Section 5.4.2 for more details.

Remark 2.5.1. *Apart from the source term method mentioned above, there is an alternative approach/ variations to deal with the local controllability of the nonlinear heat equation* (2.131)*. First, we fix a given element $\hat{y} \in L^2(0,T;L^2(0,L))$ and consider the following problem*

$$\begin{cases} y_t - y_{xx} = f(\hat{y}), & \text{in } (0,T)\times(0,L), \\ y_x(t,0) = q(t), \quad y_x(t,L) = 0, & \text{for } t \in (0,T), \\ y(0,x) = y_0(x), & \text{in } (0,L). \end{cases} \tag{2.134}$$

Here, the term $f(\hat{y})$ is appearing in the equation as a source term. If we are able to prove null controllability of this system (2.134)*, then we can conclude small-time local null controllability of the nonlinear system* (2.131) *by using a fixed-point argument. Moreover, to prove null controllability of* (2.134)*, one may introduce a cost functional (with the source term $f(\hat{y})$) and try minimizing it to deduce the Euler-Lagrange equation, which gives an equivalent condition for null controllability (similar to* (2.120) *but with a source term $f(\hat{y})$). This technique has been addressed in many works, see for instance the articles [EGGP12, EGG16, ES18] and the lecture note [Erv14].*

Chapter 3

Linearized compressible Navier-Stokes system (barotropic and non-barotropic)

This chapter is taken from the article [Kum24]:

"J. Kumbhakar. *Null controllability of one-dimensional barotropic and non-barotropic linearized compressible Navier-Stokes system using one boundary control*, 2023. doi: 10.48550/arXiv.2301.04080."

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Abstract

In this chapter, we study boundary null controllability properties of the linearized compressible Navier-Stokes equations in the interval $(0,2\pi)$ for both barotropic and non-barotropic fluids using only one boundary control. We consider all the possible cases of the act of control for both systems (density, velocity and temperature). These controls are acting on the boundary and are given as the difference of the values at $x=0$ and $x=2\pi$. In this setup, using a boundary control acting only in density, we first prove null controllability of both the barotropic and non-barotropic systems at large time in the spaces $(\dot{L}^2(0,2\pi))^2$ and $(\dot{L}^2(0,2\pi))^3$ respectively (where the dot represents functions with mean value zero). When the control is acting only in the velocity component, we prove null controllability at large time in the spaces $\dot{H}^1_{\text{per}}(0,2\pi)\times\dot{L}^2(0,2\pi)$ and $\dot{H}^1_{\text{per}}(0,2\pi)\times(\dot{L}^2(0,2\pi))^2$ respectively. Further, in both cases, we prove that these null controllability results are sharp with respect to the regularity of the initial states in velocity/ temperature case, and time in the density case. Finally, for both barotropic and non-barotropic fluids, we prove that, under some assumptions, the system cannot be approximately controllable at any time, whether there is a control acting in density, velocity or temperature.

3.1 Introduction and main results

3.1.1 Linearized compressible Navier-Stokes system in 1d

Let $I=(0,+\infty)$ be the time interval and $\Omega\subset\mathbb{R}$ be a spatial domain. For a compressible, isentropic (barotropic) fluid, that is, when the pressure depends only on the density and the temperature is constant, the Navier-Stokes system in $I\times\Omega$ consists of the equation of continuity and the momentum equation

$$\rho_t(t,x)+(\rho u)_x(t,x)=0,$$
$$\rho(t,x)[u_t(t,x)+u(t,x)u_x(t,x)]+p^b_x(t,x)-(\lambda+2\mu)u_{xx}(t,x)=0,$$

where ρ denotes the density of the fluid, u is the velocity. The constants λ,μ are called the viscosity coefficients that satisfy $\mu>0$ and $\lambda+\mu\geq 0$. The pressure p^b satisfies the following constitutive equation in $I\times\Omega$

$$p^b(t,x)=a\rho^\gamma(t,x),\quad (t,x)\in I\times\Omega,$$

for some constants $a>0$ and $\gamma\geq 1$. In the case of non-barotropic fluids, that is, when the pressure is a function of both density and temperature of the fluid, the Navier-Stokes system consists of the equation of continuity, the momentum equation, and an additional thermal energy equation

$$c_v\rho(t,x)[\theta_t(t,x)+u(t,x)\theta_x(t,x)]+\theta(t,x)p^{nb}_\theta(t,x)u_x(t,x)-\kappa\theta_{xx}(t,x)-(\lambda+2\mu)u_x^2(t,x)=0,$$

where θ is the temperature of the fluid, c_v is the specific heat constant, and κ is the heat conductivity constant. For an ideal gas, Boyles law gives the pressure $p^{nb}(t,x)=R\rho(t,x)\theta(t,x)$ in $I\times\Omega$ with R as the universal gas constant. See [Fei04, Chapter 1] for more about compressible flows.

3.1.2 The barotropic case

Let $T>0$ be a finite time. We first consider the Navier-Stokes system for compressible, isentropic (barotropic) fluids linearized around some constant steady state (Q_0,V_0) with $Q_0>0$ and $V_0>0$

$$\begin{cases}\rho_t(t,x)+V_0\rho_x(t,x)+Q_0u_x(t,x)=0, & \text{in } (0,T)\times(0,2\pi),\\ u_t(t,x)-\dfrac{\lambda+2\mu}{Q_0}u_{xx}(t,x)+V_0u_x(t,x)+a\gamma Q_0^{\gamma-2}\rho_x(t,x)=0, & \text{in } (0,T)\times(0,2\pi).\end{cases}\tag{3.1}$$

The initial conditions are

$$\rho(0,x)=\rho_0(x),\quad u(0,x)=u_0(x),\quad x\in(0,2\pi).\tag{3.2}$$

We will consider two different problems, based on the act of control, by imposing any one of the following boundary conditions on the system (3.1).

- **Control in density:**

$$\rho(t,0) = \rho(t,2\pi) + p(t), \quad u(t,0) = u(t,2\pi), \quad u_x(t,0) = u_x(t,2\pi), \quad t \in (0,T). \tag{3.3}$$

- **Control in velocity:**

$$\rho(t,0) = \rho(t,2\pi), \quad u(t,0) = u(t,2\pi) + q(t), \quad u_x(t,0) = u_x(t,2\pi), \quad t \in (0,T), \tag{3.4}$$

where p and q are controls acting on the boundary and are given as the difference of the values at $x = 0$ and $x = 2\pi$.

Definition 3.1.1. *Let H be a Hilbert space. We say the system (3.1)-(3.2)-(3.3) (resp. (3.1)-(3.2)-(3.4)) is*

- ***null controllable*** *at time T in the space H if, for any $(\rho_0, u_0) \in H$, there exists a control $p \in L^2(0,T)$ (resp. $q \in L^2(0,T)$) such that the associated solution satisfies*

$$(\rho(T), u(T)) = (0,0).$$

- ***approximately controllable*** *at time T in the space H if, for any $(\rho_0, u_0), (\rho_T, u_T) \in H$ and any $\epsilon > 0$, there exists a control $p \in L^2(0,T)$ (resp. $q \in L^2(0,T)$) such that the associated solution satisfies*

$$\|(\rho(T), u(T)) - (\rho_T, u_T)\|_H \leq \epsilon.$$

Our main goal in this article is to study null controllability of the system (3.1) at a given time $T > 0$ with the initial condition (3.2) and one of the boundary conditions (3.3) and (3.4).

Before stating our main results, we first define the positive constants

$$\mu_0 := \frac{\lambda + 2\mu}{Q_0}, \quad b := a\gamma Q_0^{\gamma-2}. \tag{3.5}$$

We also introduce the Sobolev space for any $s > 0$

$$H^s_{\text{per}}(0,2\pi) = \left\{ \varphi \ : \ \varphi = \sum_{n\in\mathbb{Z}} c_n e^{inx}, \ \sum_{n\in\mathbb{Z}} |n|^{2s} |c_n|^2 < \infty \right\},$$

with the norm

$$\|\varphi\|_{H^s_{\text{per}}(0,2\pi)} := \left(\sum_{n\in\mathbb{Z}} (1 + |n|^2)^s |c_n|^2 \right)^{\frac{1}{2}}.$$

For $s > 0$, we denote $H^{-s}_{\text{per}}(0,2\pi)$ to be the dual of the Sobolev space $H^s_{\text{per}}(0,2\pi)$ with respect to the pivot space $L^2(0,2\pi)$. We also define the space

$$\dot{L}^2(0,2\pi) := \left\{ \varphi \in L^2(0,2\pi) \ : \ \int_0^{2\pi} \varphi(x)dx = 0 \right\}$$

and

$$\dot{H}^s_{\text{per}}(0,2\pi) := \left\{ \varphi \in H^s_{\text{per}}(0,2\pi) \ : \ \int_0^{2\pi} \varphi(x)dx = 0 \right\}.$$

We also denote $\dot{H}^{-s}_{\text{per}}(0,2\pi)$ as the dual of $\dot{H}^s_{\text{per}}(0,2\pi)$ with respect to the pivot space $L^2(0,2\pi)$. Note that, if the system (3.1)-(3.2)-(3.3) is null controllable at time T by using a boundary control p, then integrating both equations in (3.1), we get a compatibility condition on the initial states

$$a\gamma Q_0^{\gamma-2} \int_0^{2\pi} \rho_0(x)dx = V_0 \int_0^{2\pi} u_0(x)dx = -a\gamma Q_0^{\gamma-2} V_0 \int_0^T p(t)dt.$$

If the system (3.1)-(3.2)-(3.4) is null controllable at time T by using a boundary control q, then also we will get a similar compatibility condition on the initial states. Since every initial state (ρ_0, u_0) in

$(L^2(0,2\pi))^2$ will not satisfy this compatibility condition, we will work on the Hilbert space $(\dot{L}^2(0,2\pi))^2$ to avoid this difficulty.

When a boundary control q is acting in the velocity component, it is known in [CM15] that the system (3.1)-(3.2)-(3.4) is null controllable at time $T > \frac{2\pi}{V_0}$ provided that the initial state is regular enough, in particular, lies in the space $\dot{H}^{s+1}_{\text{per}}(0,2\pi) \times \dot{H}^{s}_{\text{per}}(0,2\pi)$ for $s > \frac{9}{2}$. In the first part of our article, we generalize this result (with respect to the regularity of initial states). In fact, we prove null controllability of (3.1)-(3.2)-(3.4) at time $T > \frac{2\pi}{V_0}$ in the optimal space $\dot{H}^1_{\text{per}}(0,2\pi) \times \dot{L}^2(0,2\pi)$ (see Theorem 3.1.2). In addition, we also prove null controllability of the system (3.1) at time $T > \frac{2\pi}{V_0}$ in $(\dot{L}^2(0,2\pi))^2$ when there is a boundary control p acting in the density component and that the null controllability fails when the time is small, in particular, when $0 < T < \frac{2\pi}{V_0}$ (see Theorem 3.1.1). These results requires certain restrictions on the coefficients appearing in the system (3.1); otherwise the system is not even approximately controllable (see Proposition 3.1.1). To be more precise, if the coefficients Q_0, V_0, μ_0, b (defined by (3.5)) satisfy $\frac{2\sqrt{bQ_0 - V_0^2}}{\mu_0} \in \mathbb{N}$, then the associated adjoint operator A^* (defined by (3.19)) admits an eigenvalue with algebraic multiplicity and geometric multiplicity both are equal to 2, failing the unique continuation property (see the proof of Proposition 3.1.1 for details). However, if $\frac{2\sqrt{bQ_0 - V_0^2}}{\mu_0} \notin \mathbb{N}$, then all the eigenvalues of A^* have geometric multiplicity 1 and in this case, we can achieve null controllability of the system (3.1) by using one boundary control acting either in density or in velocity.

The first main results concerning the null controllability of the system (3.1) are stated below.

Theorem 3.1.1 (Control in density)**.** *The following statements hold:*

(i) Let us assume that $\frac{2\sqrt{bQ_0 - V_0^2}}{\mu_0} \notin \mathbb{N}$. *Then, the system* (3.1)-(3.2)-(3.3) *is null controllable at any time* $T > \frac{2\pi}{V_0}$ *in the space* $(\dot{L}^2(0,2\pi))^2$.

(ii) If $0 < T < \frac{2\pi}{V_0}$, *the system* (3.1)-(3.2)-(3.3) *cannot be null controllable at* T *in the space* $(\dot{L}^2(0,2\pi))^2$.

Theorem 3.1.2 (Control in velocity)**.** *The following statements hold:*

(i) Let us assume that $\frac{2\sqrt{bQ_0 - V_0^2}}{\mu_0} \notin \mathbb{N}$. *Then, the system* (3.1)-(3.2)-(3.4) *is null controllable at any time* $T > \frac{2\pi}{V_0}$ *in the space* $\dot{H}^1_{\text{per}}(0,2\pi) \times \dot{L}^2(0,2\pi)$.

(ii) If $0 \le s < 1$, *the system* (3.1)-(3.2)-(3.4) *cannot be null controllable at any time* $T > 0$ *in the space* $\dot{H}^s_{\text{per}}(0,2\pi) \times \dot{L}^2(0,2\pi)$.

Remark 3.1.1. *Following the proof of Theorem 3.1.1 - Part* (ii)*, lack of null controllability of the system* (3.1)-(3.2)-(3.4) *cannot be obtained when the time is small, in particular, when* $0 < T < \frac{2\pi}{V_0}$. *However, the lack of controllability at small time may be possible to obtain by constructing a Gaussian beam, as mentioned in [Mai15, Theorem 1.5] for the interior control case. Further, null controllability of the system* (3.1) *at time* $T = \frac{2\pi}{V_0}$ *is inconclusive in both cases, whether there is a control acting in density or velocity.*

The following result shows that the restriction on the coefficients $\left(\frac{2\sqrt{bQ_0 - V_0^2}}{\mu_0} \notin \mathbb{N}\right)$ is necessary and sufficient to achieve null controllability of the system (3.1).

Proposition 3.1.1. *If* $\frac{2\sqrt{bQ_0 - V_0^2}}{\mu_0} \in \mathbb{N}$, *the system* (3.1)-(3.2)-(3.3) (*resp.* (3.1)-(3.2)-(3.4)) *is not approximately controllable at any time* $T > 0$ *in the space* $(L^2(0,2\pi))^2$.

We note here that, due to the backward uniqueness property of the system (3.1), null controllability at time T will give us the approximate controllability at that time T for both the systems (3.1)-(3.2)-(3.3) and (3.1)-(3.2)-(3.4), see Section 3.4.2 for more details.

3.1.3 The non-barotropic case

We next consider the Navier-Stokes system for compressible non-barotropic fluids linearized around some constant steady state (Q_0, V_0, ψ_0) with $Q_0, V_0, \psi_0 > 0$

$$\begin{cases} \rho_t(t,x) + V_0\rho_x(t,x) + Q_0 u_x(t,x) = 0, & \text{in } (0,T)\times(0,2\pi), \\ u_t(t,x) - \dfrac{\lambda+2\mu}{Q_0}u_{xx}(t,x) + \dfrac{R\psi_0}{Q_0}\rho_x(t,x) + V_0 u_x(t,x) + R\theta_x(t,x) = 0, & \text{in } (0,T)\times(0,2\pi), \\ \theta_t(t,x) - \dfrac{\kappa}{Q_0 c_v}\theta_{xx}(t,x) + \dfrac{R\psi_0}{c_v}u_x(t,x) + V_0\theta_x(t,x) = 0, & \text{in } (0,T)\times(0,2\pi). \end{cases} \tag{3.6}$$

The initial conditions are

$$\rho(0,x) = \rho_0(x), \quad u(0,x) = u_0(x), \quad \theta(0,x) = \theta_0(x), \quad x \in (0,2\pi). \tag{3.7}$$

In this case, we will consider three different problems, based on the act of control, by imposing any one of the following boundary conditions on the system (3.6).

• **Control in density:**

$$\rho(t,0) = \rho(t,2\pi)+p(t),\ u(t,0) = u(t,2\pi),\ u_x(t,0) = u_x(t,2\pi),\ \theta(t,0) = \theta(t,2\pi),\ \theta_x(t,0) = \theta_x(t,2\pi). \tag{3.8}$$

• **Control in velocity:**

$$\rho(t,0) = \rho(t,2\pi),\ u(t,0) = u(t,2\pi)+q(t),\ u_x(t,0) = u_x(t,2\pi),\ \theta(t,0) = \theta(t,2\pi),\ \theta_x(t,0) = \theta_x(t,2\pi). \tag{3.9}$$

• **Control in temperature:**

$$\rho(t,0) = \rho(t,2\pi),\ u(t,0) = u(t,2\pi),\ u_x(t,0) = u_x(t,2\pi),\ \theta(t,0) = \theta(t,2\pi) + r(t),\ \theta_x(t,0) = \theta_x(t,2\pi). \tag{3.10}$$

for $t \in (0,T)$, where p, q and r are controls acting on the boundary and are given as the difference of the values at $x = 0$ and $x = 2\pi$.

In this case also, we want to prove null controllability of the system (3.6) at a given time $T > 0$ depending on the act of the control. Similar to the barotropic case, we will work on the Hilbert space $(\dot{L}^2(0,2\pi))^3$ to avoid the compatibility conditions on the initial states.

Definition 3.1.2. *Let H be a Hilbert space. We say the system* (3.6)-(3.7)-(3.8) (*resp.* (3.6)-(3.7)-(3.9), (3.6)-(3.7)-(3.10)) *is*

- ***null controllable*** *at time T in the space H if, for any $(\rho_0, u_0, \theta_0) \in H$, there exists a control $p \in L^2(0,T)$ (resp. $q, r \in L^2(0,T)$) such that the associated solution satisfies*

$$(\rho(T), u(T), \theta(T)) = (0,0,0).$$

- ***approximately controllable*** *at time T in the space H if, for any given (ρ_0, u_0, θ_0),$(\rho_T, u_T, \theta_T) \in H$ and any $\epsilon > 0$, there exists a control $p \in L^2(0,T)$ (resp. $q, r \in L^2(0,T)$) such that the associated solution satisfies*

$$\|(\rho(T), u(T), \theta(T)) - (\rho_T, u_T, \theta_T)\|_H \le \epsilon.$$

We next study mainly the null controllability of the system (3.6) at a given time $T > 0$ starting from the initial condition (3.7) and with one of the boundary conditions (3.8)-(3.9) and (3.10). Since the additional thermal energy equation satisfied by θ do not have any coupling with the density ρ, we can expect similar controllability results like the barotropic case. However, in this case, we have two parabolic equations with coefficients $\frac{\lambda+2\mu}{Q_0}$ and $\frac{\kappa}{Q_0 c_v}$ and therefore by looking at [FCGBdT10, LdT13, AKBGBdT14], one question arises naturally:

"*Under what conditions on these coefficients, the system* (3.6) *is null controllable?*"

In fact, we will prove that there exist coefficients for which the system (3.6) may not even be approximately controllable at any time $T > 0$ in $(L^2(0,2\pi))^2$. However, under some stronger assumptions on the diffusion coefficients, we can prove null controllability of (3.6) at any given time $T > \frac{2\pi}{V_0}$ in appropriate spaces (see Theorem 3.1.3).

Before going any further, we first denote the (positive) diffusion coefficients for the non-barotropic system

$$\lambda_0 := \frac{\lambda + 2\mu}{Q_0}, \quad \kappa_0 := \frac{\kappa}{Q_0 c_v}, \tag{3.11}$$

and define the set

$$\mathcal{S} := \left\{ (\lambda_0, \kappa_0) \ : \ \sqrt{\frac{\lambda_0}{\kappa_0}} \notin \mathbb{Q} \right\}. \tag{3.12}$$

We denote the same constant $\frac{\lambda+2\mu}{Q_0}$ by $lambda_0$ instead of mu_0 to distinguish it from the barotropic case. Also, the reason behind introducing such a set $\mathcal{S}$ is explained at the end of this section. First we will state our next main results which concerns null and approximate controllability of the system (3.6).

Theorem 3.1.3. *Let us assume that $(\lambda_0, \kappa_0) \in \mathcal{S}$ be such that there exists a $M > 0$ with the property that*

$$\left| \sqrt{\frac{\lambda_0}{\kappa_0}} - \frac{a}{b} \right| > \frac{1}{b^M} \tag{3.13}$$

holds for all rational numbers $\frac{a}{b}$. We further assume that all the eigenvalues of A^* (defined by (3.75)) have geometric multiplicity equal to 1. Then,

(i) the system (3.6)-(3.7)-(3.8) is null controllable at any time $T > \frac{2\pi}{V_0}$ in the space $(\dot{L}^2(0,2\pi))^3$.

(ii) the systems (3.6)-(3.7)-(3.9) and (3.6)-(3.7)-(3.10) are null controllable at any time $T > \frac{2\pi}{V_0}$ in the space $\dot{H}^1_{\text{per}}(0,2\pi) \times (\dot{L}^2(0,2\pi))^2$.

Proposition 3.1.2. *The following statements hold:*

(i) The system (3.6)-(3.7)-(3.8) is not null controllable at small time $0 < T < \frac{2\pi}{V_0}$ in the space $(\dot{L}^2(0,2\pi))^3$.

(ii) The systems (3.6)-(3.7)-(3.9) and (3.6)-(3.7)-(3.10) are not null controllable at any time $T > 0$ in the space $\dot{H}^s_{\text{per}}(0,2\pi) \times (\dot{L}^2(0,2\pi))^2$ for any $0 \leq s < 1$.

Remark 3.1.2. *Similar to the barotropic case (Remark 3.1.1), lack of null controllability of the system (3.6)-(3.7)-(3.9) or (3.6)-(3.7)-(3.10) is open when the time is small, in particular, when $0 < T < \frac{2\pi}{V_0}$. Moreover, null controllability of the system (3.6) at time $T = \frac{2\pi}{V_0}$ is inconclusive in all cases, whether there is a control act in density, velocity or temperature.*

Like the barotropic case, null controllability at some time T implies approximate controllability at that time T of the system (3.6), thanks to the backward uniqueness property of (3.6) (Section 3.4.2), and the following result shows that the restriction $(\lambda_0, \kappa_0) \in \mathcal{S}$ is not sufficient to conclude null controllability of the system (3.6).

Proposition 3.1.3. *There exist constants $(\lambda_0, \kappa_0) \in \mathcal{S}$ and $Q_0, V_0, \psi_0, R, c_v > 0$ for which the systems (3.6)-(3.7)-(3.8), (3.6)-(3.7)-(3.9) and (3.6)-(3.7)-(3.10) are not approximately controllable at any time $T > 0$ in the space $(L^2(0,2\pi))^3$.*

Remark 3.1.3. *Similar to the barotropic case, there exist constants for which the operator A^* (defined by (3.75)) has eigenvalues with geometric multiplicity greater than 1 (see Remark 3.3.2). However, characterization of these constants is quite difficult due to the complicated cubic characteristic polynomial (3.87).*

3.1.4 An Ingham-type inequality

One of the main ingredients to prove the null controllability results for both barotropic and non-barotropic systems (Theorem 3.1.1 - Theorem 3.1.2 - Theorem 3.1.3) is the following Ingham-type inequality; the proof of this inequality is given in the next chapter (see Section 4.5).

Lemma 3.1.1. *Let $\{\nu_n^h\}_{n\in\mathbb{Z}}$ and $\{\nu_n^p\}_{n\in\mathbb{Z}}$ be two sequences in $\mathbb{C}$ with the following properties: there exists $N \in \mathbb{N}$, such that*

(H1) for all $n, l \in \mathbb{Z}$, $\nu_n^h \neq \nu_l^h$ unless $n = l$;

(H2) $\nu_n^h = \beta + \tau n i + e_n$ for all $|n| \geq N$;

where $\tau > 0, \beta \in \mathbb{C}$ and $\{e_n\}_{|n|\geq N} \in \ell_2$.

Also, there exist constants $A_0 \geq 0$, $B_0 \geq \delta$ with $\delta > 0$ and some $\epsilon > 0, r > 1$ for which $\{\nu_n^p\}_{n\in\mathbb{Z}}$ satisfies

(P1) for all $n, l \in \mathbb{Z}$, $\nu_n^p \neq \nu_l^p$ unless $n = l$;

(P2) $\frac{-\mathrm{Re}(\nu_n^p)}{|\mathrm{Im}(\nu_n^p)|} \geq \widehat{c}$ for some $\widehat{c} > 0$ and for all $|n| \geq N$;

(P3) $\left|\nu_n^p - \nu_l^p\right| \geq \delta\,|n^r - l^r|$ for all $n \neq l$ with $|n|, |l| \geq N$ and

(P4) $\epsilon(A_0 + B_0\,|n|^r) \leq \left|\nu_n^p\right| \leq A_0 + B_0\,|n|^r$ for all $|n| \geq N$.

We also assume that the families are disjoint, i.e.,

$$\left\{\nu_n^h, n \in \mathbb{Z}\right\} \cap \left\{\nu_n^p, n \in \mathbb{Z}\right\} = \emptyset.$$

Then, for any time $T > \frac{2\pi}{\tau}$, there exists a positive constant C depending only on T such that

$$\int_0^T \left|\sum_{n\in\mathbb{Z}} a_n e^{\nu_n^p t} + \sum_{n\in\mathbb{Z}} b_n e^{\nu_n^h t}\right|^2 dt \geq C\left(\sum_{n\in\mathbb{Z}} |a_n|^2 e^{2\mathrm{Re}(\nu_n^p)T} + \sum_{n\in\mathbb{Z}} |b_n|^2\right), \tag{3.14}$$

for all sequences $\{a_n\}_{n\in\mathbb{Z}}$ and $\{b_n\}_{n\in\mathbb{Z}}$ in ℓ_2.

Remark 3.1.4. *In the proof of Lemma 3.1.1, we have used the following parabolic and hyperbolic Ingham inequalities for the families $(e^{\nu_n^p t})_{n\in\mathbb{Z}}$ and $(e^{\nu_n^h t})_{n\in\mathbb{Z}}$ respectively:*

$$\int_0^T \left|\sum_{n\in\mathbb{Z}} a_n e^{\nu_n^p t}\right|^2 dt \geq C_1 \sum_{n\in\mathbb{Z}} |a_n|^2\, e^{2\mathrm{Re}(\nu_n^p)T}, \quad \text{for any } T > 0, \tag{3.15}$$

$$C_2 \sum_{n\in\mathbb{Z}} |b_n|^2 \leq \int_0^T \left|\sum_{n\in\mathbb{Z}} b_n e^{\nu_n^h t}\right|^2 dt \leq C_3 \sum_{n\in\mathbb{Z}} |b_n|^2, \quad \text{for any } T > \frac{2\pi}{\tau}, \tag{3.16}$$

for some $C_i > 0, i = 1, 2, 3$. If the sequence $(\nu_n^h)_{n\in\mathbb{Z}}$ satisfy hypotheses (H1)-(H2), then the hyperbolic Ingham inequality (3.16) can be deduced from the proof of Ingham [Ing36]; see for instance [CMRR14, Proposition 3.1]. On the other hand, the proof of parabolic Ingham inequality (3.15) requires the existence of a biorthogonal family $(q_k)_{k\in\mathbb{Z}} \subset L^2(0,T)$ of $(e^{\nu_n^p t})_{n\in\mathbb{Z}}$ with the estimate $\|q_k\|_{L^2(0,T)} \leq C_\epsilon e^{\epsilon \mathrm{Re}(\nu_n^p)}$ for some $C_\epsilon > 0$ depending on some small parameter ϵ, see for instance [LZ02, Proposition 3.2] and [CMRR14, Proposition 3.2-3.3]; see also [Han91, Theorem 1.1] for the existence of a biorthogonal family in this setup. Note that, the hypotheses (P1)–(P4) can be relaxed to the following:

$$\begin{cases} \mathrm{Re}(-\nu_n^p) \geq \hat{c}\left|\nu_n^p\right|, \quad \left|\nu_n^p - \nu_m^p\right| \geq \delta\,|n - m|, \quad \forall n, m \in \mathbb{Z}, \\ \sum_{n\in\mathbb{Z}} \frac{1}{|\nu_n^p|} < \infty, \end{cases} \tag{3.17}$$

for some $\hat{c}, \delta > 0$. In this setup, we refer to [FCGBdT10, Proposition 3.4] for a proof of the inequality (3.15); see also [LZ02, Proposition 3.2] for a version of (3.15) when the sequence $(\nu_n^p)_{n\in\mathbb{Z}} \subset \mathbb{R}$. Moreover, when the eigenvalues $(\nu_n^p)_{n\in\mathbb{Z}}$ fails to satisfy the gap condition (hypothesis (P3)) but admits a good approximation (by rational numbers), there exists a biorthogonal sequence to the family $(e^{\nu_n^p t})_{n\in\mathbb{Z}}$ in $L^2(0,T)$ with the required estimate (see for instance [LdT13, Lemma 2]), giving the inequality (3.15) in this case also. As a consequence, the combined parabolic-hyperbolic Ingham-type inequality (3.14) can also be deduced under these new assumptions on the sequence $(\nu_n^p)_{n\in\mathbb{Z}}$.

Notations: For any vector v, we denote its transpose by $\mathrm{v}^\dagger$ (instead of v^T). Throughout the article, $C > 0$ denotes a generic constant that may depend on the time T.

Proving null controllability of the systems (3.1) and (3.6) using a boundary control is equivalent to proving an observability inequality for the corresponding adjoint systems. Spectrum of the associated linearized operators (for the adjoint systems) and the above Ingham-type inequality (3.14) plays a crucial role to prove such observability inequalities. For the system (3.1) (barotropic fluids), spectrum of the associated adjoint operator consists of two branches of complex eigenvalues, namely, the hyperbolic and parabolic branches. The hyperbolic branch has eigenvalues with the real part converging to $-\frac{bQ_0}{\mu_0}$, whereas real part of the parabolic branch diverges to $-\infty$. We have obtained explicit expressions of the eigenvalues and eigenfunctions in terms of a Riesz basis (See Lemma 3.2.3 for details). For the non-barotropic fluids (that is, system (3.6)), we get three branches of complex eigenvalues; one is of the hyperbolic type, and two are parabolic types. Similar to the barotropic case, the real part of the hyperbolic branch converges to $-\frac{R\psi_0}{\lambda_0}$ and real parts of both the parabolic branches diverge to $-\infty$. In this case, we have obtained explicit expressions of eigenfunctions and asymptotic behavior of the eigenvalues (Lemma 3.3.3). We also proved that the eigenfunctions form a Riesz basis in $(\dot{L}^2(0,2\pi))^2$ for the barotropic system (Proposition 3.2.3) and in $(\dot{L}^2(0,2\pi))^3$ for the non-barotropic system (Proposition 3.3.4). Then, by writing the solutions to the corresponding adjoint systems in terms of the eigenfunctions, the null controllability results have been proved using the combined parabolic-hyperbolic Ingham type inequality (3.14).

A vast amount of literature is available on the controllability of Navier-Stokes equations for incompressible fluids. For instance, one can see the works of Coron [Cor96], Coron and Fursikov [CF96], Fursikov and Imanuvilov [FE96, FE99], Imanuvilov [Ima98, Ima01], Fernández-Cara et al. [FCGIP04a, FCGIP06], Guerrero [Gue06], Coron and Guerrero [CG09], Chapouly [Cha09], Coron and Lissy [CL14], Badra, Ervedoza and Guerrero [BEG16], Coron, Marbach and Sueur [CMS20]. In comparison, for compressible fluids, less works are available on the Navier-Stokes system's controllability. In this context, we first mention the work of Ervedoza et al. [EGGP12], where the authors established local exact controllability of one dimensional (nonlinear) compressible Navier-Stokes system at a large time T in the space $H^3(0,L) \times H^3(0,L)$ using two boundary controls. This result has been improved in [ES18] where the null controllability is achieved in the space $H^1(0,L) \times H^1(0,L)$. However, studying the controllability of the (nonlinear) compressible Navier-Stokes system using only one boundary control is challenging and it is an interesting open problem. In this article, we focus only on the linearized system and study controllability properties.

It is known in [CRR12] that, for barotropic fluids, the one-dimensional compressible Navier-Stokes system linearized around $(Q_0, 0)$ (with $Q_0 > 0$) cannot be null controllable at any time $T > 0$ by using a boundary control or a localized distributed control. For the linearized system around (Q_0, V_0) (with $Q_0, V_0 > 0$), the authors in [CMRR14] proved null controllability of the Navier-Stokes equations (with homogeneous periodic boundary conditions) for viscous, compressible isothermal barotropic fluids at time T (large) in the space $\dot{H}^1_{\text{per}}(0,2\pi) \times L^2(0,2\pi)$, when there is an interior control acting only in the velocity equation. They also proved that the space $\dot{H}^1_{\text{per}}(0,2\pi) \times L^2(0,2\pi)$ is optimal in the sense that if one choose the initial states from $\dot{H}^s_{\text{per}}(0,2\pi) \times L^2(0,2\pi)$ with $0 \le s < 1$, the linearized system cannot be null controllable at any time $T > 0$. In the case of linearization around (Q_0, V_0) with $Q_0, V_0 > 0$, the compressible Navier-Stokes system (3.1) is equivalent (in some sense) to the transformed system in [MRR13]. Using a moving distributed control, the authors in [MRR13] proved the null controllability of a one-dimensional structurally damped wave equation in the space

$H^{s+2} \times H^s$ for $s > \frac{15}{2}$. There is a generalization to this result in higher dimensions by Chaves-Silva, Rosier, and Zuazua [CSRZ14b]. Inspired by the work of Martin, Rosier and Rouchon [MRR13], Chowdhury and Mitra in [CM15] proved the null controllability of the same compressible Navier-Stokes system linearized around (Q_0, V_0) at time T (large) by using a boundary control acting on the velocity component through periodic conditions, provided the initial states are regular enough, more precisely, in the space $\dot{H}^{1+s}_{\text{per}}(0, 2\pi) \times \dot{H}^s_{\text{per}}(0, 2\pi)$ with $s > 4.5$. However, the question of null controllability at a large time T in the space $\dot{H}^{1+s}_{\text{per}}(0, 2\pi) \times \dot{H}^s_{\text{per}}(0, 2\pi)$ with $s \leq 4.5$ was unaddressed in [CM15], and up to the author's knowledge, there has been no improvement of this result. In this chapter, we have answered this question (see Theorem 3.1.2). In fact, we have proved null controllability of the linearized system (3.1)-(3.2)-(3.4) at large time T in the space $\dot{H}^1_{\text{per}}(0, 2\pi) \times \dot{L}^2(0, 2\pi)$ by using one boundary control acting in the velocity component. We have also proved that our result is optimal in the sense that the system (3.1)-(3.2)-(3.4) cannot be null controllable at any $T > 0$ by a boundary control (acting in velocity) when the initial states belong to the space $\dot{H}^s_{\text{per}}(0, 2\pi) \times \dot{L}^2(0, 2\pi)$ with $0 \leq s < 1$. On the other hand, when a control is acting only in the density component through periodic boundary conditions, we have established null controllability of the linearized system (3.1)-(3.2)-(3.3) at large time T in the space $(\dot{L}^2(0, 2\pi))^2$ and that null controllability fails at small time T. In this context, it is worth mentioning that the authors in [CM15] could only proved null controllability in the space $\dot{H}^{1+s}_{\text{per}}(0, 2\pi) \times \dot{H}^s_{\text{per}}(0, 2\pi)$ with $s > 4.5$ is because of the biorthogonal estimate (corresponding to the hyperbolic family $(e^{\nu_n^h t})_{n \in \mathbb{Z}}$) of order $|k|^4$ (see Proposition 3.2 in [CM15]), which forces the initial state to be more regular. However, in our case, we have used the Ingham-type inequality (3.14) which do not require any biorthogonal estimate of the family $(e^{\nu_n^h t})_{n \in \mathbb{Z}}$, giving the optimal space for null controllability of (3.1). Furthermore, null controllability of the system (3.1) under the assumption $\frac{2\sqrt{bQ_0 - V_0^2}}{\mu_0} \in \mathbb{N}$ was not addressed properly in [CM15, Remark 3.4] and we have proved that, under this assumption, the system (3.1) fails to satisfy the unique continuation property; as a result the system (3.1) cannot be approximately controllable in $(L^2(0, 2\pi))^2$ at any time $T > 0$.

For the non-barotropic fluids, it is known in [Mai15] that the compressible Navier-Stokes system linearized around $(Q_0, 0, \psi_0)$ (with $Q_0, \psi_0 > 0$) is not null controllable at any time $T > 0$ by using a boundary control or a localized distributed control. For the linearization around (Q_0, V_0, ψ_0) with $Q_0, V_0, \psi_0 > 0$, it is only known that the system is not null controllable at small time by a localized interior control or a boundary control acting on the velocity component (see [Mai15, Theorem 1.5] for instance). To the author's knowledge, no controllability result is known for the linearized system around (Q_0, V_0, ψ_0), that is, the system (3.6), when the time is large, which is studied for the first time in this article. On the other hand, for nonlinear system, we mention the work of [Mol19], where the author proved local null controllability of the nonlinear system, in dimensions 1, 2 and 3, at large time in the space $H^2(\Omega) \times H^2(\Omega) \times H^2(\Omega)$ using three controls acting on velocity and temperature on the whole boundary and density on the inflow boundary. Moreover, in one dimension, this result has been improved by choosing the initial state from $H^1(0, L) \times H^1(0, L) \times H^1(0, L)$. However, controllability of this nonlinear system using one boundary control is very difficult to study and is an open problem. In this article, we study null and approximate controllability of only the linearized version, mainly the system (3.6). Since the system (3.6) consists of a transport equation coupled with two parabolic equations, it is worth mentioning some results known for the coupled parabolic equations. In [FCGBdT10], the authors considered a 2-parabolic system with diffusion coefficients $d_1, d_2 > 0$ and with zeroth order coupling. They proved that the coupled parabolic system is (boundary) approximately controllable at time $T > 0$ if and only if $d_1 = d_2$ or $\sqrt{\frac{d_1}{d_2}} \notin \mathbb{Q}$. Moreover, they also proved that, when $d_1 = d_2$, the system is (boundary) null controllable at any time $T > 0$. If $\sqrt{\frac{d_1}{d_2}} \notin \mathbb{Q}$, the authors in [LdT13] provided an example of a system which is approximately controllable but not null controllable at any time $T > 0$. This phenomena occurs because eigenvalues of the associated operator condensate; as a consequence, fails to satisfy the gap condition, which is very crucial to obtain L^2-estimate of the biorthogonal family. However, they [LdT13] also proved that, if $d_1 = 1$ and $\sqrt{d_2} \notin \mathbb{Q}$ is such that we can approximate it as $\left|\sqrt{d_2} - \frac{a}{b}\right| > \frac{C}{b^N}$ for some $C, N > 0$ and all rational numbers $\frac{a}{b}$, then the system is null controllable at any time $T > 0$. Such approximation is referred as "Diophantine approximations". Thus our assumption in Theorem 3.1.3 seems appropriate. We refer to [AKBGBdT14] for more insights in this matter, in terms of condensation index of the eigenvalues and minimal time for null controllability of one dimensional coupled parabolic equations. In the context of controllability results for general coupled parabolic equations, we refer to the works of [Gue07, BBM20, BBGBO14, AKBGBdT11a, KBDK05, AKBGBdT11b] (and the references therein).

The main difficulty in the linearized compressible Navier-Stokes system is the presence of transport and parabolic coupling. The thermoelasticity system is also an example involving both transport and parabolic effects. Lebeau and Zuazua[LZ98] have studied distributed controllability for thermoelasticity systems. Following [LZ98], Beauchard et al. in [BKLB20] proved null controllability for some coupled transport-parabolic systems when an interior control is acting on the system. They proved null controllability at large time T in the space $L^2(0,2\pi) \times \dot{L}^2(0,2\pi)$ by one interior control acting in the density equation and in the space $\dot{H}^2(0,2\pi) \times H^2(0,2\pi)$ when only one interior control is acting in the velocity equation; see also [KL23] for an improvement of the controllability space to $\dot{H}^1(0,2\pi) \times L^2(0,2\pi)$ in the velocity (internal) control case.

In [BCDK22], Bhandari, Chowdhury, Dutta and the author considered the linearized compressible Navier-Stokes system (3.1) with Dirichlet and mixed (Periodic-Dirichlet type) boundary conditions. We proved that the system (3.1) (with Dirichlet boundary conditions) is null controllable at large time T in the space $\dot{L}^2(0,1) \times L^2(0,1)$ by using a boundary control acting only on the density part. On the other hand, when a boundary control is acting only on the velocity component, we proved that the system (3.1) (with Dirichlet-Periodic boundary conditions) is null controllable at large time T in the space $\dot{H}^{\frac{1}{2}}(0,1) \times L^2(0,1)$. We have applied the Ingham-type inequality (3.14) and the moments method to prove these controllability results. In contrast to [BCDK22], the main contribution of this article is that we prove the null controllability of the one-dimensional linearized compressible Navier-Stokes system for both barotropic and non-barotropic fluids by using only one boundary control. We consider all the possible cases of the act of control for both systems (3.1) and (3.6). Further, we obtain better regularity of the initial states for the controllability of barotropic system (3.1) compared to [CM15]. In the case of non-barotropic fluids, since the transport equation does not affect the temperature equation, it is pretty natural to obtain similar spaces of null controllability of the system (3.6). The combined parabolic-hyperbolic Ingham type inequality (Lemma 3.1.1) helps us obtain each case's best possible results (with respect to the state space). Our results cannot be obtained as a consequence of interior control results by the extension method. In addition, when the boundary control acts in the density component, we prove that both systems (3.1) and (3.6) are not null controllable at small time. The proof is inspired from [BKLB20] and is independent of that in [Mai15].

The result stated in Theorem 3.1.1 is similar to the results in [BKLB20], showing that we can achieve the space $(\dot{L}^2(0,2\pi))^2$ in the case of only one boundary control (acting in density) also. Likewise the case of interior control [CMRR14, BKLB20, KL23], we also obtain similar results for our boundary control case (acting in velocity) (Theorem 3.1.2).

The rest of this chapter is organized as follows:

- In Section 3.2, we prove all the controllability results for the barotropic system (3.1) at a large time T using a boundary control that acts either in density or velocity, that is, Theorem 3.1.1 and Theorem 3.1.2. The proof of lack of approximate controllability at any time T under the restriction on the coefficients (Proposition 3.1.1) is also included in this section.

- In Section 3.3, we consider the non-barotropic system (3.6) and give all the related controllability results based on the act of the control, namely the proofs of Theorem 3.1.3 and Proposition 3.1.2. We have also included the proof of lack of approximate controllability result at any time T (Proposition 3.1.3).

- In section 3.4, we give few comments and open questions regarding controllability results under Dirichlet or Neumann boundary conditions and the backward uniqueness property.

3.2 Controllability of the linearized compressible Navier-Stokes system (barotropic case)

3.2.1 Functional setting

Recall from (3.5) the positive constants

$$\mu_0 := \frac{\lambda + 2\mu}{Q_0}, \quad b := a\gamma Q_0^{\gamma-2}.$$

We define the inner product in the space $(L^2(0,2\pi))^2$ as follows

$$\left\langle \begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ g_2 \end{pmatrix} \right\rangle_{L^2\times L^2} := b\int_0^{2\pi} f_1(x)\overline{f_2(x)}dx + Q_0\int_0^{2\pi} g_1(x)\overline{g_2(x)}dx,$$

for $f_i, g_i \in L^2(0,2\pi), i = 1,2$. From now on-wards, the notation $\langle \cdot,\cdot \rangle_{L^2\times L^2}$ means the above inner product in $L^2 \times L^2$. We write the system (3.1) in abstract differential equation

$$U'(t) = AU(t), \quad U(0) = U_0, \quad t \in (0,T), \tag{3.18}$$

where $U := (\rho, u)^\dagger, U_0 := (\rho_0, u_0)^\dagger$ and the operator A is given by

$$A := \begin{pmatrix} -V_0\partial_x & -Q_0\partial_x \\ -b\partial_x & \mu_0\partial_{xx} - V_0\partial_x \end{pmatrix}$$

with the domain

$$\mathcal{D}(A) := H^1_{\text{per}}(0,2\pi) \times H^2_{\text{per}}(0,2\pi).$$

The adjoint of the operator A is given by

$$A^* := \begin{pmatrix} V_0\partial_x & Q_0\partial_x \\ b\partial_x & \mu_0\partial_{xx} + V_0\partial_x \end{pmatrix} \tag{3.19}$$

with the same domain $\mathcal{D}(A^*) = \mathcal{D}(A)$. The adjoint system is then given by

$$\begin{cases} -\sigma_t(t,x) - V_0\sigma_x(t,x) - Q_0 v_x(t,x) = 0, & \text{in } (0,T)\times(0,2\pi), \\ -v_t(t,x) - \mu_0 v_{xx}(t,x) - V_0 v_x(t,x) - b\sigma_x(t,x) = 0, & \text{in } (0,T)\times(0,2\pi), \\ \sigma(t,0) = \sigma(t,2\pi), \quad v(t,0) = v(t,2\pi), \quad v_x(t,0) = v_x(t,2\pi), & t \in (0,T), \\ \sigma(T,x) = \sigma_T(x), \quad v(T,x) = v_T(x), & x \in (0,2\pi). \end{cases} \tag{3.20}$$

We now write the adjoint system with source terms f and g.

$$\begin{cases} -\sigma_t(t,x) - V_0\sigma_x(t,x) - Q_0 v_x(t,x) = f, & \text{in } (0,T)\times(0,2\pi), \\ -v_t(t,x) - \mu_0 v_{xx}(t,x) - V_0 v_x(t,x) - b\sigma_x(t,x) = g, & \text{in } (0,T)\times(0,2\pi), \\ \sigma(t,0) = \sigma(t,2\pi), \quad v(t,0) = v(t,2\pi), \quad v_x(t,0) = v_x(t,2\pi), & t \in (0,T), \\ \sigma(T,x) = \sigma_T(x), \quad v(T,x) = v_T(x), & x \in (0,2\pi). \end{cases} \tag{3.21}$$

3.2.2 Well-posedness of the systems

This section devotes to the well-posedness of the system (3.1) under the boundary conditions (3.3), (3.4) and the initial conditions (3.2), and the adjoint system (3.21).

When there is no control acting on the system, we have the existence of solutions to the system (3.1) using semigroups.

Lemma 3.2.1 ([CMRR14, Lemma 2.1]). *The operator A (resp. A^*) generates a C^0-semigroup of contractions on $(L^2(0,2\pi))^2$. Moreover, for every $U_0 \in (L^2(0,2\pi))^2$ the system (3.18) admits a unique weak solution U in $C^0([0,T];(L^2(0,2\pi))^2)$ and*

$$\|U(t)\|_{(L^2(0,2\pi))^2} \leq C \|U_0\|_{(L^2(0,2\pi))^2}$$

for all $t \geq 0$.

The following lemma shows the existence of a unique weak solution to the adjoint system (3.21).

Lemma 3.2.2. *The following statements hold:*

1. *For any given source term $(f,g) \in L^2(0,T;(L^2(0,2\pi))^2)$ and given $(\sigma_T, v_T) \in (L^2(0,2\pi))^2$, the adjoint system (3.21) has a unique weak solution (σ, v) in the space*

$$C^0([0,T];L^2(0,2\pi)) \times [C^0([0,T];L^2(0,2\pi)) \cap L^2(0,T;H^1_{\text{per}}(0,2\pi))].$$

Furthermore, we have the hidden regularity property $\sigma(\cdot,2\pi) \in L^2(0,T)$.

2. *For any given $(f,g) \in L^2(0,T;H^1_{\text{per}}(0,2\pi) \times L^2(0,2\pi))$ and $(\sigma_T,v_T) \in H^{-1}_{\text{per}}(0,2\pi) \times L^2(0,2\pi)$, the system (3.20) admits a unique solution $(\sigma,v) \in C^0([0,T];H^{-1}_{\text{per}}(0,2\pi) \times L^2(0,2\pi))$.*

In particular, when $(\sigma_T,v_T) = (0,0)$, the solution (σ,v) belong to the space

$$C^0([0,T];H^1_{\text{per}}(0,2\pi)) \times [C^0([0,T];H^1_{\text{per}}(0,2\pi)) \cap L^2(0,T;H^2_{\text{per}}(0,2\pi))].$$

Proof of the first part is given in Appendix A.0.2; see also Appendix A.1 for the hidden regularity result. For the second part, we refer to [CMRR14, Proposition 2.5], see also [Gir08, Chapter 4].

Once we have the existence results of the homogeneous system (without any boundary control) associated to the system (3.1), we can now guarantee the existence of a unique solution to the system (3.1) (in the sense of transposition) when there is a boundary control p (resp. q) acting in density (resp. velocity) in the space $L^2(0,T)$. Before writing the statements, let us first define the notion of a solution in the sense of transposition.

Definition 3.2.1. *We give the following definitions based on the act of the control.*

1. *For any given initial state $(\rho_0,u_0) \in (L^2(0,2\pi))^2$ and boundary control $p \in L^2(0,T)$, a function $(\rho,u) \in L^2(0,T;(L^2(0,2\pi))^2)$ is a solution to the system (3.1)-(3.2)-(3.3) if, for any given $(f,g) \in L^2(0,T;(L^2(0,2\pi))^2)$ the following identity holds true:*

$$\begin{aligned}&\int_0^T \left\langle (\rho(t,\cdot),u(t,\cdot))^\dagger, (f(t,\cdot),g(t,\cdot))^\dagger \right\rangle_{L^2\times L^2} dt \\ &= \left\langle (\rho_0(\cdot),u_0(\cdot))^\dagger, (\sigma(0,\cdot),v(0,\cdot))^\dagger \right\rangle_{L^2\times L^2} + b\int_0^T \left[V_0\overline{\sigma(t,2\pi)} + Q_0\overline{v(t,2\pi)}\right] p(t)dt,\end{aligned}$$

where (σ,v) is the unique weak solution to the adjoint system (3.21) with $(\sigma_T,v_T) = (0,0)$.

2. *For any given initial state $(\rho_0,u_0) \in (L^2(0,2\pi))^2$ and boundary control $q \in L^2(0,T)$, a function $(\rho,u) \in L^2(0,T;H^{-1}_{\text{per}}(0,2\pi) \times L^2(0,2\pi))$ is a solution to the system (3.1)-(3.2)-(3.4) if, for any given $(f,g) \in L^2(0,T;H^1_{\text{per}}(0,2\pi) \times L^2(0,2\pi))$ the following identity holds true:*

$$\begin{aligned}&\int_0^T \left\langle (\rho(t,\cdot),u(t,\cdot))^\dagger, (f(t,\cdot),g(t,\cdot))^\dagger \right\rangle_{H^{-1}_{\text{per}}\times L^2, H^1_{\text{per}}\times L^2} dt \\ &= \left\langle (\rho_0(\cdot),u_0(\cdot))^\dagger, (\sigma(0,\cdot),v(0,\cdot))^\dagger \right\rangle_{L^2\times L^2} + Q_0\int_0^T \left[b\overline{\sigma(t,2\pi)} + V_0\overline{v(t,2\pi)} + \mu_0\overline{v_x(t,2\pi)}\right] q(t)dt,\end{aligned}$$

where (σ,v) is the unique weak solution to the adjoint system (3.21) with $(\sigma_T,v_T) = (0,0)$.

Proposition 3.2.1. *For any given initial state* $(\rho_0, u_0) \in (L^2(0, 2\pi))^2$ *and boundary control* $p \in L^2(0, T)$*, the system* (3.1)-(3.2)-(3.3) *admits a unique solution* (ρ, u) *in the space*

$$C^0([0, T]; L^2(0, 2\pi)) \times [C^0([0, T]; L^2(0, 2\pi)) \cap L^2(0, T; H^1_{\text{per}}(0, 2\pi))].$$

Proposition 3.2.2. *For any given initial state* $(\rho_0, u_0) \in (L^2(0, 2\pi))^2$ *and boundary control* $q \in L^2(0, T)$*, the system* (3.1)-(3.2)-(3.4) *admits a unique solution* (ρ, u) *in the space*

$$C^0([0, T]; H^{-1}_{\text{per}}(0, 2\pi)) \times [C^0([0, T]; H^{-1}_{\text{per}}(0, 2\pi)) \cap L^2(0, T; L^2(0, 2\pi))].$$

The proof of the first result (density case) will be similar to that given in the Appendix A.0.2. For the velocity case, the proof can be done in a standard fashion using the semigroup theory of the homogeneous system and the properties of the transport and parabolic equations, see for instance [CR13, Gir08].

3.2.3 Spectral analysis of A^*

We denote the spectrum of A^* by $\sigma(A^*)$. The following lemma gives behavior of the spectrum of the operator A^*.

Lemma 3.2.3. *The following statements hold.*

(i) $\ker(A^*) = span\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}.$

(ii) $\sup\{\text{Re}(\nu) \ : \ \nu \in \sigma(A^*), \ \nu \neq 0\} < 0.$

(iii) The spectrum of A^* *consists of the eigenvalue* $\nu_0 = 0$ *and pairs of complex eigenvalues* $\{\nu_n^h, \nu_n^p\}_{n \in \mathbb{Z}^*}$ *given as*

$$\nu_n^h = -\frac{1}{2}\left(\mu_0 n^2 - \sqrt{\mu_0^2 n^4 - 4bQ_0 n^2 - 2V_0 in}\right), \tag{3.22}$$

$$\nu_n^p = -\frac{1}{2}\left(\mu_0 n^2 + \sqrt{\mu_0^2 n^4 - 4bQ_0 n^2 - 2V_0 in}\right), \tag{3.23}$$

for all $n \in \mathbb{Z}^*$.

(iv) The eigenvalues satisfy the following properties

$$\begin{cases} \lim_{|n|\to\infty} \text{Re}(\nu_n^h) = -\omega_0, & \lim_{|n|\to\infty} \frac{\text{Re}(\nu_n^p)}{n^2} = -\mu_0 \\ \lim_{|n|\to\infty} \frac{\text{Im}(\nu_n^h)}{n} = V_0, & \lim_{|n|\to\infty} \frac{\text{Im}(\nu_n^p)}{n} = V_0 \end{cases}$$

with $\omega_0 = \frac{bQ_0}{\mu_0}$.

(v) The eigenfunctions of A^* *corresponding to* ν_n^h *and* ν_n^p *are respectively*

$$\Phi_n^h = \begin{pmatrix} \xi_n^h \\ \eta_n^h \end{pmatrix} = \begin{pmatrix} Q_0 \\ \nu_2^n - V_0 \end{pmatrix} e^{inx}, \quad \Phi_n^p = \begin{pmatrix} \xi_n^p \\ \eta_n^p \end{pmatrix} = \begin{pmatrix} \frac{Q_0}{\nu_1^n - V_0} \\ 1 \end{pmatrix} e^{inx}, \tag{3.24}$$

for $n \in \mathbb{Z}^*$*, where* $\nu_1^n = \frac{1}{in}\nu_n^p$ *and* $\nu_2^n = \frac{1}{in}\nu_n^h$ *for* $n \in \mathbb{Z}^*$.

Proof. We prove each part separately.

Part-(i). Let $\Phi = (\xi, \eta)^\dagger \in \mathcal{D}(A^*)$ be such that $A^*\Phi = 0$. This gives $V_0\xi_x + Q_0\eta_x = 0$ and $\mu_0\eta_{xx} + V_0\eta_x + b\xi_x = 0$ and therefore we have $\mu_0 V_0 \eta_{xx} + (V_0^2 - bQ_0)\eta_x = 0$. The boundary conditions $\eta(0) = \eta(2\pi)$ and $\eta_x(0) = \eta_x(2\pi)$ implies η = constant and consequently ξ = constant.

Part-(ii). Let $\Phi = (\xi,\eta)^\dagger \in \mathcal{D}(A^*)$ be the eigenfunction of A^* corresponding to the eigenvalue $\nu \neq 0$. Then, we have

$$\left\langle A^* \begin{pmatrix}\xi\\ \eta\end{pmatrix}, \begin{pmatrix}\xi\\ \eta\end{pmatrix}\right\rangle_{L^2\times L^2} = \left\langle \nu \begin{pmatrix}\xi\\ \eta\end{pmatrix}, \begin{pmatrix}\xi\\ \eta\end{pmatrix}\right\rangle_{L^2\times L^2},$$

that is,

$$\begin{aligned} &bV_0 \int_0^{2\pi} \overline{\xi(x)}\xi_x(x)dx + bQ_0 \int_0^{2\pi} \overline{\xi(x)}\eta_x(x)dx + \mu_0 Q_0 \int_0^{2\pi} \overline{\eta(x)}\eta_{xx}(x)dx \\ &+ Q_0V_0 \int_0^{2\pi} \overline{\eta(x)}\eta_x(x)dx + bQ_0 \int_0^{2\pi} \xi_x(x)\overline{\eta(x)}dx = \nu b \int_0^{2\pi} |\xi(x)|^2\, dx + \nu Q_0 \int_0^{2\pi} |\eta(x)|^2\, dx. \end{aligned}$$

An integration by parts yields

$$\mathrm{Re}(\nu) = -\frac{\mu_0 Q_0 \left\|\eta_x\right\|^2_{L^2(0,2\pi)}}{b\left\|\xi\right\|^2_{L^2(0,2\pi)} + Q_0\left\|\eta\right\|^2_{L^2(0,2\pi)}} < 0,$$

which proves part (ii), since η cannot be constant for $nu \neq 0$, thanks to the first part.

Parts-(iii),(v). We denote

$$\varphi_n(x) := e^{inx}, \quad n \in \mathbb{Z}.$$

Then the set $\left\{\begin{pmatrix}\varphi_n\\ 0\end{pmatrix}, \begin{pmatrix}0\\ \varphi_n\end{pmatrix}\ ;\ n\in\mathbb{Z}\right\}$ forms an orthogonal basis of $(L^2(0,2\pi))^2$. Let us define

$$E_n := \begin{pmatrix}\varphi_n & 0\\ 0 & \varphi_n\end{pmatrix}, \quad \text{and } \Phi_n := (\xi_n,\eta_n)^\dagger,$$

for all $n\in\mathbb{Z}$. Then, we have the following relation

$$A^*E_n\Phi_n = E_nR_n\Phi_n, \quad n\in\mathbb{Z}, \tag{3.25}$$

where the matrix R_n for $n\in\mathbb{Z}$ is given by

$$R_n := \begin{pmatrix} V_0 in & Q_0 in\\ bin & -\mu_0 n^2 + V_0 in\end{pmatrix}, \quad n\in\mathbb{Z}. \tag{3.26}$$

Thus, if (α_n,ν_n) is an eigenpair of R_n, then $(E_n\alpha_n,\nu_n)$ will be an eigenpair of A^*. Therefore, it remains to find the eigenvalues and eigenvectors of the matrix R_n for $n\in\mathbb{Z}$. The characteristics equation of R_n is

$$\nu^2 - (-\mu_0 n^2 + 2V_0 in)\nu - \mu_0 V_0 in^3 - V_0^2 n^2 + bQ_0 n^2 = 0, \tag{3.27}$$

for all $n\in\mathbb{Z}$. Therefore, the eigenvalues of the matrix R_n are

$$\nu_n^h := \frac{1}{2}\left(-\mu_0 n^2 + 2V_0 in + \sqrt{\mu_0^2 n^4 - 4bQ_0 n^2}\right), \quad \nu_n^p := \frac{1}{2}\left(-\mu_0 n^2 + 2V_0 in - \sqrt{\mu_0^2 n^4 - 4bQ_0 n^2}\right),$$

for all $n\in\mathbb{Z}$. Note that, 0 cannot be an eigenvalue of the matrix R_n for all $n\in\mathbb{Z}^*$ and V_0 cannot be an eigenvalue of R_n for all $n\in\mathbb{Z}$, because $b,Q_0,\mu_0,V_0 > 0$. Let us denote $\nu_1^n := \frac{1}{in}\nu_n^p$ and $\nu_2^n := \frac{1}{in}\nu_n^h$. To find the eigenvectors of the matrix R_n, we first consider the equation

$$R_n\alpha_n^h = \nu_n^h\alpha_n^h, \quad \text{for } n\in\mathbb{Z},$$

where $\alpha_n^h := (\alpha_1^n,\alpha_2^n)^\dagger$, that is,

$$(V_0 in - \nu_n^h)\alpha_1^n + Q_0 in\alpha_2^n = 0, \quad bin\alpha_1^n + (-\mu_0 n^2 + V_0 in - \nu_n^h)\alpha_2^n = 0, \tag{3.28}$$

for all $n \in \mathbb{Z}$. One solution is given by

$$\alpha_n^h = \begin{pmatrix} \alpha_1^n \\ \alpha_2^n \end{pmatrix} := \begin{pmatrix} Q_0 \\ \nu_2^n - V_0 \end{pmatrix}, \quad n \in \mathbb{Z}. \tag{3.29}$$

We next consider the equation

$$R_n \alpha_n^p = \nu_n^p \alpha_n^p, \quad \text{for } n \in \mathbb{Z},$$

where $\alpha_n^p := (\beta_1^n, \beta_2^n)^\dagger$, that is,

$$(V_0 in - \nu_n^p)\beta_1^n + Q_0 in \beta_2^n = 0, \quad bin\beta_1^n + (-\mu_0 n^2 + V_0 in - \nu_n^p)\beta_2^n = 0, \tag{3.30}$$

for all $n \in \mathbb{Z}$. One solution is given by

$$\alpha_n^p = \begin{pmatrix} \beta_1^n \\ \beta_2^n \end{pmatrix} := \begin{pmatrix} \frac{Q_0}{\nu_1^n - V_0} \\ 1 \end{pmatrix}, \quad n \in \mathbb{Z}. \tag{3.31}$$

Thus, the eigenvectors of R_n corresponding to the eigenvalues ν_n^h and ν_n^p are respectively

$$\alpha_n^h = \begin{pmatrix} \alpha_1^n \\ \alpha_2^n \end{pmatrix} = \begin{pmatrix} Q_0 \\ \nu_2^n - V_0 \end{pmatrix}, \quad \alpha_n^p = \begin{pmatrix} \beta_1^n \\ \beta_2^n \end{pmatrix} = \begin{pmatrix} \frac{Q_0}{\nu_1^n - V_0} \\ 1 \end{pmatrix}, \quad n \in \mathbb{Z}.$$

Hence, the eigenvalues of the operator A^* are $\nu_0 := 0$ and

$$\nu_n^h := \frac{1}{2}\left(-\mu_0 n^2 + 2V_0 in + \sqrt{\mu_0^2 n^4 - 4bQ_0 n^2}\right), \quad \nu_n^p := \frac{1}{2}\left(-\mu_0 n^2 + 2V_0 in - \sqrt{\mu_0^2 n^4 - 4bQ_0 n^2}\right),$$

for $n \in \mathbb{Z}^*$ and the corresponding eigenfunctions are respectively

$$\Phi_n^h := \begin{pmatrix} \xi_n^h \\ \eta_n^h \end{pmatrix} = E_n \alpha_n^h = \alpha_n^h e^{inx}, \quad \Phi_n^p := \begin{pmatrix} \xi_n^p \\ \eta_n^p \end{pmatrix} = E_n \alpha_n^p = \alpha_n^p e^{inx},$$

for all $n \in \mathbb{Z}^*$ and $x \in (0, 2\pi)$. This proves parts (iii) and (v).

Part-(iv). Follows immediately from the expression of the eigenvalues ν_n^h and ν_n^p, given by (3.22)-(3.23). Indeed, we can write

$$\nu_n^h = -\frac{2bQ_0}{\mu_0 + \sqrt{\mu_0^2 - \frac{4bQ_0}{n^2}}} + V_0 in, \text{ and } \nu_n^p = -\frac{n^2}{2}\left(\mu_0 + \sqrt{\mu_0^2 - \frac{4bQ_0}{n^2}}\right) + V_0 in$$

for $n \in \mathbb{Z}^*$.

□

From the expression of the eigenvalues given by (3.22)-(3.23), we can further deduce several important properties, which are given by the following Lemma:

Lemma 3.2.4 (Properties of the eigenvalues). *Let $n, l \in \mathbb{Z}^*$. Then,*

(i) $\nu_n^h = \nu_l^h$ if and only if $n = l$.

(ii) if $n_1 := \frac{2\sqrt{bQ_0 - V_0^2}}{\mu_0} \in \mathbb{N}$, then $\nu_{n_1}^p = \nu_{-n_1}^p$ and $\nu_n^p \neq \nu_l^p$ for remaining $n, l \in \mathbb{Z}^*$ with $n \neq l$.

(iii) if $n_0 := \frac{2\sqrt{bQ_0}}{\mu_0} \in \mathbb{N}$, then $\nu_j^h = \nu_j^p$ for $j = \pm n_0$ and $\nu_n^h \neq \nu_l^p$ for all $n, l \in \mathbb{Z}^* \setminus \{\pm n_0\}$.

(iv) if $\frac{2\sqrt{bQ_0}}{\mu_0}, \frac{2\sqrt{bQ_0 - V_0^2}}{\mu_0} \notin \mathbb{N}$, then all the eigenvalues of A^* are simple.

Proof. We prove each part separately.

Part-(i). Let us denote $n_0 := \frac{2\sqrt{bQ_0}}{\mu_0}$. Then, $\mathrm{Im}(\nu_n^h) = V_0 n$ for all $|n| \geq n_0$ and therefore $\nu_n^h \neq \nu_l^h$ for all $|n|, |l| \geq n_0$ with $n \neq l$. For $1 \leq |n| < n_0$, we have $\nu_n^h = -\frac{\mu_0}{2}n^2 + i(V_0 n + \frac{1}{2}\sqrt{4bQ_0 n^2 - \mu_0^2 n^4})$. Since $\mathrm{Im}(\nu_n^h) \neq \mathrm{Im}(\nu_{-n}^h)$ for all $1 \leq |n| < n_0$, we readily have $\nu_n^h \neq \nu_l^h$ for all $1 \leq |n|, |l| < n_0$.

Part-(ii). Note that $\mathrm{Im}(\nu_n^p) = V_0 n$ for all $|n| \geq n_0$ and therefore $\nu_n^p \neq \nu_l^p$ for all $|n|, |l| \geq n_0$ with $n \neq l$. For $1 \leq |n| < n_0$, we have $\nu_n^p = -\frac{\mu_0}{2}n^2 + i(V_0 n - \frac{1}{2}\sqrt{4bQ_0 n^2 - \mu_0^2 n^4})$. Then, $\mathrm{Im}(\nu_n^p) = -\mathrm{Im}(\nu_{-n}^p)$ for all $1 \leq |n| < n_0$, which implies $\nu_n^p = \nu_{-n}^p$ holds only if $V_0 n - \frac{1}{2}\sqrt{4bQ_0 n^2 - \mu_0^2 n^4} = 0$, that is, when $n = \frac{2\sqrt{bQ_0 - V_0^2}}{\mu_0}$. Moreover, $\mathrm{Re}(\nu_n^p) \neq \mathrm{Re}(\nu_l^p)$ for remaining values of $n, l \in \mathbb{Z}^* (n \neq l)$, implying $\nu_n^p \neq \nu_l^p$.

Part-(iii). Let $n, l \in \mathbb{Z}^*$ with $|n|, |l| > n_0$. Since $\mathrm{Im}(\nu_n^h) = \mathrm{Im}(\nu_n^p) = V_0 n$, therefore $\mathrm{Im}(\nu_n^h) = \mathrm{Im}(\nu_l^p)$ is true if and only if $n = l$ and $\mathrm{Re}(\nu_n^h) \neq \mathrm{Re}(\nu_n^p)$. This proves that $\nu_n^h \neq \nu_l^p$ for all $|n|, |l| > n_0$. For $1 \leq |n|, |l| < n_0$, $\mathrm{Re}(\nu_n^h) = \mathrm{Re}(\nu_n^p) = -\frac{\mu_0}{2}n^2$ and therefore $\mathrm{Re}(\nu_n^h) = \mathrm{Re}(\nu_l^p)$ holds if and only if $n = \pm l$. On the other hand, $\mathrm{Im}(\nu_n^h) \neq \mathrm{Im}(\nu_l^p)$ for $n = \pm l$, which implies $\nu_n^h \neq \nu_l^p$ for all $1 \leq |n|, |l| < n_0$. Let $1 \leq |n| \leq n_0$ and $|l| > n_0$. Then, $\nu_n^h = -\frac{\mu_0}{2}n^2 + i(V_0 n + \frac{1}{2}\sqrt{4bQ_0 n^2 - \mu_0^2 n^4})$ and $\nu_l^p = -\frac{\mu_0}{2}l^2 - \frac{1}{2}\sqrt{\mu_0^2 l^4 - 4bQ_0 l^2} + V_0 il$. Thus $\nu_n^h = \nu_l^p$ implies $\frac{1}{2}\sqrt{\mu_0^2 l^4 - 4bQ_0 l^2} = -\frac{\mu_0}{2}(l^2 - n^2) < 0$, which is not possible. Therefore, the only possible case is $|n| = |l| = n_0$ and in this case, we have $\nu_n^h = \nu_l^p$, which proves part (iii)

Part-(iv). Follows from parts (i), (ii) and (iii).

This completes the proof. □

From this Lemma, we note that when $n_0 = \frac{2\sqrt{bQ_0}}{\mu_0} \in \mathbb{N}$, the matrix R_j admits an eigenvalue $\nu_j := -\frac{\mu_0 j^2}{2} + iV_0 j$ of multiplicity 2 with the eigenvectors $\alpha_j := \begin{pmatrix} Q_0 \\ \nu_2^j - V_0 \end{pmatrix}$ for $j = \pm n_0$. Let $\tilde{\alpha}_j = (\tilde{\alpha}_1^j, \tilde{\alpha}_2^j)$ be the generalized eigenvector corresponding to ν_j for $j = \pm n_0$, then we have the following set of relations:

$$\begin{cases} (iV_0 j - \nu_j)\tilde{\alpha}_1^j + iQ_0 j\tilde{\alpha}_2^j = Q_0, \\ ibj\tilde{\alpha}_1^j + (-\mu_0 j^2 + iV_0 j - \nu_j)\tilde{\alpha}_2^j = \nu_2^j - V_0, \end{cases} \tag{3.32}$$

for $j = \pm n_0$. Thus, if $\frac{2\sqrt{bQ_0}}{\mu_0} \in \mathbb{N}$, the operator A^* admits generalized eigenfunction corresponding to the eigenvalue $\nu_j^h = \nu_j^p = \nu_j$ for $j = \pm n_0$. We denote the generalized eigenfunction corresponding to ν_j by $\tilde{\Phi}_j := \tilde{\alpha}_j e^{ijx}$ for $j = \pm n_0$. Also, recall that the set of eigenfunctions corresponding to the eigenvalue $\nu_0 = 0$ is $\left\{\Phi_0 := \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ \tilde{\Phi}_0 := \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$. Then, with the above mentioned properties of the eigenvalues, we can prove that the set of (generalized) eigenfunctions of A^* form a Riesz basis of $(L^2(0, 2\pi))^2$.

Proposition 3.2.3. *If* $\frac{2\sqrt{bQ_0}}{\mu_0} \in \mathbb{N}$, *the set of (generalized) eigenfunctions*

$$\mathcal{E}(A^*) := \left\{\Phi_n^h, \ \Phi_n^p \ : \ n \in \mathbb{Z}^* \setminus \{\pm n_0\}; \ \Phi_j, \ \tilde{\Phi}_j \ : \ j = 0, \pm n_0\right\}$$

form a Riesz basis in $(L^2(0, 2\pi))^2$. *In particular, when* $\frac{2\sqrt{bQ_0}}{\mu_0} \notin \mathbb{N}$, *the set of eigenfunctions*

$$\left\{\Phi_n^h, \ \Phi_n^p \ : \ n \in \mathbb{Z}^*\right\}$$

of A^* *form a Riesz basis in* $(\dot{L}^2(0, 2\pi))^2$.

Proof. Denote $\Psi_n(x) := \begin{pmatrix} Q_0 \\ 0 \end{pmatrix} e^{inx}$, $\tilde{\Psi}_n(x) := \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{inx}$ for $n \in \mathbb{Z}$. Then, the set of generalized eigenfunctions $\left\{\Phi_n^h,\ \Phi_n^p\ :\ n \in \mathbb{Z}^* \setminus \{\pm n_0\};\ \Phi_j,\ \tilde{\Phi}_j\ :\ j = 0, \pm n_0\right\}$ of A^* is quadratically close to the orthogonal basis $\left\{\Psi_n,\ \tilde{\Psi}_n;\ n \in \mathbb{Z}\right\}$ in $(L^2(0,2\pi))^2$. Indeed, we have for a large $N \in \mathbb{N}$

$$\sum_{|n|>N} \left(\left\| \Phi_n^h - \Psi_n \right\|^2_{(L^2(0,2\pi))^2} + \left\| \Phi_n^p - \tilde{\Psi}_n \right\|^2_{(L^2(0,2\pi))^2} \right) \le C \sum_{|n|>N} \frac{1}{|n|^2} < \infty,$$

thanks to the fact that $\left|\nu_2^n - V_0\right| \le \frac{C}{|n|}$ and $\left|\nu_1^n - V_0\right| \ge C\,|n|$ for large n. Since the set $\left\{\Psi_n, \tilde{\Psi}_n\ ;\ n \in \mathbb{Z}\right\}$ is an orthogonal basis of $(L^2(0,2\pi))^2$, this Proposition is now an immediate consequence of the result of Bao-Zhu Guo [Guo01, Theorem 6.3]. □

3.2.4 Observation estimates

As mentioned in the introduction, we need to prove certain observability inequalities to achieve null controllability of the system (3.1) and to do so, we need lower bound estimates of the corresponding observation terms (when the control is acting in density or velocity). Looking at the Definition 3.2.1 of the solution to (3.1) (in the sense of transposition), let us first define the observation operators associated to the system (3.1) as follows:

- The observation operator $\mathcal{B}_\rho^* : \mathcal{D}(A^*) \to \mathbb{C}$ to the system (3.1)-(3.2)-(3.3) is defined by

$$\mathcal{B}_\rho^* \Phi := V_0 \xi(2\pi) + Q_0 \eta(2\pi), \quad \text{for } \Phi := (\xi, \eta) \in \mathcal{D}(A^*). \tag{3.33}$$

- The observation operator $\mathcal{B}_u^* : \mathcal{D}(A^*) \to \mathbb{C}$ to the system (3.1)-(3.2)-(3.4) is defined by

$$\mathcal{B}_u^* \Phi := b\xi(2\pi) + V_0 \eta(2\pi) + \mu_0 \eta_x(2\pi), \quad \text{for } \Phi := (\xi, \eta) \in \mathcal{D}(A^*). \tag{3.34}$$

Recall that $\mathcal{E}(A^*)$ denotes the set of all (generalized) eigenfunctions of A^*. The following result proves that these observation terms are non-zero for all $\Phi \in \mathcal{E}(A^*) \setminus \{\Phi_0, \tilde{\Phi}_j,\ j = 0, \pm n_0\}$, and have positive lower bounds for all $n \in \mathbb{Z}^*$.

Lemma 3.2.5. *For all $\Phi_\nu \in \mathcal{E}(A^*) \setminus \{\Phi_0, \tilde{\Phi}_j,\ j = 0, \pm n_0\}$, the observation operators satisfy $\mathcal{B}_\rho^* \Phi_\nu \neq 0$ and $\mathcal{B}_u^* \Phi_\nu \neq 0$. Moreover, we have the following estimates*

$$\left|\mathcal{B}_\rho^* \Phi_n^h\right| \ge C, \quad \left|\mathcal{B}_\rho^* \Phi_n^p\right| \ge C, \tag{3.35}$$

$$\left|\mathcal{B}_u^* \Phi_n^h\right| \ge \frac{C}{|n|}, \quad \left|\mathcal{B}_u^* \Phi_n^p\right| \ge C\,|n|, \tag{3.36}$$

for some $C > 0$ and all $n \in \mathbb{Z}^*$.

Proof. Recall from the proof of Lemma 3.2.3 that eigenvectors $(\alpha_1^n, \alpha_2^n)^\dagger$ and $(\beta_1^n, \beta_2^n)^\dagger$ of the matrix R_n satisfies the following equations:

$$(V_0 in - \nu_n^h)\alpha_1^n + Q_0 in \alpha_2^n = 0, \quad bin\alpha_1^n + (-\mu_0 n^2 + V_0 in - \nu_n^h)\alpha_2^n = 0, \tag{3.37}$$

$$(V_0 in - \nu_n^p)\beta_1^n + Q_0 in \beta_2^n = 0, \quad bin\beta_1^n + (-\mu_0 n^2 + V_0 in - \nu_n^p)\beta_2^n = 0, \tag{3.38}$$

for $n \in \mathbb{Z}$. Also, recall the expressions of $\nu_1^n = \frac{1}{in}\nu_n^p$ and $\nu_2^n = \frac{1}{in}\nu_n^h$. We will use these equation to conclude the proof of this result. Note that

$$\mathcal{B}_\rho^* \Phi_n^h = V_0 \xi_n^h(2\pi) + Q_0 \eta_n^h(2\pi) = V_0 \alpha_1^n + Q_0 \alpha_2^n = \nu_2^n \alpha_1^n \neq 0,$$
$$\mathcal{B}_\rho^* \Phi_n^p = V_0 \xi_n^p(2\pi) + Q_0 \eta_n^p(2\pi) = V_0 \beta_1^n + Q_0 \beta_2^n = \nu_1^n \beta_1^n \neq 0,$$

for all $n \in \mathbb{Z}^*$, thanks to the first equations of (3.37)-(3.38). The estimates on $\mathcal{B}_\rho^* \Phi_n^h$ and $\mathcal{B}_\rho^* \Phi_n^p$ follows directly from the above expressions.

For the parabolic frequencies, we have

$$\mathcal{B}_u^*\Phi_n^h = b\xi_n^h(2\pi) + V_0\eta_n^h(2\pi) + \mu_0(\eta_n^h)_x(2\pi) = b\alpha_1^n + (V_0 + \mu_0 in)\alpha_2^n = \nu_2^n\alpha_2^n \neq 0,$$
$$\mathcal{B}_u^*\Phi_n^p = b\xi_n^p(2\pi) + V_0\eta_n^p(2\pi) + \mu_0(\eta_n^p)_x(2\pi) = b\beta_1^n + (V_0 + \mu_0 in)\beta_2^n = \nu_1^n\beta_2^n \neq 0,$$

for all $n \in \mathbb{Z}^*$, thanks to the second equations in (3.37)-(3.38). Since $\left|\alpha_2^n\right| \geq \frac{C}{|n|}$ and ν_2^n is bounded (away from zero) for all $n \in \mathbb{Z}^*$, the estimate on $\mathcal{B}_u^*\Phi_n^h$ and $\mathcal{B}^*\Phi_n^p$ follows directly from the above expressions. □

Remark 3.2.1. *For the generalized eigenfunction $\tilde{\Phi}_j \in \mathcal{E}(A^*)$ ($j = \pm n_0$), we can choose $\tilde{\alpha}_1^j$ and $\tilde{\alpha}_2^j$ accordingly so that $\mathcal{B}_\rho^*\tilde{\Phi}_j = V_0\tilde{\alpha}_1^j + Q_0\tilde{\alpha}_2^j \neq 0$ and $\mathcal{B}_u^*\tilde{\Phi}_j = b\tilde{\alpha}_1^j + (V_0 + \mu_0 ij)\tilde{\alpha}_2^j \neq 0$ for $j = \pm n_0$.*

3.2.5 Observability inequalities

In this section, we prove our main null controllability results of the system (3.1), namely Theorem 3.1.1 and Theorem 3.1.2. We first state two results which are equivalent to null controllability of the system (3.1) using controls acting in density and velocity respectively. The proofs are standard (see for instance [MZ04, Section 2.3.4],[Zua07, Section 4.3]), so we skip the details.

Theorem 3.2.1. *Let $T > 0$ be given. Then, the system (3.1)-(3.2)-(3.3) is null controllable at time T in the space $(\dot{L}^2(0, 2\pi))^2$ if and only if the inequality*

$$\left\|(\sigma(0), v(0))^\dagger\right\|^2_{(\dot{L}^2(0,2\pi))^2} \leq C\int_0^T |V_0\sigma(t, 2\pi) + Q_0 v(t, 2\pi)|^2\, dt \tag{3.39}$$

holds for all solutions $(\sigma, v)^\dagger$ of the adjoint system (3.20) with terminal data $(\sigma_T, v_T)^\dagger \in \mathcal{D}(A^*)$.

Theorem 3.2.2. *Let $T > 0$ be given. Then, the system (3.1)-(3.2)-(3.4) is null controllable at time T in the space $\dot{H}^1_{\text{per}}(0, 2\pi) \times \dot{L}^2(0, 2\pi)$ if and only if the inequality*

$$\left\|(\sigma(0), v(0))^\dagger\right\|^2_{\dot{H}^{-1}_{\text{per}}(0,2\pi)\times\dot{L}^2(0,2\pi)} \leq C\int_0^T |b\sigma(t, 2\pi) + V_0 v(t, 2\pi) + \mu_0 v_x(t, 2\pi)|^2\, dt \tag{3.40}$$

holds for all solutions $(\sigma, v)^\dagger$ of the adjoint system (3.20) with terminal data $(\sigma_T, v_T)^\dagger \in \mathcal{D}(A^*)$.

The inequalities (3.39) and (3.40) are referred as observability inequalities for the systems (3.1)-(3.2)-(3.3) and (3.1)-(3.2)-(3.4) respectively. To prove these inequalities, we will use the Ingham-type inequality (3.14) to obtain a lower bound of the observation terms (given in the right hand sides of (3.39) and (3.40)) together with the upper bounds of norms of $(\sigma(0), v(0))^\dagger$ in the respective spaces.

Let $\frac{2\sqrt{bQ_0 - V_0^2}}{\mu_0} \notin \mathbb{N}$. We first assume that $\frac{2\sqrt{bQ_0}}{\mu_0} \notin \mathbb{N}$, that is, all the eigenvalues of A^* are simple (Lemma 3.2.4-(iv)), and prove null controllability of the system (3.1) (Theorem 3.1.1-Part(i) and Theorem 3.1.2-Part(i)). In the case of multiple eigenvalues (when $\frac{2\sqrt{bQ_0}}{\mu_0} \in \mathbb{N}$), we give a detailed proof of Theorem 3.1.1-Part(i) at the end of this section. The proof of Theorem 3.1.2-Part(i) in the presence of multiple eigenvalues will be similar to that of Theorem 3.1.1-Part(i) and so we give some comments at the end of this section.

3.2.5.1 The case of simple eigenvalues

Let us assume that $\frac{2\sqrt{bQ_0 - V_0^2}}{\mu_0}, \frac{2\sqrt{bQ_0}}{\mu_0} \notin \mathbb{N}$ and let $(\sigma_T, v_T)^\dagger \in (\dot{L}^2(0, 2\pi))^2$. Since the set of eigenfunctions $\{\Phi_n^h, \Phi_n^p \ ; \ n \in \mathbb{Z}^*\}$ forms a Riesz basis in $(\dot{L}^2(0, 2\pi))^2$ (thanks to Proposition 3.2.3), therefore any $(\sigma_T, v_T)^\dagger \in (\dot{L}^2(0, 2\pi))^2$ can be written as

$$(\sigma_T, v_T)^\dagger = \sum_{n\in\mathbb{Z}^*}\left(a_n^h\Phi_n^h + a_n^p\Phi_n^p\right),$$

for some $(a_n^h)_{n\in\mathbb{Z}^*}, (a_n^p)_{n\in\mathbb{Z}^*} \in \ell_2$. Then the solution to the adjoint system (3.20) is

$$(\sigma(t,x), v(t,x))^\dagger = \sum_{n\in\mathbb{Z}^*} a_n^h e^{\nu_n^h (T-t)} \Phi_n^h + \sum_{n\in\mathbb{Z}^*} a_n^p e^{\nu_n^p (T-t)} \Phi_n^p,$$

for $(t,x) \in (0,T)\times(0,2\pi)$. Thus, we get

$$\sigma(t,x) = Q_0 \sum_{n\in\mathbb{Z}^*} a_n^h e^{\nu_n^h (T-t)} e^{inx} + \sum_{n\in\mathbb{Z}^*} a_n^p e^{\nu_n^p (T-t)} \frac{Q_0}{\nu_1^n - V_0} e^{inx},$$

and

$$v(t,x) = \sum_{n\in\mathbb{Z}^*} a_n^h e^{\nu_n^h (T-t)} (\nu_2^n - V_0) e^{inx} + \sum_{n\in\mathbb{Z}^*} a_n^p e^{\nu_n^p (T-t)} e^{inx},$$

for all $(t,x) \in (0,T)\times(0,2\pi)$.

Estimates on the norms of $(\sigma(0), v(0))^\dagger$: We have

$$\begin{aligned} \left\|(\sigma(0), v(0))^\dagger\right\|^2_{(\dot{L}^2(0,2\pi))^2} &\leq C \left[\sum_{n\in\mathbb{Z}^*} \left|a_n^h\right|^2 \left(1 + \left|\nu_2^n - V_0\right|^2\right) e^{2\mathrm{Re}(\nu_n^h)T} \left\|e^{inx}\right\|^2_{\dot{L}^2(0,2\pi)} \right. \\ &\qquad \left. + \sum_{n\in\mathbb{Z}^*} \left|a_n^p\right|^2 \left(\frac{1}{\left|\nu_1^n - V_0\right|^2} + 1 \right) e^{2\mathrm{Re}(\nu_n^p)T} \left\|e^{inx}\right\|^2_{\dot{L}^2(0,2\pi)} \right] \\ &\leq C \left[\sum_{n\in\mathbb{Z}^*} \left|a_n^h\right|^2 + \sum_{n\in\mathbb{Z}^*} \left|a_n^p\right|^2 e^{2\mathrm{Re}(\nu_n^p)T} \right], \end{aligned} \tag{3.41}$$

since the sequences $1 + \left|\nu_2^n - V_0\right|^2$ and $1 + \frac{1}{\left|\nu_1^n - V_0\right|^2}$ are bounded for all $n \in \mathbb{Z}^*$. We similarly have

$$\left\|(\sigma(0), v(0))^\dagger\right\|^2_{\dot{H}^{-1}_{\mathrm{per}}(0,2\pi)\times\dot{L}^2(0,2\pi)} \leq C \left[\sum_{n\in\mathbb{Z}^*} \left|a_n^h\right|^2 \frac{1}{|n|^2} + \sum_{n\in\mathbb{Z}^*} \left|a_n^p\right|^2 e^{2\mathrm{Re}(\nu_n^p)T} \right], \tag{3.42}$$

since the sequences $\nu_2^n - V_0, \frac{1}{\nu_1^n - V_0} \sim_{+\infty} \frac{1}{n}$. We now find the lower bounds of the respective observation terms and prove our main null controllability results for the barotropic case. We use the Ingham-type inequality (Lemma 3.1.1) to obtain these bounds. First, we show that the eigenvalues $(\nu_n^h)_{n\in\mathbb{Z}^*}$ and $(\nu_n^p)_{n\in\mathbb{Z}^*}$ satisfy all the hypotheses of Lemma 3.1.1. Recall the set of eigenvalues $(\nu_n^h)_{n\in\mathbb{Z}^*}$ and $(\nu_n^p)_{n\in\mathbb{Z}^*}$ of the operator A^*:

$$\begin{aligned} \nu_n^h &= -\frac{1}{2}\left(\mu_0 n^2 - \sqrt{\mu_0^2 n^4 - 4bQ_0 n^2} - 2V_0 in\right), \\ \nu_n^p &= -\frac{1}{2}\left(\mu_0 n^2 + \sqrt{\mu_0^2 n^4 - 4bQ_0 n^2} - 2V_0 in\right), \end{aligned}$$

for $n \in \mathbb{Z}^*$.

- Due to the assumption on the coefficients $(\frac{2\sqrt{bQ_0}}{\mu_0}, \frac{2\sqrt{bQ_0 - V_0^2}}{\mu_0} \notin \mathbb{N})$, we have $\nu_n^h \neq \nu_l^h$, $\nu_n^p \neq \nu_l^p$ for all $n, l \in \mathbb{Z}^*$ with $n \neq l$ and the families are disjoint, that is, $\{\nu_n^h, n \in \mathbb{Z}^*\} \cap \{\nu_n^p, n \in \mathbb{Z}^*\} = \emptyset$, thanks to Lemma 3.2.4.
- We now rewrite ν_n^h as

$$\nu_n^h = -\omega_0 + V_0 in - \omega_0 \frac{\mu_0 n^2 - \sqrt{\mu_0^2 n^4 - 4bQ_0 n^2}}{\mu_0 n^2 + \sqrt{\mu_0^2 n^4 - 4bQ_0 n^2}}, \quad |n| \geq n_0.$$

  This shows that the family $(\nu_n^h)_{n\in\mathbb{Z}^*}$ satisfies hypothesis (H2) of Lemma 3.1.1 with $\beta = -\omega_0, \tau = V_0$ and $e_n = -\omega_0 \frac{\mu_0 n^2 - \sqrt{\mu_0^2 n^4 - 4bQ_0 n^2}}{\mu_0 n^2 + \sqrt{\mu_0^2 n^4 - 4bQ_0 n^2}}$ for $|n| \geq n_0$. Note that $|e_n| \leq \frac{C}{|n|^2}$ and therefore $(e_n)_{|n|\geq n_0} \in \ell_2$.

- On the other hand, we have for all $|n| \geq n_0$

$$\frac{-\mathrm{Re}(\nu_n^p)}{\left|\mathrm{Im}(\nu_n^p)\right|} = \frac{1}{2}\frac{\mu_0 n^2 + \sqrt{\mu_0^2 n^4 - 4bQ_0 n^2}}{V_0 n} \geq \frac{\mu_0}{2V_0},$$

  which verifies hypothesis (P2) of Lemma 3.1.1.

- We now compute for $|n|, |l| \geq n_0$ with $n \neq l$

$$\begin{aligned}\left|\nu_n^p - \nu_l^p\right|^2 &= \frac{1}{4}\left(\mu_0(n^2 - l^2) + \sqrt{\mu_0^2 n^4 - 4bQ_0 n^2} - \sqrt{\mu_0^2 l^4 - 4bQ_0 l^2}\right)^2 + V_0^2(n-l)^2 \\ &\geq \frac{1}{4}\left(\mu_0(n^2 - l^2) + \mu_0 n^2\sqrt{1 - \frac{4bQ_0}{\mu_0^2 n^2}} - \mu_0 l^2\sqrt{1 - \frac{4bQ_0}{\mu_0^2 l^2}}\right)^2.\end{aligned}$$

  Let $|n| > |l|$. Then we have $\mu_0 n^2\sqrt{1 - \frac{4bQ_0}{\mu_0^2 n^2}} > \mu_0 l^2\sqrt{1 - \frac{4bQ_0}{\mu_0^2 l^2}}$, and this implies

$$\left|\nu_n^p - \nu_l^p\right|^2 \geq \frac{\mu_0^2}{4}(n^2 - l^2)^2 \quad\Longrightarrow\quad \left|\nu_n^p - \nu_l^p\right| \geq \frac{\mu_0}{2}(n^2 - l^2).$$

  We similarly have for $|n| < |l|$

$$\left|\nu_n^p - \nu_l^p\right| \geq \frac{\mu_0}{2}(l^2 - n^2).$$

  This proves that $(\nu_n^p)_{|n|\geq n_0}$ satisfies hypothesis (P3) of Lemma 3.1.1 with $r = 2$ and $\delta = \frac{\mu_0}{2}$.

- Finally, we have for $|n| \geq n_0$

$$\begin{aligned}\left|\nu_n^p\right|^2 &= \frac{1}{4}\left(\mu_0 n^2 + \sqrt{\mu_0^2 n^4 - 4bQ_0 n^2}\right)^2 + V_0^2 n^2 \\ &= \frac{\mu_0^2}{4}n^4\left(1 + \sqrt{1 - \frac{4bQ_0}{\mu_0^2 n^2}}\right)^2 + V_0^2 n^2,\end{aligned}$$

  and therefore

$$\frac{\mu_0^2}{4}n^4 \leq \left|\nu_n^p\right|^2 \leq \frac{\mu_0^2}{2}n^4, \quad \forall\, |n| \geq n_0.$$

  This proves that the family $(\nu_n^p)_{|n|\geq n_0}$ satisfies hypothesis (P4) of Lemma 3.1.1 with $\epsilon = \frac{1}{\sqrt{2}}$, $A_0 = 0$ and $B_0 = \frac{\mu_0}{\sqrt{2}} > \delta$.

We are now ready to prove the null controllability results of the system (3.1) in the case of simple eigenvalues.

Proof of Theorem 3.1.1-Part (i): Let $T > \frac{2\pi}{V_0}$. Thanks to Theorem 3.2.1, it is enough to prove the observability inequality (3.39), that is,

$$\int_0^T |V_0\sigma(t,2\pi) + Q_0 v(t,2\pi)|^2\, dt \geq C\left\|(\sigma(0), v(0))^\dagger\right\|^2_{(\dot{L}^2(0,2\pi))^2},$$

for all $(\sigma_T, v_T)^\dagger \in \mathcal{D}(A^*)$. Recall the operator $\mathcal{B}_\rho^*$ given by (3.33). Then, we can write the observation term as

$$\int_0^T |V_0\sigma(t,2\pi) + Q_0 v(t,2\pi)|^2\, dt = \int_0^T \left|\sum_{n\in\mathbb{Z}^*} a_n^h e^{\nu_n^h(T-t)}\mathcal{B}_\rho^*\Phi_n^h + \sum_{n\in\mathbb{Z}^*} a_n^p e^{\nu_n^p(T-t)}\mathcal{B}_\rho^*\Phi_n^p\right|^2 dt.$$

Using the combined parabolic-hyperbolic Ingham type inequality (3.14) (Lemma 3.1.1) and the observation estimates (3.35), we obtain

$$\int_0^T |V_0\sigma(t,2\pi)+Q_0v(t,2\pi)|^2\,dt \geq C\left[\sum_{n\in\mathbb{Z}^*}\left|a_n^h\right|^2 e^{2\mathrm{Re}(\nu_n^h)T}\left|\mathcal{B}_\rho^*\Phi_n^h\right|^2+\sum_{n\in\mathbb{Z}^*}\left|a_n^p\right|^2 e^{2\mathrm{Re}(\nu_n^p)T}\left|\mathcal{B}_\rho^*\Phi_n^p\right|^2\right]$$
$$\geq C\left[\sum_{n\in\mathbb{Z}^*}\left|a_n^h\right|^2+\sum_{n\in\mathbb{Z}^*}\left|a_n^p\right|^2 e^{2\mathrm{Re}(\nu_n^p)T}\right]$$

This estimate together with the norm estimate (3.41), the observability inequality (3.39) follows. This completes the proof in the case of simple eigenvalues. □

Proof of Theorem 3.1.2-Part (i): Let $T>\frac{2\pi}{V_0}$. Similar to the density case, it is enough to prove the observability inequality (3.40), that is,

$$\int_0^T |b\sigma(t,2\pi)+V_0v(t,2\pi)+\mu_0v_x(t,2\pi)|^2\,dt \geq C\left\|(\sigma(0),v(0))^\dagger\right\|^2_{\dot{H}^{-1}_{\mathrm{per}}(0,2\pi)\times\dot{L}^2(0,2\pi)},$$

for all $(\sigma_T,v_T)^\dagger\in\mathcal{D}(A^*)$. We have

$$\int_0^T |b\sigma(t,2\pi)+V_0v(t,2\pi)+\mu_0v_x(t,2\pi)|^2\,dt = \int_0^T\left|\sum_{n\in\mathbb{Z}^*}a_n^h e^{\nu_n^h(T-t)}\mathcal{B}_u^*\Phi_n^h+\sum_{n\in\mathbb{Z}^*}a_n^p e^{\nu_n^p(T-t)}\mathcal{B}_u^*\Phi_n^p\right|^2 dt,$$

where $\mathcal{B}_u^*$ is defined in (3.34). Using the combined parabolic-hyperbolic Ingham type inequality (3.14) (Lemma 3.1.1), we obtain

$$\int_0^T |b\sigma(t,2\pi)+V_0v(t,2\pi)+\mu_0v_x(t,2\pi)|^2\,dt$$
$$\geq C\left[\sum_{n\in\mathbb{Z}^*}\left|a_n^h\right|^2 e^{2\mathrm{Re}(\nu_n^h)T}\left|\mathcal{B}_u^*\Phi_n^h\right|^2+\sum_{n\in\mathbb{Z}^*}\left|a_n^p\right|^2 e^{2\mathrm{Re}(\nu_n^p)T}\left|\mathcal{B}_u^*\Phi_n^p\right|^2\right]$$
$$\geq C\left[\sum_{n\in\mathbb{Z}^*}\left|a_n^h\right|^2\frac{1}{|n|^2}+\sum_{n\in\mathbb{Z}^*}\left|a_n^p\right|^2|n|^2 e^{2\mathrm{Re}(\nu_n^p)T}\right],$$

thanks to the estimate (3.36). Combining this estimate and (3.42), we deduce that

$$\int_0^T |b\sigma(t,2\pi)+V_0v(t,2\pi)+\mu_0v_x(t,2\pi)|^2\,dt \geq C\left\|(\sigma(0),v(0))^\dagger\right\|_{\dot{H}^{-1}_{\mathrm{per}}(0,2\pi)\times\dot{L}^2(0,2\pi)}.$$

This proves the observability inequality (3.40) and hence the proof is complete for simple eigenvalues. □

3.2.5.2 The case of multiple eigenvalues

In this section, we prove null controllability of the system (3.1) in the presence of multiple eigenvalues. The proof will be similar in both cases (control acting in density or velocity), so we present a detailed proof for the density case and give brief details for the velocity case. The proof is inspired from [KL05, Section 4.4] and [CMRR14, Section 4.2] and throughout the proof, we assume the conditions $n_0=\frac{2\sqrt{bQ_0}}{\mu_0}\in\mathbb{N}$ and $\frac{2\sqrt{bQ_0-V_0^2}}{\mu_0}\notin\mathbb{N}$. Then, we only have two multiple eigenvalues $\nu_{n_0}^h=\nu_{n_0}^p=:\nu_{n_0}$ and $\nu_{-n_0}^h=\nu_{-n_0}^p=:\nu_{-n_0}$ with the generalized eigenfunctions $\{\Phi_{n_0},\tilde{\Phi}_{n_0}\}$ and $\{\Phi_{-n_0},\tilde{\Phi}_{-n_0}\}$ respectively, where $\Phi_j:=(\xi_j,\eta_j)^\dagger$ and $\tilde{\Phi}_j:=(\tilde{\xi}_j,\tilde{\eta}_j)^\dagger$ for $j=\pm n_0$.

Control in density. Let $(\sigma_T,v_T)^\dagger\in(\dot{L}^2(0,2\pi))^2$. We decompose it as

$$(\sigma_T,v_T)^\dagger=(\sigma_{T,1},v_{T,1})^\dagger+(\sigma_{T,2},v_{T,2})^\dagger, \tag{3.43}$$

where

$$(\sigma_{T,1}, v_{T,1})^{\dagger} = \sum_{j=\pm n_0} (a_j \Phi_j + \tilde{a}_j \tilde{\Phi}_j)$$

and

$$(\sigma_{T,2}, v_{T,2})^{\dagger} = \sum_{n\in\mathbb{Z}^*\setminus\{\pm n_0\}} (a_n^h \Phi_n^h + a_n^p \Phi_n^p).$$

Let $(\sigma_1, v_1)^{\dagger}$ and $(\sigma_2, v_2)^{\dagger}$ be the solutions of the adjoint system (3.20) with the terminal data $(\sigma_{T,1}, v_{T,1})^{\dagger}$ and $(\sigma_{T,2}, v_{T,2})^{\dagger}$ respectively. Then, we have

$$(\sigma_1, v_1)^{\dagger} = \sum_{j=\pm n_0} e^{\nu_j (T-t)} \left(a_j \Phi_j + (T-t) \tilde{a}_j \tilde{\Phi}_j \right) \tag{3.44}$$

and

$$(\sigma_2, v_2)^{\dagger} = \sum_{n\in\mathbb{Z}^*\setminus\{\pm n_0\}} \left(a_n^h e^{\nu_n^h (T-t)} \Phi_n^h + a_n^p e^{\nu_n^p (T-t)} \Phi_n^p \right) \tag{3.45}$$

In the expression of $(\sigma_2, v_2)^{\dagger}$, all eigenvalues are simple, so we have the following observability inequality

$$\int_0^T |V_0 \sigma_2(t, 2\pi) + Q_0 v_2(t, 2\pi)|^2 \, dt \geq C \left\| (\sigma_2(0), v_2(0))^{\dagger} \right\|^2_{(\dot{L}^2(0,2\pi))^2}. \tag{3.46}$$

Note that $V_0 \sigma_1(t, 2\pi) + Q_0 v_1(t, 2\pi) = \sum_{j=\pm n_0} e^{\nu_j (T-t)} \left(a_j \mathcal{B}_\rho^* \Phi_j + (T-t) \tilde{a}_j \mathcal{B}_\rho^* \tilde{\Phi}_j \right)$. We first add the term $e^{\nu_{n_0} (T-t)} \left(a_{n_0} \mathcal{B}_\rho^* \Phi_{n_0} + (T-t) \tilde{a}_{n_0} \mathcal{B}_\rho^* \tilde{\Phi}_{n_0} \right)$ in the above inequality. Denote

$$\mathcal{Y}(t) := V_0 \sigma_2(t, 2\pi) + Q_0 v_2(t, 2\pi) + e^{\nu_{n_0} (T-t)} \left(a_{n_0} \mathcal{B}_\rho^* \Phi_{n_0} + (T-t) \tilde{a}_{n_0} \mathcal{B}_\rho^* \tilde{\Phi}_{n_0} \right)$$

and

$$\mathcal{Z}(t) := \mathcal{Y}(t) - \frac{1}{2\delta} \int_{-\delta}^{\delta} e^{\nu_{n_0} s} \mathcal{Y}(t+s) ds$$

for $t \in (\delta, T-\delta)$ with $\delta > 0$ (chosen later accordingly). Then, we have the following estimate (see [KL05, Section 4.4] for details).

$$\int_\delta^{T-\delta} |\mathcal{Z}(t)|^2 \, dt \leq C \int_0^T |\mathcal{Y}(t)|^2 \, dt. \tag{3.47}$$

We now prove that

$$\int_\delta^{T-\delta} |\mathcal{Z}(t)|^2 \, dt \geq C \left\| (\sigma_2(0), v_2(0))^{\dagger} \right\|^2_{(\dot{L}^2(0,2\pi))^2}. \tag{3.48}$$

From the expression of $\mathcal{Y}(t)$, we can get

$$\begin{aligned}\mathcal{Z}(t) = &\sum_{n\in\mathbb{Z}^*\setminus\{\pm n_0\}} a_n^h e^{\nu_n^h (T-t)} \mathcal{B}_\rho^* \Phi_n^h \left(1 - \frac{\sinh((\nu_n^h - \nu_{n_0})\delta)}{(\nu_n^h - \nu_{n_0})\delta} \right) \\ &+ \sum_{n\in\mathbb{Z}^*\setminus\{\pm n_0\}} a_n^p e^{\nu_n^p (T-t)} \mathcal{B}_\rho^* \Phi_n^p \left(1 - \frac{\sinh((\nu_n^p - \nu_{n_0})\delta)}{(\nu_n^p - \nu_{n_0})\delta} \right)\end{aligned}$$

Since $\inf_{n\in\mathbb{Z}^*\setminus\{\pm n_0\}} \left| \nu_n^h - \nu_{n_0} \right| > 0$ and $\inf_{n\in\mathbb{Z}^*\setminus\{\pm n_0\}} \left| \nu_n^p - \nu_{n_0} \right| > 0$, we have (for appropriate $\delta > 0$)

$$\inf_{n\in\mathbb{Z}^*\setminus\{\pm n_0\}} \left| 1 - \frac{\sinh((\nu_n^h - \nu_{n_0})\delta)}{(\nu_n^h - \nu_{n_0})\delta} \right| > 0, \text{ and } \inf_{n\in\mathbb{Z}^*\setminus\{\pm n_0\}} \left| 1 - \frac{\sinh((\nu_n^p - \nu_{n_0})\delta)}{(\nu_n^p - \nu_{n_0})\delta} \right| > 0.$$

Since $T > \frac{2\pi}{V_0}$, we can choose δ small enough such that $T - 2\delta > \frac{2\pi}{V_0}$. Applying Ingham-type inequality (3.14) (for simple eigenvalues), we obtain

$$\begin{aligned}\int_{\delta}^{T-\delta} |\mathcal{Z}(t)|^2\, dt &\geq C \left[\sum_{n\in\mathbb{Z}^*\setminus\{\pm n_0\}} \left|a_n^h\right|^2 e^{2\mathrm{Re}(\nu_n^h)T} \left|\mathcal{B}_\rho^* \Phi_n^h\right|^2 + \sum_{n\in\mathbb{Z}^*\setminus\{\pm n_0\}} \left|a_n^p\right|^2 e^{2\mathrm{Re}(\nu_n^p)T} \left|\mathcal{B}_\rho^* \Phi_n^p\right|^2 \right] \\ &\geq C \left\|(\sigma_2(0), v_2(0))^\dagger\right\|^2_{(\dot{L}^2(0,2\pi))^2}.\end{aligned}$$

Therefore, using the estimate (3.47), we obtain

$$\int_0^T |\mathcal{Y}(t)|^2\, dt \geq C \left\|(\sigma_2(0), v_2(0))^\dagger\right\|^2_{(\dot{L}^2(0,2\pi))^2}. \tag{3.49}$$

Since $T > \frac{2\pi}{V_0}$, we can choose $\epsilon > 0$ such that $T - \epsilon > \frac{2\pi}{V_0}$. Therefore we can write

$$\int_\epsilon^T |\mathcal{Y}(t)|^2\, dt \geq C \left\|(\sigma_2(\epsilon), v_2(\epsilon))^\dagger\right\|^2_{(\dot{L}^2(0,2\pi))^2}$$

and thus

$$\int_0^T |\mathcal{Y}(t)|^2\, dt \geq C \int_\epsilon^T |\mathcal{Y}(t)|^2\, dt \geq C \left\|(\sigma_2(\epsilon), v_2(\epsilon))^\dagger\right\|^2_{(\dot{L}^2(0,2\pi))^2}. \tag{3.50}$$

Thanks to the well-posedness result (Lemma 3.2.2) of the adjoint system (3.20), we have

$$\int_0^\epsilon |V_0\sigma_2(t,2\pi) + Q_0 v_2(t,2\pi)|^2\, dt \leq C \left\|(\sigma_2(\epsilon), v_2(\epsilon))^\dagger\right\|^2_{(\dot{L}^2(0,2\pi))^2}. \tag{3.51}$$

From equations (3.50) and (3.51), we deduce that

$$\int_0^T |\mathcal{Y}(t)|^2\, dt \geq C \int_0^\epsilon |V_0\sigma_2(t,2\pi) + Q_0 v_2(t,2\pi)|^2\, dt \tag{3.52}$$

Using this inequality, we obtain

$$\begin{aligned}&\int_0^\epsilon \left| e^{\nu_{n_0}(T-t)} \left(a_{n_0} \mathcal{B}_\rho^* \Phi_{n_0} + (T-t)\tilde{a}_{n_0} \mathcal{B}_\rho^* \tilde{\Phi}_{n_0} \right) \right|^2 dt \\ &\leq C \int_0^\epsilon |\mathcal{Y}(t)|^2\, dt + C \int_0^\epsilon |V_0\sigma_2(t,2\pi) + Q_0 v_2(t,2\pi)|^2\, dt \\ &\leq C \int_0^T |\mathcal{Y}(t)|^2\, dt\end{aligned} \tag{3.53}$$

We now prove that

$$\int_0^\epsilon \left| e^{\nu_{n_0}(T-t)} \left(a_{n_0} \mathcal{B}_\rho^* \Phi_{n_0} + (T-t)\tilde{a}_{n_0} \mathcal{B}_\rho^* \tilde{\Phi}_{n_0} \right) \right|^2 dt \geq C \left(\left|a_{n_0}\right|^2 + \left|\tilde{a}_{n_0}\right|^2 \right) \tag{3.54}$$

Denote the finite dimensional space

$$\mathcal{X} := \mathrm{span}\left\{\Phi_{n_0}, \tilde{\Phi}_{n_0}\right\}$$

and define norms on $\mathcal{X}$:

$$\begin{aligned}\left\|(\hat{\sigma}_{T,1}, \hat{v}_{T,1})^\dagger\right\|_1^2 &:= \int_0^\epsilon \left| e^{\nu_{n_0}(T-t)} \left(a_{n_0} \mathcal{B}_\rho^* \Phi_{n_0} + (T-t)\tilde{a}_{n_0} \mathcal{B}_\rho^* \tilde{\Phi}_{n_0} \right) \right|^2 dt, \\ \left\|(\hat{\sigma}_{T,1}, \hat{v}_{T,1})^\dagger\right\|_2^2 &:= \left\|(\hat{\sigma}_1(0), \hat{v}_1(0))^\dagger\right\|^2_{(\dot{L}^2(0,2\pi))^2},\end{aligned}$$

where $(\hat{\sigma}_1(t), \hat{v}_1(t))^\dagger = e^{\nu_{n_0}(T-t)} \left(a_{n_0}\Phi_{n_0} + (T-t)\tilde{a}_{n_0}\tilde{\Phi}_{n_0} \right)$ for $t \in (0,T)$ is the solution of the adjoint system with terminal data $(\hat{\sigma}_{T,1}, \hat{v}_{T,1})^\dagger \in \mathcal{X}$. In fact, $\left\|(\hat{\sigma}_{T,1}, \hat{v}_{T,1})^\dagger\right\|_1 = 0$ implies $\mathcal{B}_\rho^*\Phi_{n_0} = \mathcal{B}_\rho^*\tilde{\Phi}_{n_0} = 0$.

This gives $\Phi_{n_0} = \tilde{\Phi}_{n_0} = 0$ (thanks to Lemma 3.2.5 - Remark 3.2.1) and hence $(\hat{\sigma}_{T,1}, \hat{v}_{T,1}) = (0,0)$. Also, $(\hat{\sigma}_1(0), \hat{v}_1(0)) = (0,0)$ implies $\Phi_{n_0} = \tilde{\Phi}_{n_0} = 0$ and consequently $(\hat{\sigma}_1, \hat{v}_1) = (0,0)$.
Since any two norms in a finite dimensional space are equivalent, we can write

$$\int_0^{\epsilon} \left| e^{\nu_{n_0}(T-t)} \left(a_{n_0} \mathcal{B}_\rho^* \Phi_{n_0} + (T-t) \tilde{a}_{n_0} \mathcal{B}_\rho^* \tilde{\Phi}_{n_0} \right) \right|^2 dt \geq C \left\| (\hat{\sigma}_1(0), \hat{v}_1(0))^\dagger \right\|^2_{(\dot{L}^2(0,2\pi))^2},$$

proving the inequality (3.54). Hence, using (3.53), we finally obtain

$$\int_0^T |\mathcal{Y}(t)|^2 \, dt \geq C \left\| (\hat{\sigma}_1(0), \hat{v}_1(0))^\dagger \right\|^2_{(\dot{L}^2(0,2\pi))^2}. \tag{3.55}$$

This inequality, together with (3.49) implies

$$\begin{aligned} \int_0^T |\mathcal{Y}(t)|^2 \, dt &\geq C \left[\left\| (\hat{\sigma}_1(0), \hat{v}_1(0))^\dagger \right\|^2_{(\dot{L}^2(0,2\pi))^2} + \left\| (\sigma_2(0), v_2(0))^\dagger \right\|^2_{(\dot{L}^2(0,2\pi))^2} \right] \\ &\geq C \left\| (\sigma_2(0) + \hat{\sigma}_1(0), v_2(0) + \hat{v}_1(0))^\dagger \right\|^2_{(\dot{L}^2(0,2\pi))^2}. \end{aligned}$$

Proceeding in a similar way, we can add the term $e^{\nu_{-n_0}(T-t)} \left(a_{-n_0} \mathcal{B}_\rho^* \Phi_{-n_0} + (T-t) \tilde{a}_{-n_0} \mathcal{B}_\rho^* \tilde{\Phi}_{-n_0} \right)$ and obtain the desired observability inequality

$$\int_0^T |V_0 \sigma(t, 2\pi) + Q_0 v(t, 2\pi)|^2 \, dt \geq C \left\| (\sigma(0), v(0))^\dagger \right\|^2_{(\dot{L}^2(0,2\pi))^2}.$$

This completes the proof of Theorem 3.1.1-Part (i) in the case of multiple eigenvalues. □

Control in velocity. The proof of Theorem 3.1.2-Part (i) (control acting in the velocity component) in the case of multiple eigenvalues can be done in a similar way as above. The only missing part is the following admissibility condition (see the inequality (3.51))

$$\int_0^{\epsilon} |b\sigma_2(t, 2\pi) + V_0 v_2(t, 2\pi) + \mu_0 (v_2)_x(t, 2\pi)|^2 \, dt \leq C \left\| (\sigma_2(\epsilon), v_2(\epsilon))^\dagger \right\|_{\dot{H}^{-1}_{\text{per}}(0,2\pi) \times \dot{L}^2(0,2\pi)}. \tag{3.56}$$

The terminal data $(\sigma_2, v_2) \in \dot{H}^{-1}_{\text{per}}(0, 2\pi) \times \dot{L}^2(0, 2\pi)$ is less regular and so one cannot expect that the observation term $b\sigma_2(\cdot, 2\pi) + V_0 v_2(\cdot, 2\pi) + \mu_0 (v_2)_x(\cdot, 2\pi) \in L^2(0, \epsilon)$ for some $\epsilon > 0$. This is the main difficulty of boundary controllability in comparison with the distributed controllability. In this context, we refer to [CMRR14, Equation (4.43)], where one can easily have the admissibility condition due to the internal control. However, in our setup, we can obtain a slightly modified estimate (weak admissibility) to (3.56) as follows:

$$\int_0^{\frac{\epsilon}{2}} |b\sigma_2(t, 2\pi) + V_0 v_2(t, 2\pi) + \mu_0 (v_2)_x(t, 2\pi)|^2 \, dt \leq C \left\| (\sigma_2(\epsilon), v_2(\epsilon))^\dagger \right\|^2_{\dot{H}^{-1}_{\text{per}}(0,2\pi) \times \dot{L}^2(0,2\pi)}. \tag{3.57}$$

Using this inequality (3.57) and proceeding similarly as before, we can obtain the required observability inequality (3.40) in the presence of multiple eigenvalues. Thus, the only technical part is to prove the inequality (3.57), which we prove below:
Recall the expression of $(\sigma_2, v_2)^\dagger$ given by (3.45). We compute

$$\begin{aligned} &\int_0^{\frac{\epsilon}{2}} |b\sigma_2(t, 2\pi) + V_0 v_2(t, 2\pi) + \mu_0 (v_2)_x(t, 2\pi)|^2 \, dt \\ &\leq \int_0^{\frac{\epsilon}{2}} \left| \sum_{n \in \mathbb{Z}^* \setminus \{\pm n_0\}} \left(a_n^h e^{\nu_n^h (T-t)} \mathcal{B}_u^* \Phi_n^h + a_n^p e^{\nu_n^p (T-t)} \mathcal{B}_u^* \Phi_n^p \right) \right|^2 dt \\ &\leq \int_0^{\frac{\epsilon}{2}} \left| \sum_{n \in \mathbb{Z}^* \setminus \{\pm n_0\}} a_n^h e^{\nu_n^h (T-t)} \mathcal{B}_u^* \Phi_n^h \right|^2 dt + \int_0^{\frac{\epsilon}{2}} \left| \sum_{n \in \mathbb{Z}^* \setminus \{\pm n_0\}} a_n^p e^{\nu_n^p (T-t)} \mathcal{B}_u^* \Phi_n^p \right|^2 dt \end{aligned}$$

Note that

$$\int_0^{\frac{\epsilon}{2}} \left| \sum_{n\in\mathbb{Z}^*\setminus\{\pm n_0\}} a_n^h e^{\nu_n^h(T-t)} \mathcal{B}_u^* \Phi_n^h \right|^2 dt \leq C \sum_{n\in\mathbb{Z}^*\setminus\{\pm n_0\}} \left| a_n^h \mathcal{B}_u^* \Phi_n^h \right|^2 \leq C \sum_{n\in\mathbb{Z}^*\setminus\{\pm n_0\}} \frac{\left|a_n^h\right|^2}{|n|^2}, \tag{3.58}$$

thanks to the inequality (3.16) (right side). Note that the estimate $\left|\mathcal{B}_u^*\Phi_n^h\right| \leq \frac{C}{|n|}$ follows due to the fact that $\mathcal{B}_u^*\Phi_n^h = \nu_2^n \alpha_2^n$ for all $n \in \mathbb{Z}^*$ (see the proof of Lemma 3.2.5). For the parabolic part, we apply Hölder's inequality to obtain

$$\begin{aligned}
&\int_0^{\frac{\epsilon}{2}} \left| \sum_{n\in\mathbb{Z}^*\setminus\{\pm n_0\}} a_n^p e^{\nu_n^p(T-t)} \mathcal{B}_u^* \Phi_n^p \right|^2 dt \\
&\leq \sum_{n\in\mathbb{Z}^*\setminus\{\pm n_0\}} \left|a_n^p\right|^2 e^{2\mathrm{Re}(\nu_n^p)(T-\epsilon)} \sum_{n\in\mathbb{Z}^*\setminus\{\pm n_0\}} \left|\mathcal{B}_u^*\Phi_n^p\right|^2 e^{-2\mathrm{Re}(\nu_n^p)(T-\epsilon)} \int_0^{\frac{\epsilon}{2}} e^{2\mathrm{Re}(\nu_n^p)(T-t)} dt \\
&\leq \sum_{n\in\mathbb{Z}^*\setminus\{\pm n_0\}} \left|a_n^p\right|^2 e^{2\mathrm{Re}(\nu_n^p)(T-\epsilon)} \sum_{n\in\mathbb{Z}^*\setminus\{\pm n_0\}} \left|\mathcal{B}_u^*\Phi_n^p\right|^2 e^{\mathrm{Re}(\nu_n^p)\epsilon} \\
&\leq C \sum_{n\in\mathbb{Z}^*\setminus\{\pm n_0\}} \left|a_n^p\right|^2 e^{2\mathrm{Re}(\nu_n^p)(T-\epsilon)},
\end{aligned}$$

as we have $\mathrm{Re}(\nu_n^p) < 0$ for all $n \in \mathbb{Z}^*$. Combining these two estimates, we obtain

$$\begin{aligned}
&\int_0^{\frac{\epsilon}{2}} |b\sigma_2(t,2\pi) + V_0 v_2(t,2\pi) + \mu_0 (v_2)_x(t,2\pi)|^2 \, dt \\
&\qquad \leq C \sum_{n\in\mathbb{Z}^*\setminus\{\pm n_0\}} \frac{\left|a_n^h\right|^2}{|n|^2} + C \sum_{n\in\mathbb{Z}^*\setminus\{\pm n_0\}} \left|a_n^p\right|^2 e^{2\mathrm{Re}(\nu_n^p)(T-\epsilon)}
\end{aligned} \tag{3.59}$$

On the other hand (recall the expression given by (3.45) and (3.24)), we have

$$\begin{aligned}
&\left\|(\sigma_2(\epsilon), v_2(\epsilon))^\dagger\right\|^2_{\dot{H}^{-1}_{\mathrm{per}}(0,2\pi)\times\dot{L}^2(0,2\pi)} \\
&= \sum_{n\in\mathbb{Z}^*\setminus\{\pm n_0\}} \left(\frac{b}{|n|^2} \left| a_n^h e^{\nu_n^h(T-\epsilon)} Q_0 + a_n^p e^{\nu_n^p(T-\epsilon)} \frac{Q_0}{\nu_1^n - V_0} \right|^2 + Q_0 \left| a_n^h e^{\nu_n^h(T-\epsilon)} (\nu_2^n - V_0) + a_n^p e^{\nu_n^p(T-\epsilon)} \right|^2 \right)
\end{aligned}$$

Since $\nu_1^n - V_0 \sim_{+\infty} n$ and $\nu_2^n - V_0 \sim_{+\infty} \frac{1}{n}$, we deduce that for N large enough

$$\begin{aligned}
&\sum_{|n|>N} \left(\frac{b}{|n|^2} \left| a_n^h e^{\nu_n^h(T-\epsilon)} Q_0 + a_n^p e^{\nu_n^p(T-\epsilon)} \frac{Q_0}{\nu_1^n - V_0} \right|^2 + Q_0 \left| a_n^h e^{\nu_n^h(T-\epsilon)} (\nu_2^n - V_0) + a_n^p e^{\nu_n^p(T-\epsilon)} \right|^2 \right) \\
&\qquad \geq C \sum_{|n|>N} \frac{\left|a_n^h\right|^2}{|n|^2} + C \sum_{|n|>N} \left|a_n^p\right|^2 e^{2\mathrm{Re}(\nu_n^p)(T-\epsilon)}.
\end{aligned} \tag{3.60}$$

This can be seen from the following inequality:

$$\frac{1}{|n|^2} \left| z_n + \frac{w_n}{n} \right|^2 + \left| \frac{z_n}{n} + w_n \right|^2 \geq C \left(\frac{|z_n|^2}{|n|^2} + |w_n|^2 \right), \tag{3.61}$$

for all complex sequences $(z_n)_{|n|>N}$ and $(w_n)_{|n|>N}$, where $N \in \mathbb{N}$ is arbitrarily large number. Indeed, if $z_n, w_n \neq 0$ for all $|n| > N$, then we can write

$$\frac{1}{|n|^2} \left| z_n + \frac{w_n}{n} \right|^2 + \left| \frac{z_n}{n} + w_n \right|^2 = \frac{|z_n|^2}{|n|^2} \left| 1 + \frac{w_n}{n z_n} \right|^2 + |w_n|^2 \left| \frac{z_n}{n w_n} + 1 \right|^2, \quad |n| > N.$$

If $\inf_{|n|>N}\left|1+\frac{w_n}{nz_n}\right| > 0$ and $\inf_{|n|>N}\left|\frac{z_n}{nw_n}+1\right| > 0$, the inequality (3.61) is obvious. Let us assume $\inf_{|n|>N}\left|1+\frac{w_n}{nz_n}\right| = 0$, then we can write (up to a subsequence) $1+\frac{w_n}{nz_n} = \delta_n$ where $\delta_n \to 0$ as $n \to \infty$. This implies $\frac{w_n}{z_n} = n(-1+\delta_n)$ for all $n \in \mathbb{N}$ and therefore

$$\left|\frac{z_n}{n}+w_n\right|^2 = |z_n|^2\left|\frac{1}{n}+\frac{w_n}{z_n}\right|^2 = |z_n|^2\left|\frac{1}{n}-n(-1+\delta_n)\right|^2 \geq C\,|n|^2\,|z_n|^2 .$$

On the other hand, we have $|n|^2\,|z_n|^2 = \frac{|n|^2}{2}\,|z_n|^2 + \frac{|n|^2}{2}\,|z_n|^2 \geq \frac{|n|^2}{2}\,|z_n|^2 + C\,|w_n|^2$, proving the inequality (3.61). Similarly, we can prove (3.61) in the case $\inf_{|n|>N}\left|\frac{z_n}{nw_n}+1\right| = 0$.

Now, adding finitely many terms in the estimate (3.60) (or, one can include these finitely many terms in $(\sigma_1, v_1)^\dagger$ part), we get that

$$\left\|(\sigma_2(\epsilon), v_2(\epsilon))^\dagger\right\|^2_{\dot{H}^{-1}_{\text{per}}(0,2\pi)\times\dot{L}^2(0,2\pi)} \geq C\left(\sum_{n\in\mathbb{Z}^*\setminus\{\pm n_0\}} \frac{\left|a^h_n\right|^2}{|n|^2} + \sum_{n\in\mathbb{Z}^*\setminus\{\pm n_0\}} \left|a^p_n\right|^2 e^{2\text{Re}(\nu^p_n)(T-\epsilon)}\right). \tag{3.62}$$

With this, the inequality (3.57) follows. □

3.2.6 Lack of null controllability for less regular initial states

We first write the following result, the proof of which is standard and so we skip the details (see Theorem 3.2.2).

Proposition 3.2.4. *Let $0 \leq s < 1$ and $T > 0$ be given. Then, the system (3.1)-(3.2)-(3.4) is null controllable at time T in the space $\dot{H}^s_{\text{per}}(0,2\pi)\times\dot{L}^2(0,2\pi)$ if and only if the inequality*

$$\left\|(\sigma(0), v(0))^\dagger\right\|^2_{\dot{H}^{-s}_{\text{per}}(0,2\pi)\times\dot{L}^2(0,2\pi)} \leq C\int_0^T \left|b\sigma(t,2\pi)+V_0 v(t,2\pi)+\mu_0 v_x(t,2\pi)\right|^2 dt \tag{3.63}$$

holds for all solutions $(\sigma, v)^\dagger$ of the adjoint system (3.20) with terminal data $(\sigma_T, v_T)^\dagger \in \mathcal{D}(A^*)$.

To prove Theorem 3.1.2-Part (ii), it is enough to find a sequence of terminal data $(\sigma^n_T, v^n_T)_{n\in\mathbb{Z}^*} \in \mathcal{D}(A^*)$ for which the observability inequality (3.63) fails. We will show below that the eigenfunctions corresponding to the hyperbolic branch of eigenvalues helps us disprove this observability inequality.

3.2.6.1 Proof of Theorem 3.1.2-Part (ii)

For $(\sigma^n_T, v^n_T)^\dagger = \Phi^h_n$, the solution to the adjoint system (3.20) is

$$(\sigma^n(t,x), v^n(t,x))^\dagger = e^{\nu^h_n(T-t)}\Phi^h_n(x),$$

for $(t,x) \in (0,T)\times(0,2\pi)$ and $n \in \mathbb{Z}^*$. Recall the expression of Φ^h_n from (3.24). For all $n \in \mathbb{Z}^*$, we have the following estimate

$$\left\|\Phi^h_n\right\|_{\dot{H}^{-s}_{\text{per}}(0,2\pi)\times\dot{L}^2(0,2\pi)} \geq \frac{C}{|n|^s},$$

and therefore

$$\left\|(\sigma^n(0), v^n(0))^\dagger\right\|^2_{\dot{H}^{-s}_{\text{per}}(0,2\pi)\times\dot{L}^2(0,2\pi)} \geq \frac{C}{|n|^{2s}}$$

for all $n \in \mathbb{Z}^*$, since $\text{Re}(\nu^h_n)$ is bounded. On the other hand, we have the upper bound of the observation term

$$\int_0^T \left|b\sigma^n(t,2\pi)+V_0 v^n(t,2\pi)+\mu_0 v^n_x(t,2\pi)\right|^2 dt \leq \frac{C}{|n|^2},$$

for all $n \in \mathbb{Z}^*$ (see (3.58) for instance). Thus, if the observability inequality (3.63) holds, then one must have

$$\frac{C}{|n|^{2s}} \leq \frac{C}{|n|^2} \implies |n|^{2-2s} \leq C,$$

which is not possible due to our assumption $0 \leq s < 1$. This completes the proof. □

3.2.7 Lack of controllability at small time

We prove that the system (3.6) is not null controllable in $\dot{L}^2(0,2\pi)$ when the time is small, that is, Theorem 3.1.1-Part (ii). We construct an approximate solution for the corresponding transport equation. The idea of constructing an approximate solution for the transport equation was addressed in [BKLB20], where the authors proved a lack of null controllability result at a small time in the case of an interior control (acting only on the transport equation). Very recently, in [CDM23, Section 6], this approach has been applied to a coupled transport-elliptic system in the case of a boundary control (acts in density). We will follow mainly the proof given in [CDM23] to prove our lack of null controllability result when the time is small.

3.2.7.1 Proof of Theorem 3.1.1-Part (ii)

Let $0 < T < \frac{2\pi}{V_0}$. We first consider the transport equation

$$\begin{cases} \tilde{\sigma}_t(t,x) + V_0\tilde{\sigma}_x(t,x) - \dfrac{bQ_0}{\mu_0}\tilde{\sigma}(t,x) = 0, & (t,x) \in (0,T)\times(0,2\pi), \\ \tilde{\sigma}(t,0) = \tilde{\sigma}(t,2\pi), \quad t \in (0,T), \\ \tilde{\sigma}(T,x) = \tilde{\sigma}_T(x), \quad x \in (0,2\pi) \end{cases} \tag{3.64}$$

with $\tilde{\sigma}_T \in \dot{L}^2(0,2\pi)$. Since $V_0T < 2\pi$, there exists a nontrivial function $\tilde{\sigma}_T \in C^\infty(0,2\pi)$ with $\operatorname{supp}(\tilde{\sigma}_T) \subset (V_0T, 2\pi)$ such that the solution $\tilde{\sigma}$ of (3.64) satisfies $\tilde{\sigma}(t,0) = \tilde{\sigma}(t,2\pi) = 0$ for all $t \in (0,T)$, but $\tilde{\sigma}$ is not identically zero in $(0,T)\times(0,2\pi)$. Let $N > 0$ be a fixed integer. We define the polynomial

$$P^N(x) := \prod_{\substack{l=-N \\ l\neq 0}}^{N} (x-l), \quad x \in (0,2\pi) \tag{3.65}$$

and the function

$$\tilde{\sigma}_T^N := P^N\left(-i\frac{d}{dx}\right)\tilde{\sigma}_T. \tag{3.66}$$

Since $\tilde{\sigma}_T \in \dot{L}^2(0,2\pi)$, we can write

$$\tilde{\sigma}_T(x) := \sum_{n\in\mathbb{Z}^*} a_n e^{inx}, \quad x \in (0,2\pi),$$

where $(a_n)_{n\in\mathbb{Z}^*} \in \ell_2$. Using the definition of P^N given by (3.65), we get from (3.66) that

$$\tilde{\sigma}_T^N(x) = \sum_{n\in\mathbb{Z}^*} a_n \prod_{\substack{l=-N \\ l\neq 0}}^{N} \left(-i\frac{d}{dx} - l\right) e^{inx} = \sum_{n\in\mathbb{Z}^*} a_n \prod_{\substack{l=-N \\ l\neq 0}}^{N} (n-l)\, e^{inx} = \sum_{n\in\mathbb{Z}^*} a_n P^N(n) e^{inx},$$

for $x \in (0,2\pi)$. Note that $P^N(n) = 0$ for all $0 < |n| \le N$ and therefore

$$\tilde{\sigma}_T^N(x) = \sum_{|n|\ge N+1} a_n P^N(n) e^{inx}, \quad x \in (0,2\pi).$$

With this $\tilde{\sigma}_T^N$, let us now consider the following system

$$\begin{cases} \tilde{\sigma}_t + V_0\tilde{\sigma}_x = \dfrac{bQ_0}{\mu_0}\tilde{\sigma}, & \text{in } (0,T)\times(0,2\pi), \\ \tilde{\sigma}(t,0) = \tilde{\sigma}(t,2\pi), & \text{for } t \in (0,T), \\ \tilde{\sigma}(T,x) = \tilde{\sigma}_T^N(x), & \text{in } (0,2\pi). \end{cases} \tag{3.67}$$

Since $\operatorname{supp}(\tilde{\sigma}_T^N) \subset \operatorname{supp}(\tilde{\sigma}_T) \subset (V_0T, 2\pi)$, the solution $\tilde{\sigma}^N$ of (3.67) satisfies $\tilde{\sigma}^N(t,0) = \tilde{\sigma}^N(t,2\pi) = 0$ for all $t \in (0,T)$. We now consider the following adjoint system

$$\begin{cases} \sigma_t + V_0\sigma_x + Q_0 v_x = 0, & \text{in } (0,T)\times(0,2\pi),\\ v_t - \mu_0 v_{xx} + V_0 v_x + b\sigma_x = 0, & \text{in } (0,T)\times(0,2\pi),\\ \sigma(t,0) = \sigma(t,2\pi), & \text{for } t\in(0,T),\\ v(t,0) = v(t,2\pi), \quad v_x(t,0) = v_x(t,2\pi), & \text{for } t\in(0,T),\\ \sigma(T,x) = \tilde{\sigma}_T^N(x), \quad v(T,x) = v_T^N(x), & \text{in } (0,2\pi), \end{cases} \tag{3.68}$$

where we choose v_T^N such that

$$(\tilde{\sigma}_T^N, v_T^N)^\dagger = \sum_{|n|\geq N+1} \tilde{a}_n^h \Phi_n^h$$

with $\tilde{a}_n^h Q_0 := a_n P^N(n)$ for all $|n| \geq N+1$. We write the solutions to the systems (3.67) and (3.68) respectively as

$$\tilde{\sigma}^N(t,x) = \sum_{|n|\geq N+1} a_n P^N(n) e^{(V_0 in - \frac{bQ_0}{\mu_0})(T-t)} e^{inx}, \tag{3.69}$$

$$\sigma^N(t,x) = \sum_{|n|\geq N+1} a_n P^N(n) e^{\nu_n^h (T-t)} e^{inx}, \tag{3.70}$$

$$v^N(t,x) = \sum_{|n|\geq N+1} a_n P^N(n) \frac{\nu_2^n - V_0}{Q_0} e^{\nu_n^h (T-t)} e^{inx}, \tag{3.71}$$

for $(t,x) \in [0,T]\times[0,2\pi]$. We prove that the solution component σ^N of (3.68) approximates the solution $\tilde{\sigma}^N$ of (3.67). Indeed,

$$\begin{aligned} &\left\|\sigma^N(\cdot,x) - \tilde{\sigma}^N(\cdot,x)\right\|^2_{L^2(0,T)}\\ &\leq \sum_{|n|\geq N+1} |a_n|^2 \left|P^N(n)\right|^2 \left\| e^{\nu_n^h(T-t)} - e^{\left(V_0 in - \frac{bQ_0}{\mu_0}\right)(T-t)}\right\|^2_{L^2(0,T)}\\ &\leq \sum_{|n|\geq N+1} |a_n|^2 \left|P^N(n)\right|^2 \left\| e^{V_0 in(T-t)} e^{-\frac{\mu_0 n}{2}(n - \sqrt{n^2 - \frac{4bQ_0}{\mu_0^2}})(T-t)} - e^{(V_0 in - \frac{bQ_0}{\mu_0})(T-t)}\right\|^2_{L^2(0,T)}\\ &\leq \sum_{|n|\geq N+1} |a_n|^2 \left|P^N(n)\right|^2 \left\| e^{-\frac{\mu_0 n}{2}\left(n - \sqrt{n^2 - \frac{4bQ_0}{\mu_0^2}}\right)(T-t)} - e^{-\frac{bQ_0}{\mu_0}(T-t)}\right\|^2_{L^2(0,T)}\\ &\leq \sum_{|n|\geq N+1} \frac{1}{|n|^2} |a_n|^2 \left|P^N(n)\right|^2, \end{aligned}$$

for all $x \in [0,2\pi]$ and therefore

$$\left\|\sigma^N(\cdot,x) - \tilde{\sigma}^N(\cdot,x)\right\|^2_{L^2(0,T)} \leq \frac{C}{|N|^2} \sum_{|n|\geq N+1} |a_n|^2 \left|P^N(n)\right|^2,$$

for all $x \in [0,2\pi]$. We also find L^2- estimate of the solution component v^N. We have for all $x \in [0,2\pi]$

$$\begin{aligned} \left\|v^N(\cdot,x)\right\|^2_{L^2(0,T)} &\leq \sum_{|n|\geq N+1} |a_n|^2 \left|P^N(n)\right|^2 \frac{\left|\nu_2^n - V_0\right|^2}{Q_0^2} \left\| e^{\nu_n^h(T-t)}\right\|^2_{\dot{L}^2(0,T)}\\ &\leq C \sum_{|n|\geq N+1} |a_n|^2 \left|P^N(n)\right|^2 \frac{1}{|n|^2}\\ &\leq \frac{C}{|N|^2} \sum_{|n|\geq N+1} |a_n|^2 \left|P^N(n)\right|^2. \end{aligned}$$

Let us now suppose that the following observability inequality holds

$$\int_0^T \left|V_0\sigma^N(t,2\pi)+Q_0v^N(t,2\pi)\right|^2 dt \geq C\left\|(\sigma^N(0),v^N(0))\right\|^2_{(\dot{L}^2(0,2\pi))^2}. \tag{3.72}$$

Then, we have

$$\begin{aligned}
&\left\|(\sigma^N(0),v^N(0))\right\|^2_{(\dot{L}^2(0,2\pi))^2}\\
&\leq C\int_0^T \left|V_0\sigma^N(t,2\pi)+Q_0v^N(t,2\pi)\right|^2 dt\\
&\leq C\int_0^T \left(V_0^2\left|(\sigma^N(t,2\pi)-\tilde{\sigma}^N(t,2\pi))\right|^2+V_0^2\left|\tilde{\sigma}^N(t,2\pi)\right|^2+Q_0^2\left|v^N(t,2\pi)\right|^2\right)dt\\
&\leq \frac{C}{N^2}\sum_{|n|\geq N+1}|a_n|^2\left|P^N(n)\right|^2,
\end{aligned}$$

as we have $\tilde{\sigma}^N(t,0)=0=\tilde{\sigma}^N(t,2\pi)$ for all $t\in(0,T)$. Thus we get

$$\left\|\sigma^N(0)\right\|^2_{\dot{L}^2(0,2\pi)} \leq \left\|(\sigma^N(0),v^N(0))^\dagger\right\|^2_{(\dot{L}^2(0,2\pi))^2} \leq \frac{C}{N^2}\sum_{|n|\geq N+1}|a_n|^2\left|P^N(n)\right|^2 \leq \frac{C}{N^2}\left\|\sigma^N(0)\right\|^2_{\dot{L}^2(0,2\pi)},$$

since $\mathrm{Re}(\nu_n^h)$ is bounded. Therefore, $1\leq \frac{C}{N^2}$ for all N and hence the above inequality cannot hold. This is a contradiction and therefore the observability inequality (3.72) cannot hold. This completes the proof. □

3.2.8 Lack of approximate controllability

In this section, we prove that the system (3.1) is not approximately controllable at any time $T>0$ in $(L^2(0,2\pi))^2$ when we have the restriction on the coefficients $\frac{2\sqrt{bQ_0-V_0^2}}{\mu_0}\in\mathbb{N}$ (that is, Proposition 3.1.1). We present the proof of Proposition 3.1.1 in the case when there is a boundary control acting in density component. The proof will be similar for the velocity control case and so we omit the details.

3.2.8.1 Proof of Proposition 3.1.1

Let $T>0$ be given and $\frac{2\sqrt{bQ_0-V_0^2}}{\mu_0}\in\mathbb{N}$. To prove this result (in the density case), it is enough to find a terminal data $(\sigma_T,v_T)\in\mathcal{D}(A^*)$ such that the associated solution (σ,v) of (3.20) fails to satisfy the following unique continuation property:

$$V_0\sigma(t,2\pi)+Q_0v(t,2\pi)=0 \ \text{ implies } \ (\sigma,v)=(0,0),$$

see for instance [Cor07, Theorem 2.43]. Let us denote $n_1:=\frac{2\sqrt{bQ_0-V_0^2}}{\mu_0}$ and the eigenvalue $\nu_{n_1}^p=\nu_{-n_1}^p=:\nu_{n_1}$. The eigenfunctions of A^* corresponding to this multiple eigenvalue ν_{n_1} are $\Phi_{n_1}^p=\begin{pmatrix}\frac{Q_0}{\nu_1^{n_1}-V_0}\\ 1\end{pmatrix}e^{in_1x}$ and $\Phi_{-n_1}^p=\begin{pmatrix}\frac{Q_0}{\nu_1^{-n_1}-V_0}\\ 1\end{pmatrix}e^{-in_1x}$ (see (3.24) in Lemma 3.2.3). We now choose the terminal data as

$$(\sigma_T,v_T)^\dagger=C\Phi_{n_1}^p+D\Phi_{-n_1}^p,$$

where C,D are (complex) constants that will be chosen later. The solution of (3.20) is then given by

$$(\sigma(t),v(t))^\dagger=e^{\nu_{n_1}(T-t)}\left(C\Phi_{n_1}^p+D\Phi_{-n_1}^p\right),\quad t\in(0,T).$$

Recall the operator $\mathcal{B}_\rho^*$ given by (3.33). We get

$$V_0\sigma(t,2\pi)+Q_0v(t,2\pi)=e^{\nu_{n_1}(T-t)}\left(C\mathcal{B}_\rho^*\Phi_{n_1}^p+D\mathcal{B}_\rho^*\Phi_{-n_1}^p\right),\quad t\in(0,T).$$

If we take $C=-\mathcal{B}_\rho^*\Phi_{-n_1}^p$ and $D=\mathcal{B}_\rho^*\Phi_{n_1}^p$, then $C,D\neq 0$ (thanks to Lemma 3.2.5) and for these choice of C,D, we have $V_0\sigma(t,2\pi)+Q_0v(t,2\pi)=0$ for all $t\in(0,T)$ but $(\sigma,v)\neq(0,0)$. This completes the proof. □

3.3 Controllability of the linearized compressible Navier-Stokes system (non-barotropic case)

3.3.1 Functional setting

Recall the positive constants (equation (3.11))

$$\lambda_0 := \frac{\lambda + 2\mu}{Q_0}, \text{ and } \kappa_0 := \frac{\kappa}{Q_0 c_\nu},$$

and from now on-wards, we re-denote c_ν by c_0 to distinguish it from the eigenvalue ν.

We define the inner product in the space $(L^2(0,2\pi))^3$ as follows

$$\left\langle \begin{pmatrix} f_1 \\ g_1 \\ h_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ g_2 \\ h_2 \end{pmatrix} \right\rangle_{L^2\times L^2\times L^2} := R\psi_0 \int_0^{2\pi} f_1(x)\overline{f_2(x)}dx + Q_0^2 \int_0^{2\pi} g_1(x)\overline{g_2(x)}dx + \frac{Q_0^2 c_0}{\psi_0}\int_0^{2\pi} h_1(x)\overline{h_2(x)}dx,$$

for $f_i, g_i, h_i \in L^2(0,2\pi), i = 1,2,3$. From now on-wards, the notation $\langle \cdot, \cdot \rangle_{L^2\times L^2\times L^2}$ means the above inner product in $L^2 \times L^2 \times L^2$. We write the system (3.6) in abstract differential equation

$$U'(t) = AU(t), \quad U(0) = U_0, \quad t \in (0,T), \tag{3.73}$$

where $U := (\rho, u, \theta)^\dagger, U_0 := (\rho_0, u_0, \theta_0)^\dagger$ and the operator A is given by

$$A := \begin{pmatrix} -V_0\partial_x & -Q_0\partial_x & 0 \\ -\frac{R\psi_0}{Q_0}\partial_x & \lambda_0\partial_{xx} - V_0\partial_x & -R\partial_x \\ 0 & -\frac{R\psi_0}{c_0}\partial_x & \kappa_0\partial_{xx} - V_0\partial_x \end{pmatrix}$$

with the domain

$$\mathcal{D}(A) := H^1_{\text{per}}(0,2\pi) \times (H^2_{\text{per}}(0,2\pi))^2. \tag{3.74}$$

The adjoint of the operator A is given by

$$A^* := \begin{pmatrix} V_0\partial_x & Q_0\partial_x & 0 \\ \frac{R\psi_0}{Q_0}\partial_x & \lambda_0\partial_{xx} + V_0\partial_x & R\partial_x \\ 0 & \frac{R\psi_0}{c_0}\partial_x & \kappa_0\partial_{xx} + V_0\partial_x \end{pmatrix} \tag{3.75}$$

with the same domain $\mathcal{D}(A^*) = \mathcal{D}(A)$. The adjoint system is given by

$$\begin{cases} -\sigma_t - V_0\sigma_x - Q_0 v_x = 0, & \text{in } (0,T)\times(0,2\pi), \\ -v_t - \lambda_0 v_{xx} - \dfrac{R\psi_0}{Q_0}\sigma_x - V_0 v_x - R\varphi_x = 0, & \text{in } (0,T)\times(0,2\pi), \\ -\varphi_t - \kappa_0\varphi_{xx} - \dfrac{R\psi_0}{c_0} v_x - V_0\varphi_x = 0, & \text{in } (0,T)\times(0,2\pi), \\ \sigma(t,0) = \sigma(t,2\pi), & \text{for } t \in (0,T), \\ v(t,0) = v(t,2\pi), \quad v_x(t,0) = v_x(t,2\pi), & \text{for } t \in (0,T), \\ \varphi(t,0) = \varphi(t,2\pi), \quad \varphi_x(t,0) = \varphi_x(t,2\pi), & \text{for } t \in (0,T), \\ \sigma(T,x) = \sigma_T(x), \quad v(T,x) = v_T(x), \quad \varphi(T,x) = \varphi_T(x), & \text{in } (0,2\pi), \end{cases} \tag{3.76}$$

where $(\sigma_T, v_T, \varphi_T)$ is a terminal state. We also write the following system with source terms f, g, and h.

$$\begin{cases} -\sigma_t - V_0\sigma_x - Q_0 v_x = f, & \text{in } (0,T)\times(0,2\pi), \\ -v_t - \lambda_0 v_{xx} - \dfrac{R\psi_0}{Q_0}\sigma_x - V_0 v_x - R\varphi_x = g, & \text{in } (0,T)\times(0,2\pi), \\ -\varphi_t - \kappa_0\varphi_{xx} - \dfrac{R\psi_0}{c_0} v_x - V_0\varphi_x = h, & \text{in } (0,T)\times(0,2\pi), \\ \sigma(t,0) = \sigma(t,2\pi), & \text{for } t\in(0,T), \\ v(t,0) = v(t,2\pi), \quad v_x(t,0) = v_x(t,2\pi), & \text{for } t\in(0,T), \\ \varphi(t,0) = \varphi(t,2\pi), \quad \varphi_x(t,0) = \varphi_x(t,2\pi), & \text{for } t\in(0,T), \\ \sigma(T,x) = \sigma_T(x), \quad v(T,x) = v_T(x), \quad \varphi(T,x) = \varphi_T(x), & \text{in } (0,2\pi). \end{cases} \tag{3.77}$$

3.3.2 Well-posedness of the systems

We first state the following well-posedness result of the system (3.6) when there is no control input.

Lemma 3.3.1. *The operator A (resp. A^*) generates a C^0-semigroup of contractions on $(L^2(0,2\pi))^3$. Moreover, for any given $U_0 \in (L^2(0,2\pi))^3$, the system (3.73) admits a unique weak solution U in the space $C^0([0,T];(L^2(0,2\pi))^3)$ and*

$$\|U(t)\|_{(L^2(0,2\pi))^3} \le C\,\|U_0\|_{(L^2(0,2\pi))^3}$$

for all $t \ge 0$.

For the sake of completeness, we give a proof of this result in Appendix A.1.1. As a consequence of this result, we have the following existence results:

Lemma 3.3.2. *The following statements hold:*

1. *For any given $(f,g,h) \in L^2(0,T;(L^2(0,2\pi))^3)$ and $(\sigma_T, v_T, \varphi_T) \in (L^2(0,2\pi))^3$, the adjoint system (3.77) has a unique weak solution (σ, v, φ) in the space*

   $$C^0([0,T];L^2(0,2\pi)) \times [C^0([0,T];L^2(0,2\pi)) \cap L^2(0,T;H^1_{\text{per}}(0,2\pi))]^2.$$

   Moreover, we have the hidden regularity property $\sigma(\cdot,2\pi) \in L^2(0,T)$.

2. *For any given $(f,g,h) \in L^2(0,T;H^1_{\text{per}}(0,2\pi)\times(L^2(0,2\pi))^2)$ and $(\sigma_T, v_T, \varphi_T) \in H^{-1}_{\text{per}}(0,2\pi)\times(L^2(0,2\pi))^2$, the adjoint system (3.77) has a unique weak solution (σ, v, φ) in*

   $$C^0([0,T];H^{-1}_{\text{per}}(0,2\pi)\times(L^2(0,2\pi))^2).$$

   In particular, when $(\sigma_T, v_T, \varphi_T) = (0,0,0)$, the solution (σ, v, φ) belong to the space

   $$C^0([0,T];H^1_{\text{per}}(0,2\pi)) \times [C^0([0,T];H^1_{\text{per}}(0,2\pi)) \cap L^2(0,T;H^2_{\text{per}}(0,2\pi))]^2.$$

The proof of this lemma can be proved similarly as in the barotropic case (Lemma 3.2.2), see for instance [Mai15]. We now define the notion of a solution to the system (3.6) in the sense of transposition when a boundary control is present in the system.

Definition 3.3.1. *We give the following definitions of solutions based on the act of the controls.*

- *For any given initial state $(\rho_0, u_0, \theta_0) \in (L^2(0,2\pi))^3$ and boundary control $p \in L^2(0,T)$, we say $(\rho, u, \theta) \in L^2(0,T;(L^2(0,2\pi))^3)$ is a solution to the system (3.6)-(3.7)-(3.8) if, for every $(f,g,h) \in L^2(0,T;(L^2(0,2\pi))^3)$ the following identity holds:*

  $$\int_0^T \left\langle (\rho(t,\cdot), u(t,\cdot), \theta(t,\cdot))^\dagger, (f(t,\cdot), g(t,\cdot), h(t,\cdot))^\dagger \right\rangle_{L^2\times L^2\times L^2}$$
  $$= \left\langle (\rho_0, u_0, \theta_0)^\dagger, (\sigma(0,\cdot), v(0,\cdot), \varphi(0,\cdot))^\dagger \right\rangle_{L^2\times L^2\times L^2} + R\psi_0 \int_0^T \left[V_0\overline{\sigma(t,2\pi)} + Q_0\overline{v(t,2\pi)} \right] p(t)dt,$$

  where (σ, v, φ) is the weak solution to the adjoint system (3.77) with $(\sigma_T, v_T, \varphi_T) = (0,0,0)$.

- *For any given initial state* $(\rho_0, u_0, \theta_0) \in (L^2(0,2\pi))^3$ *and boundary control* $q \in L^2(0,T)$*, we say* $(\rho, u, \theta) \in L^2(0,T; H^{-1}_{\text{per}}(0,2\pi)) \times L^2(0,T;(L^2(0,2\pi))^2)$ *is a solution to the system* (3.6)-(3.7)-(3.9) *if, for any* $(f,g,h) \in L^2(0,T; H^1_{\text{per}}(0,2\pi)) \times L^2(0,T;(L^2(0,2\pi))^2)$ *the following identity holds:*

$$\int_0^T \left\langle (\rho(t,\cdot), u(t,\cdot), \theta(t,\cdot))^\dagger, (f(t,\cdot), g(t,\cdot), h(t,\cdot))^\dagger \right\rangle_{H^{-1}_{\text{per}}\times L^2\times L^2, H^1_{\text{per}}\times L^2 \times L^2} dt$$
$$= \left\langle (\rho_0(\cdot), u_0(\cdot), \theta_0(\cdot))^\dagger, (\sigma(0,\cdot), v(0,\cdot), \varphi(0,\cdot))^\dagger \right\rangle_{L^2\times L^2\times L^2}$$
$$+ Q_0 \int_0^T \left[R\psi_0 \overline{\sigma(t,2\pi)} + \lambda_0 Q_0 \overline{v_x(t,2\pi)} + Q_0 V_0 \overline{v(t,2\pi)} + RQ_0 \overline{\varphi(t,2\pi)} \right] q(t) dt,$$

where (σ, v, φ) *is the weak solution to the adjoint system* (3.77) *with* $(\sigma_T, v_T, \varphi_T) = (0,0,0)$.

- *For any given initial state* $(\rho_0, u_0, \theta_0) \in (L^2(0,2\pi))^3$ *and boundary control* $r \in L^2(0,T)$*, we say* $(\rho, u, \theta) \in L^2(0,T; H^{-1}_{\text{per}}(0,2\pi)) \times L^2(0,T;(L^2(0,2\pi))^2)$ *is a solution to the system* (3.6)-(3.7)-(3.9) *if, for any* $(f,g,h) \in L^2(0,T; H^1_{\text{per}}(0,2\pi)) \times L^2(0,T;(L^2(0,2\pi))^2)$ *the following identity holds:*

$$\int_0^T \left\langle (\rho(t,\cdot), u(t,\cdot), \theta(t,\cdot))^\dagger, (f(t,\cdot), g(t,\cdot), h(t,\cdot))^\dagger \right\rangle_{H^{-1}_{\text{per}}\times L^2\times L^2, H^1_{\text{per}}\times L^2 \times L^2} dt$$
$$= \left\langle (\rho_0(\cdot), u_0(\cdot), \theta_0(\cdot))^\dagger, (\sigma(0,\cdot), v(0,\cdot), \varphi(0,\cdot))^\dagger \right\rangle_{L^2\times L^2\times L^2}$$
$$+ Q_0^2 \int_0^T \left[R\overline{v(t,2\pi)} + \frac{c_0 V_0}{\psi_0} \overline{\varphi(t,2\pi)} + \frac{c_0 \kappa_0}{\psi_0} \overline{\varphi_x(t,2\pi)} \right] r(t) dt,$$

where (σ, v, φ) *is the weak solution to the adjoint system* (3.77) *with* $(\sigma_T, v_T, \varphi_T) = (0,0,0)$.

We now write the following well-posedness results for the system (3.6) based on the act of the boundary control.

Proposition 3.3.1. *For any given initial state* $(\rho_0, u_0, \theta_0) \in (L^2(0,2\pi))^3$ *and boundary control* $p \in L^2(0,T)$*, the system* (3.6)-(3.7)-(3.8) *admits a unique solution* (ρ, u, θ) *in the space*

$$C^0([0,T]; L^2(0,2\pi)) \times [C^0([0,T]; L^2(0,2\pi)) \cap L^2(0,T; H^1_{\text{per}}(0,2\pi))]^2.$$

Proposition 3.3.2. *For any given initial state* $(\rho_0, u_0, \theta_0) \in (L^2(0,2\pi))^3$ *and boundary control* $q \in L^2(0,T)$*, the system* (3.6)-(3.7)-(3.9) *admits a unique solution* (ρ, u, θ) *in the space*

$$C^0([0,T]; H^{-1}_{\text{per}}(0,2\pi)) \times [C^0([0,T]; H^{-1}_{\text{per}}(0,2\pi)) \cap L^2(0,T; (L^2(0,2\pi)))]^2.$$

Proposition 3.3.3. *For any given initial state* $(\rho_0, u_0, \theta_0) \in (L^2(0,2\pi))^3$ *and boundary control* $r \in L^2(0,T)$*, the system* (3.6)-(3.7)-(3.10) *admits a unique solution* (ρ, u, θ) *in the space*

$$C^0([0,T]; H^{-1}_{\text{per}}(0,2\pi)) \times [C^0([0,T]; H^{-1}_{\text{per}}(0,2\pi)) \cap L^2(0,T; (L^2(0,2\pi)))]^2.$$

The proofs of Proposition 3.3.1, Proposition 3.3.2 and Proposition 3.3.3 can be done in a similar way ([BCDK22, Theorem 2.4] and [CR13, Gir08]) like the barotropic case and so we skip the proofs.

3.3.3 Spectral Analysis of A^*

Let $\sigma(A^*)$ denotes the spectrum of the operator A^*. We first write the following lemma.

Lemma 3.3.3. *The following statements hold:*

(i) $\ker(A^*) = \textit{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\}$.

(ii) $\sup\{\text{Re}(\nu) \ : \ \nu \in \sigma(A^*), \ \nu \neq 0\} < 0$.

(iii) The spectrum of A^* consists of the eigenvalue 0 and three branches of complex eigenvalues

$$\{\nu_n^h, \nu_n^{p_1}, \nu_n^{p_2}\}_{n\in\mathbb{Z}^*}$$

with the asymptotic expressions given as

$$\nu_n^h = V_0 in - \overline{\omega} + O(|n|^{-2}), \tag{3.78}$$
$$\nu_n^{p_1} = -\lambda_0 n^2 + V_0 in + O(1), \tag{3.79}$$
$$\nu_n^{p_2} = -\kappa_0 n^2 + V_0 in + O(1), \tag{3.80}$$

for all $|n|$ large, where $\overline{\omega} = \frac{R\psi_0}{\lambda_0}$.

(iv) The eigenfunctions of A^* corresponding to ν_n^h and $\nu_n^{p_1}, \nu_n^{p_2}$ are respectively

$$\Phi_n^h = \begin{pmatrix} \xi_n^h \\ \eta_n^h \\ \zeta_n^h \end{pmatrix} = \begin{pmatrix} \alpha_1^n \\ \alpha_2^n \\ \alpha_3^n \end{pmatrix} e^{inx}, \quad \Phi_n^{p_1} = \begin{pmatrix} \xi_n^{p_1} \\ \eta_n^{p_1} \\ \zeta_n^{p_1} \end{pmatrix} = \begin{pmatrix} \beta_1^n \\ \beta_2^n \\ \beta_3^n \end{pmatrix} e^{inx}, \quad \Phi_n^{p_2} = \begin{pmatrix} \xi_n^{p_2} \\ \eta_n^{p_2} \\ \zeta_n^{p_2} \end{pmatrix} = \begin{pmatrix} \gamma_1^n \\ \gamma_2^n \\ \gamma_3^n \end{pmatrix} e^{inx}, \tag{3.81}$$

for all $n \in \mathbb{Z}^*$, with the constants α_i^n, β_i^n and γ_i^n $(i = 1, 2, 3)$ given as

$$\begin{cases} \alpha_1^n = RQ_0, \quad \alpha_2^n = -R(V_0 - \nu_3^n), \quad \alpha_3^n = (\lambda_0 in + V_0 - \nu_3^n)(V_0 - \nu_3^n) - R\psi_0 \\ \beta_1^n = -\frac{RQ_0}{V_0 - \nu_1^n}, \quad \beta_2^n = R, \quad \beta_3^n = \frac{1}{V_0 - \nu_1^n}[R\psi_0 - (\lambda_0 in + V_0 - \nu_1^n)(V_0 - \nu_1^n)] \\ \gamma_1^n = (\lambda_0 in + V_0 - \nu_2^n)(\kappa_0 in + V_0 - \nu_2^n) - \frac{R^2\psi_0}{c_0}, \quad \gamma_2^n = -\frac{R\psi_0}{Q_0}(\kappa_0 in + V_0 - \nu_2^n), \quad \gamma_3^n = \frac{R^2\psi_0^2}{Q_0 c_0}, \end{cases} \tag{3.82}$$

for all $n \in \mathbb{Z}^*$, where ν_1^n, ν_2^n and ν_3^n are roots of the cubic polynomial

$$\begin{aligned} \nu^3 - [(\lambda_0 + \kappa_0)in + 3V_0]\nu^2 - [\lambda_0\kappa_0 n^2 - 2(\lambda_0 + \kappa_0)V_0 in - 3V_0^2 + \frac{R^2\psi_0}{c_0} + R\psi_0]\nu \\ +\lambda_0\kappa_0 V_0 n^2 - (\lambda_0 + \kappa_0)V_0^2 in - V_0^3 + \frac{R^2\psi_0}{c_0}V_0 + R\psi_0\kappa_0 in + R\psi_0 V_0 = 0, \end{aligned} \tag{3.83}$$

for all $n \in \mathbb{Z}^*$.

Remark 3.3.1. *We have the asymptotic expressions of $\alpha_i^n, \beta_i^n, \gamma_i^n$, $i = 1, 2, 3$ as follows.*

$$\begin{cases} \alpha_1^n \sim_{+\infty} 1, \quad \alpha_2^n \sim_{+\infty} \frac{1}{|n|}, \quad \alpha_3^n \sim_{+\infty} \frac{1}{|n|}, \\ \beta_1^n \sim_{+\infty} \frac{1}{|n|}, \quad \beta_2^n \sim_{+\infty} 1, \quad \beta_3^n \sim_{+\infty} \frac{1}{|n|}, \\ \gamma_1^n \sim_{+\infty} \frac{1}{|n|}, \quad \gamma_2^n \sim_{+\infty} \frac{1}{|n|}, \quad \gamma_3^n \sim_{+\infty} 1. \end{cases} \tag{3.84}$$

Proof. We will prove each part separately.

Part-(i). Follows immediately from the fact that $A^*(\xi, \eta, \zeta)^\dagger = 0$ implies (ξ, η, ζ) =constant.

Part-(ii). Let $\Phi = (\xi, \eta, \zeta)^\dagger \in \mathcal{D}(A^*)$ be the eigenfunction of A^* corresponding to the eigenvalue $\nu \neq 0$. Then, we have

$$\left\langle A^* \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} \right\rangle_{L^2\times L^2\times L^2} = \left\langle \nu \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} \right\rangle_{L^2\times L^2\times L^2},$$

that is,

$$R\psi_0 V_0 \int_0^{2\pi} \overline{\xi(x)}\xi_x(x)dx + R\psi_0 Q_0 \int_0^{2\pi} \overline{\xi(x)}\eta_x(x)dx + \lambda_0 Q_0^2 \int_0^{2\pi} \overline{\eta(x)}\eta_{xx}(x)dx$$

$$
\begin{aligned}
&+ Q_0^2 V_0 \int_0^{2\pi} \overline{\eta(x)} \eta_x(x) dx + R\psi_0 Q_0 \int_0^{2\pi} \xi_x(x) \overline{\eta(x)} dx + RQ_0^2 \int_0^{2\pi} \overline{\eta(x)} \zeta_x(x) dx \\
&+ \frac{Q_0^2 c_0}{\psi_0} \kappa_0 \int_0^{2\pi} \overline{\zeta(x)} \zeta_{xx}(x) dx + \frac{Q_0^2 c_0}{\psi_0} V_0 \int_0^{2\pi} \overline{\zeta(x)} \zeta_x(x) dx + RQ_0^2 \int_0^{2\pi} \eta_x(x) \overline{\zeta(x)} dx \\
&= \nu R\psi_0 \int_0^{2\pi} |\xi(x)|^2 \, dx + \nu Q_0^2 \int_0^{2\pi} |\eta(x)|^2 \, dx + \nu \frac{Q_0^2 c_0}{\psi_0} \int_0^{2\pi} |\zeta(x)|^2 \, dx.
\end{aligned}
$$

An integration by parts yields

$$
\mathrm{Re}(\nu) = - \frac{\lambda_0 Q_0^2 \|\eta_x\|^2_{L^2(0,2\pi)} + \frac{Q_0^2 c_0}{\psi_0} \kappa_0 \|\zeta_x\|^2_{L^2(0,2\pi)}}{R\psi_0 \|\xi\|^2_{L^2(0,2\pi)} + Q_0^2 \|\eta\|^2_{L^2(0,2\pi)} + \frac{Q_0^2 c_0}{\psi_0} \|\zeta\|^2_{L^2(0,2\pi)}} < 0,
$$

which proves part (ii).

Parts (iii)-(iv). We denote

$$
\varphi_n(x) := e^{inx}, \quad n \in \mathbb{Z}.
$$

Then the set $\left\{ \begin{pmatrix} \varphi_n \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \varphi_n \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \varphi_n \end{pmatrix} \right\}$ forms an orthogonal basis of $(L^2(0,2\pi))^3$. Let us define

$$
E_n := \begin{pmatrix} \varphi_n & 0 & 0 \\ 0 & \varphi_n & 0 \\ 0 & 0 & \varphi_n \end{pmatrix}, \quad \text{and } \Phi_n := (\xi_n, \eta_n, \zeta_n)^\dagger,
$$

for all $n \in \mathbb{Z}$. Then, we have the following relation

$$
A^* E_n \Phi_n = in E_n R_n \Phi_n, \quad n \in \mathbb{Z}, \tag{3.85}
$$

where

$$
R_n := \begin{pmatrix} V_0 & Q_0 & 0 \\ \frac{R\psi_0}{Q_0} & \lambda_0 in + V_0 & R \\ 0 & \frac{R\psi_0}{c_0} & \kappa_0 in + V_0 \end{pmatrix}, \quad n \in \mathbb{Z}. \tag{3.86}
$$

Thus, if (α_n, ν_n) is an eigenpair of R_n, then $(E_n\alpha_n, in\nu_n)$ will be an eigenpair of A^*. Therefore, it remains to find the eigenvalues and eigenvectors of the matrix R_n for $n \in \mathbb{Z}$. The characteristics equation of R_n is

$$
\begin{aligned}
\nu^3 - [(\lambda_0 + \kappa_0) in + 3V_0]\nu^2 - [\lambda_0\kappa_0 n^2 - 2(\lambda_0 + \kappa_0)V_0 in - 3V_0^2 + \frac{R^2\psi_0}{c_0} + R\psi_0]\nu \\
+ \lambda_0\kappa_0 V_0 n^2 - (\lambda_0 + \kappa_0)V_0^2 in - V_0^3 + \frac{R^2\psi_0}{c_0} V_0 + R\psi_0\kappa_0 in + R\psi_0 V_0 = 0,
\end{aligned} \tag{3.87}
$$

for all $n \in \mathbb{Z}$.

Claim 1. 0 cannot be a root of the polynomial (3.87) for any $n \in \mathbb{Z}$.

Proof of Claim 1. Let $\nu = 0$ be a root of (3.87). Then, there exists some $n \in \mathbb{Z}$ such that

$$
\lambda_0\kappa_0 V_0 n^2 - (\lambda_0 + \kappa_0)V_0^2 in - V_0^3 + \frac{R^2\psi_0}{c_0} V_0 + R\psi_0\kappa_0 in + R\psi_0 V_0 = 0,
$$

which implies

$$
\lambda_0\kappa_0 n^2 - V_0^2 + \frac{R^2\psi_0}{c_0} + R\psi_0 = 0, \text{ and } (\lambda_0 + \kappa_0)V_0^2 = R\psi_0\kappa_0.
$$

We then have

$$\lambda_0\kappa_0 n^2 = V_0^2 - \frac{R^2\psi_0}{c_0} - R\psi_0 = V_0^2 - \frac{R^2\psi_0}{c_0} - \left(\frac{\lambda_0}{\kappa_0}+1\right)V_0^2 = -\frac{R^2\psi_0}{c_0} - \frac{\lambda_0}{\kappa_0}V_0^2 < 0,$$

a contradiction. This proves our first claim.

Claim 2. V_0 cannot be a root of the polynomial (3.87) for any $n \in \mathbb{Z}^*$.

Proof of Claim 2. Observe that V_0 is a root of (3.87) if and only if $R\psi_0\kappa_0 in = 0$. Thus, for all $n \in \mathbb{Z}^*$, V_0 cannot be a root of (3.87), which proves our second claim.

For fixed $n \in \mathbb{Z}^*$, let v_1^n, v_2^n and v_3^n be the roots of this cubic polynomial. The relation between roots and coefficients are

$$\begin{cases} v_1^n + v_2^n + v_3^n = (\lambda_0+\kappa_0)in + 3V_0 \\ v_1^n v_2^n + v_2^n v_3^n + v_3^n v_1^n = -[\lambda_0\kappa_0 n^2 - 2(\lambda_0+\kappa_0)V_0 in - 3V_0^2 + \frac{R^2\psi_0}{c_0} + R\psi_0] \\ v_1^n v_2^n v_3^n = -[\lambda_0\kappa_0 V_0 n^2 - (\lambda_0+\kappa_0)V_0^2 in - V_0^3 + \frac{R^2\psi_0}{c_0}V_0 + R\psi_0\kappa_0 in + R\psi_0 V_0]. \end{cases}$$

We will find the asymptotic expressions of roots of the cubic polynomial (3.87) for large values of $|n|$. The first relation between roots and coefficients tells us that V_0 is present in at least one of the roots of the cubic polynomial (3.87). Thus, using the transformation

$$v = V_0 + \epsilon_n, \tag{3.88}$$

it is enough to find the roots of the transformed cubic equation in ϵ_n

$$\epsilon_n^3 - (\lambda_0+\kappa_0)in\epsilon_n^2 - \left(\lambda_0\kappa_0 n^2 + \frac{R^2\psi_0}{c_0} + R\psi_0\right)\epsilon_n + R\psi_0\kappa_0 in = 0 \tag{3.89}$$

for all $n \in \mathbb{Z}^*$. We use the transformation $\epsilon_n = in\tilde{\epsilon}_n$ for $n \in \mathbb{Z}^*$, to simplify the above equation and we get

$$\tilde{\epsilon}_n^3 - (\lambda_0+\kappa_0)\tilde{\epsilon}_n^2 + \left(\lambda_0 k_0 + \frac{1}{n^2}\left(\frac{R^2\psi_0}{c_0} + R\psi_0\right)\right)\tilde{\epsilon}_n - \frac{R\psi_0\kappa_0}{n^2} = 0 \tag{3.90}$$

for all $n \in \mathbb{Z}^*$. We now use the Rouche's Theorem to find the roots of this polynomial. Let us first state the Rouché's Theorem, the proof of which can be found in [Con78, Rud87, Ahl78].

Theorem 3.3.1 (Rouché's Theorem)**.** *Let $\Omega \subset \mathbb{C}$ be an open connected set and $f, g : \Omega \to \mathbb{C}$ be holomorphic on Ω. Suppose there exists $a \in \Omega$ and $R > 0$ such that $\overline{B(a,R)} \subset \Omega$ and*

$$|g(z) - f(z)| < |g(z)| \text{ for all } z \in \partial B(a,R),$$

then f and g have the same number of zeros inside $B(a,R)$.

Let $n \in \mathbb{Z}^*$. We define the functions $f, g : \mathbb{C} \to \mathbb{C}$ by

$$f(z) := z^3 - (\lambda_0+\kappa_0)z^2 + \left(\lambda_0 k_0 + \frac{1}{n^2}\left(\frac{R^2\psi_0}{c_0} + R\psi_0\right)\right)z - \frac{R\psi_0\kappa_0}{n^2}$$

and

$$g(z) := z^3 - (\lambda_0+\kappa_0)z^2 + \lambda_0 k_0 z$$

for all $z \in \mathbb{C}$. The roots of g are $0, \lambda_0$ and κ_0. We choose $R_0 := \frac{1}{2}\min\{\lambda_0, \kappa_0, |\lambda_0 - \kappa_0|\}$. Then, we have the following estimates

$$|g(z) - f(z)| = \left|\frac{1}{n^2}\left(\frac{R^2\psi_0}{c_0} + R\psi_0\right)z - \frac{R\psi_0\kappa_0}{n^2}\right|$$

$$\leq \frac{C}{n^2}(|z|+1)\begin{cases} = \frac{C}{n^2}(R_0+1), & \text{for all } z \in \partial B(0,R_0), \\ \leq \frac{C}{n^2}(\lambda_0+R_0+1), & \text{for all } z \in \partial B(\lambda_0,R_0), \\ \leq \frac{C}{n^2}(\kappa_0+R_0+1), & \text{for all } z \in \partial B(\kappa_0,R_0), \end{cases}$$

for all $n \in \mathbb{Z}^*$. On the other hand, the choice of R_0 tells us that the function g does not have any root on the sets $\partial B(0,R_0)$, $\partial B(\lambda_0,R_0)$ and $\partial B(\kappa_0,R_0)$. This shows that $\inf_{|z|=R_0}|g(z)| > 0$, $\inf_{|z-\lambda_0|=R_0}|g(z)| > 0$ and $\inf_{|z-\kappa_0|=R_0}|g(z)| > 0$. Therefore, for $|n|$ large enough, we have

$$|g(z) - f(z)| < |g(z)| \text{ for all } z \in \partial B(0,R_0) \cup \partial B(\lambda_0,R_0) \cup \partial B(\kappa_0,R_0).$$

Thus, for each $n \in \mathbb{Z}^*$, the function f has a unique root inside each of the sets $B(0,R_0)$, $B(\lambda_0,R_0)$ and $B(\kappa_0,R_0)$. We denote these roots by z_1^n, z_2^n and z_3^n respectively. We now find asymptotic expressions of these roots.

Asymptotic expression of z_1^n. Since $z_1^n \in B(0,R_0)$, we have

$$z_1^n = \frac{1}{(z_1^n-\lambda_0)(z_1^n-\kappa_0)}\left(\frac{R\psi_0\kappa_0}{n^2} - \frac{1}{n^2}\left(\frac{R^2\psi_0}{c_0}+R\psi_0\right)z_1^n\right)$$

and therefore

$$\left|z_1^n\right| \leq \frac{1}{\left|z_1^n-\lambda_0\right|\left|z_1^n-\kappa_0\right|}\left(\left|\frac{R\psi_0\kappa_0}{n^2}\right| + \left|\frac{1}{n^2}\left(\frac{R^2\psi_0}{c_0}+R\psi_0\right)z_1^n\right|\right) \leq \frac{C}{|n|^2}$$

for $|n|$ large enough. To find the asymptotic expression of z_1^n, we write $f(z_1^n)=0$ in the following way

$$\begin{aligned} z_1^n &= \frac{R\psi_0\kappa_0}{n^2}\left(\lambda_0\kappa_0 - (\lambda_0+\kappa_0)z_1^n + (z_1^n)^2 + \frac{1}{n^2}\left(\frac{R\psi_0}{c_0}+R\psi_0\right)\right)^{-1} \\ &= \frac{R\psi_0\kappa_0}{n^2}\frac{1}{\lambda_0\kappa_0}\left(1 - \frac{(\lambda_0+\kappa_0)}{\lambda_0\kappa_0}z_1^n + \frac{1}{\lambda_0\kappa_0 n^2}\left(\frac{R\psi_0}{c_0}+R\psi_0\right) + O(|n|^{-4})\right)^{-1} \\ &= \frac{\overline{\omega}}{n^2}\left(1 + \frac{(\lambda_0+\kappa_0)}{\lambda_0\kappa_0}z_1^n - \frac{1}{\lambda_0\kappa_0 n^2}\left(\frac{R\psi_0}{c_0}+R\psi_0\right) + O(|n|^{-4})\right) \\ &= \frac{\overline{\omega}}{n^2} + O(|n|^{-4}), \end{aligned}$$

since $\left|z_1^n\right| \leq \frac{C}{n^2}$ for all $|n|$ large, where $\overline{\omega} = \frac{R\psi_0}{\lambda_0}$.

Asymptotic expression of z_2^n. Since $z_2^n \in B(\lambda_0,R_0)$, we have

$$z_2^n - \lambda_0 = \frac{1}{z_2^n(z_2^n-\kappa_0)}\left(\frac{R\psi_0\kappa_0}{n^2} - \frac{1}{n^2}\left(\frac{R^2\psi_0}{c_0}+R\psi_0\right)z_1^n\right)$$

and therefore

$$\left|z_2^n-\lambda_0\right| \leq \frac{1}{\left|z_2^n\right|\left|z_2^n-\kappa_0\right|}\left(\left|\frac{R\psi_0\kappa_0}{n^2}\right| + \left|\frac{1}{n^2}\left(\frac{R^2\psi_0}{c_0}+R\psi_0\right)z_1^n\right|\right) \leq \frac{C}{|n|^2}$$

for $|n|$ large enough. Thus, we can write

$$z_2^n = \lambda_0 + O(|n|^{-2})$$

for all $|n|$ large.

Asymptotic expression of z_3^n. Following the similar approach as mentioned above, we can get

$$z_3^n = \kappa_0 + O(|n|^{-2})$$

for all $|n|$ large.

Combining all of the above, we obtain the asymptotic expressions of the roots of (3.89) as

$$\begin{cases} \epsilon_1^n := \lambda_0 in + O(|n|^{-1}), \\ \epsilon_2^n := \kappa_0 in + O(|n|^{-1}), \\ \epsilon_3^n := -\dfrac{\overline{\omega}}{in} + O(|n|^{-3}) \end{cases}$$

for all $|n|$ large. Therefore, for $n \in \mathbb{Z}^*$, eigenvalues of the matrix R_n are ν_1^n, ν_2^n and ν_3^n with the asymptotic expressions

$$\nu_1^n = \lambda_0 in + V_0 + O(|n|^{-1}), \tag{3.91}$$

$$\nu_2^n = \kappa_0 in + V_0 + O(|n|^{-1}), \tag{3.92}$$

$$\nu_3^n = V_0 - \frac{\overline{\omega}}{in} + O(|n|^{-3}), \tag{3.93}$$

for all $|n|$ large.

To find the eigenvectors of the matrix R_n, we now consider the equation

$$R_n \alpha_n = \nu_3^n \alpha_n, \quad \text{for } n \in \mathbb{Z}^*,$$

where $\alpha_n = (\alpha_1^n, \alpha_2^n, \alpha_3^n)^\dagger$, $n \in \mathbb{Z}^*$, that is,

$$(V_0 - \nu_3^n)\alpha_1^n + Q_0 \alpha_2^n = 0, \tag{3.94}$$

$$\frac{R\psi_0}{Q_0}\alpha_1^n + (\lambda_0 in + V_0 - \nu_3^n)\alpha_2^n + R\alpha_3^n = 0, \tag{3.95}$$

$$\frac{R\psi_0}{c_0}\alpha_2^n + (\kappa_0 in + V_0 - \nu_3^n)\alpha_3^n = 0, \tag{3.96}$$

for all $n \in \mathbb{Z}^*$. One solution is given by

$$\alpha_1^n = RQ_0, \quad \alpha_2^n = -R(V_0 - \nu_3^n), \quad \alpha_3^n = (\lambda_0 in + V_0 - \nu_3^n)(V_0 - \nu_3^n) - R\psi_0, \quad n \in \mathbb{Z}^*.$$

We next consider the equation

$$R_n \beta_n = \nu_1^n \beta_n, \quad \text{for } n \in \mathbb{Z}^*,$$

where $\beta_n = (\beta_1^n, \beta_2^n, \beta_3^n)^\dagger, n \in \mathbb{Z}^*$, that is,

$$(V_0 - \nu_1^n)\beta_1^n + Q_0 \beta_2^n = 0, \tag{3.97}$$

$$\frac{R\psi_0}{Q_0}\beta_1^n + (\lambda_0 in + V_0 - \nu_1^n)\beta_2^n + R\beta_3^n = 0, \tag{3.98}$$

$$\frac{R\psi_0}{c_0}\beta_2^n + (\kappa_0 in + V_0 - \nu_1^n)\beta_3^n = 0, \tag{3.99}$$

for all $n \in \mathbb{Z}^*$. One solution is given by

$$\beta_1^n = -\frac{RQ_0}{V_0 - \nu_1^n}, \quad \beta_2^n = R, \quad \beta_3^n = \frac{1}{V_0 - \nu_1^n}[R\psi_0 - (\lambda_0 in + V_0 - \nu_1^n)(V_0 - \nu_1^n)], \quad n \in \mathbb{Z}^*.$$

We finally consider the equation

$$R_n \gamma_n = \nu_2^n \gamma_n, \quad \text{for } n \in \mathbb{Z}^*,$$

where $\gamma_n = (\gamma_1^n, \gamma_2^n, \gamma_3^n)^\dagger$, $n \in \mathbb{Z}^*$, that is,

$$(V_0 - \nu_2^n)\gamma_1^n + Q_0 \gamma_2^n = 0, \tag{3.100}$$

$$\frac{R\psi_0}{Q_0}\gamma_1^n + (\lambda_0 in + V_0 - \nu_2^n)\gamma_2^n + R\gamma_3^n = 0, \tag{3.101}$$
$$\frac{R\psi_0}{c_0}\gamma_2^n + (\kappa_0 in + V_0 - \nu_2^n)\gamma_3^n = 0, \tag{3.102}$$

for all $n \in \mathbb{Z}^*$. One solution is given by

$$\gamma_1^n = (\lambda_0 in + V_0 - \nu_2^n)(\kappa_0 in + V_0 - \nu_2^n) - \frac{R^2\psi_0}{c_0}, \quad \gamma_2^n = -\frac{R\psi_0}{Q_0}(\kappa_0 in + V_0 - \nu_2^n), \quad \gamma_3^n = \frac{R^2\psi_0^2}{Q_0 c_0},$$

for $n \in \mathbb{Z}^*$. Therefore, the eigenvectors of R_n corresponding to the eigenvalues ν_3^n, ν_1^n and ν_2^n are respectively α_n, β_n and γ_n, where

$$\alpha_n = \begin{pmatrix} \alpha_1^n \\ \alpha_2^n \\ \alpha_3^n \end{pmatrix}, \quad \beta_n = \begin{pmatrix} \beta_1^n \\ \beta_2^n \\ \beta_3^n \end{pmatrix}, \quad \gamma_n = \begin{pmatrix} \gamma_1^n \\ \gamma_2^n \\ \gamma_3^n \end{pmatrix},$$

for all $n \in \mathbb{Z}^*$. Hence, the eigenvalues of the operator A^* are $\nu_n^h := in\nu_n^3, \nu_n^{p_1} := in\nu_n^2$ and $\nu_n^{p_2} := in\nu_n^1$ for all $n \in \mathbb{Z}^*$ with the asymptotic expressions

$$\begin{aligned} \nu_n^h &= V_0 in - \overline{\omega} + O(|n|^{-1}), \\ \nu_n^{p_1} &= -\lambda_0 n^2 + V_0 in + O(1), \\ \nu_n^{p_2} &= -\kappa_0 n^2 + V_0 in + O(1), \end{aligned}$$

for $|n|$ large enough and the corresponding eigenfunctions are

$$\Phi_n^h(x) := E_n(x)\alpha_n = \alpha_n e^{inx}, \quad \Phi_n^{p_1}(x) := E_n(x)\beta_n = \beta_n e^{inx}, \quad \Phi_n^{p_2}(x) := E_n(x)\gamma_n = \gamma_n e^{inx},$$

for all $n \in \mathbb{Z}^*$ and $x \in (0, 2\pi)$.

This completes the proof. □

Remark 3.3.2. *Note that, all the eigenvalues of A^* are simple at least for $|n|$ large enough. Depending on the constants $Q_0, V_0, \psi_0, \lambda_0, \kappa_0, R$ and c_0, there may be multiple eigenvalues, but that would be only finitely many of them. For example, if we take $Q_0 = V_0 = \lambda_0 = 1$ and $R\psi_0 = \frac{R^2\psi_0}{c_0} = \frac{1}{2}, \kappa_0 = 2$, then the characteristics equation (3.87) of R_n (with $n = 1$) becomes*

$$\nu^3 - (3i + 3)\nu^2 + 6in\nu + 2 - 2i = 0$$

and therefore $\nu = 1 + i$ is a root of multiplicity 3, and consequently $-1 + i$ is an eigenvalue of A^* with algebraic multiplicity 3. In this case, the proof of null controllability of the system (3.6) will be similar to the barotropic case (Section 3.2.5.2) and for the sake of completeness, we will give a brief proof in this (non-barotropic) case also.

Furthermore, there can exist (finitely many) multiple eigenvalues for different values of n. For example, if we take $Q_0 = V_0 = \lambda_0 = 1 = R\psi_0 = \frac{R^2\psi_0}{c_0} = 1$ and $\kappa_0 = 2$, then $\nu = -1$ is an eigenvalue of A^* with $n = 1$ and $n = -1$, that is, $\nu_1 = \nu_{-1} = -1$. Indeed, the polynomial equation (3.87) for $n = 1$ and $n = -1$ becomes

$$\begin{aligned} \nu^3 - (3i + 3)\nu^2 - (1 - 6i)\nu + 3 - i = 0, \\ \nu^3 - (-3i + 3)\nu^2 - (1 + 6i)\nu + 3 + i = 0, \end{aligned}$$

and the root of which are i and $-i$ respectively. In this case, as mentioned in the barotropic case, we have two independent eigenfunctions corresponding to this eigenvalue; as a consequence, the adjoint system (3.76) fails to satisfy the unique continuation property; see Section 3.3.8 for more details.

Let us assume that there exists a $n_0 \in \mathbb{N}$ such that all the eigenvalues ν_n of A^* are algebraically simple for all $|n| \geq n_0$. There can exist only finitely many multiple eigenvalues, which we re-denote as ν_j for $1 \leq j \leq j_0$ (for some $j_0 \in \mathbb{N}$). Let N_j be the algebraic multiplicity of the eigenvalue ν_j and let Φ_j denote the corresponding eigenfunction of A^* for each $j = 1, \dots, j_0$. We also denote the set of generalized eigenfunctions of A^* by $\left\{\tilde{\Phi}_{l,j} \ : \ l = 1, \dots, N_j - 1\right\}$ corresponding to the eigenvalue ν_j. Also, recall from Lemma 3.3.3–Part (i) that the set of (generalized) eigenfunctions of A^* corresponding to the eigenvalue $\nu_0 = 0$ is $\left\{\Phi_0 := \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \ \tilde{\Phi}_{1,0} := \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \ \tilde{\Phi}_{2,0} := \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right\}$. Then, one can have the following result:

Proposition 3.3.4. *The set of (generalized) eigenfunctions*

$$\mathcal{E}(A^*) := \left\{\Phi_n^h, \ \Phi_n^{p_1}, \ \Phi_n^{p_2} \ : \ |n| \geq n_0; \ \Phi_j, \tilde{\Phi}_{l,j} \ : \ 1 \leq l < N_j, \ 0 \leq j \leq j_0\right\}$$

forms a Riesz basis in $(L^2(0,2\pi))^3$. *In particular, if all the eigenvalues of* A^* *are simple, then the set of eigenfunctions*

$$\left\{\Phi_n^h, \ \Phi_n^{p_1}, \ \Phi_n^{p_2} \ : \ n \in \mathbb{Z}^*\right\}$$

forms a Riesz basis in $(\dot{L}^2(0,2\pi))^3$.

Proof. In view of the proof of Proposition 3.2.3, it is enough to find an orthogonal basis of $(L^2(0,2\pi))^3$ that is quadratically close to the set of generalized eigenfunctions of A^*. One obvious choice is the following orthogonal basis

$$\left\{\Psi_n(x) := \begin{pmatrix} RQ_0 \\ 0 \\ 0 \end{pmatrix} e^{inx}, \ \tilde{\Psi}_n(x) := \begin{pmatrix} 0 \\ R \\ 0 \end{pmatrix} e^{inx}, \ \tilde{\tilde{\Psi}}_n(x) := \begin{pmatrix} 0 \\ 0 \\ \frac{R^2\psi_0^2}{Q_0 c_0} \end{pmatrix} e^{inx} \ : \ n \in \mathbb{Z}\right\}.$$

Indeed, we have

$$\sum_{|n|\geq n_0} \left(\left\|\Phi_n^h - \Psi_n\right\|_{(L^2(0,2\pi))^3} + \left\|\Phi_n^{p_1} - \tilde{\Psi}_n\right\|_{(L^2(0,2\pi))^3} + \left\|\Phi_n^{p_2} - \tilde{\tilde{\Psi}}_n\right\|_{(L^2(0,2\pi))^3} \right) \leq C \sum_{|n|\geq n_0} \frac{1}{|n|^2} < \infty,$$

thanks to Remark 3.3.1. This completes the proof. □

3.3.4 Observation estimates

As mentioned in the barotropic case (Section 3.2.4), we need lower bound estimates of certain observation terms associated to the system (3.6). First, we define the observation operator corresponding to the system (3.6) as follows (see the Definition 3.3.1):

- The observation operator $\mathcal{B}_\rho^* : \mathcal{D}(A^*) \to \mathbb{C}$ to the system (3.6)-(3.7)-(3.8) is defined by

$$\mathcal{B}_\rho^*\Phi := V_0\xi(2\pi) + Q_0\eta(2\pi), \quad \text{for } \Phi = (\xi, \eta) \in \mathcal{D}(A^*). \tag{3.103}$$

- The observation operator $\mathcal{B}_u^* : \mathcal{D}(A^*) \to \mathbb{C}$ to the system (3.6)-(3.7)-(3.8) is defined by

$$\mathcal{B}_u^*\Phi := R\psi_0\xi(2\pi) + Q_0V_0\eta(2\pi) + \lambda_0Q_0\eta_x(2\pi) + RQ_0\zeta(2\pi), \quad \text{for } \Phi = (\xi, \eta) \in \mathcal{D}(A^*). \tag{3.104}$$

- The observation operator $\mathcal{B}_\theta^* : \mathcal{D}(A^*) \to \mathbb{C}$ to the system (3.6)-(3.7)-(3.8) is defined by

$$\mathcal{B}_\theta^*\Phi := R\eta(2\pi) + \frac{c_0V_0}{\psi_0}\zeta(2\pi) + \frac{c_0\kappa_0}{\psi_0}\zeta_x(2\pi), \quad \text{for } \Phi = (\xi, \eta) \in \mathcal{D}(A^*). \tag{3.105}$$

Recall that $\mathcal{E}(A^*)$ denotes the set of all (generalized) eigenfunctions of A^*. Then, we write the following observation estimates under the assumption that all the eigenvalues of A^* are algebraically simple.

Lemma 3.3.4. *For all eigenfunction $\Phi_\nu \in \mathcal{E}(A^*) \setminus \{\Phi_0\}$, the observation operators satisfies $\mathcal{B}_\rho^*\Phi_\nu \neq 0, \mathcal{B}_u^*\Phi_\nu \neq 0$ and $\mathcal{B}_\theta^*\Phi_\nu \neq 0$. Moreover, we have the following estimates:*

$$\left|\mathcal{B}_\rho^*\Phi_n^h\right| \geq C, \quad \left|\mathcal{B}_\rho^*\Phi_n^{p_1}\right| \geq C, \quad \left|\mathcal{B}_\rho^*\Phi_n^{p_2}\right| \geq C, \tag{3.106}$$

$$\left|\mathcal{B}_u^*\Phi_n^h\right| \geq \frac{C}{|n|}, \quad \left|\mathcal{B}_u^*\Phi_n^{p_1}\right| \geq C\,|n|, \left|\mathcal{B}_u^*\Phi_n^{p_2}\right| \geq C, \tag{3.107}$$

$$\left|\mathcal{B}_\theta^*\Phi_n^h\right| \geq \frac{C}{|n|}, \quad \left|\mathcal{B}_\theta^*\Phi_n^{p_2}\right| \geq C \quad \left|\mathcal{B}_\theta^*\Phi_n^{p_2}\right| \geq C\,|n|, \tag{3.108}$$

for some $C > 0$ and all $n \in \mathbb{Z}^*$.

Proof. Recall from the proof of Lemma 3.3.3 that $\nu_1^n, \nu_2^n, \nu_3^n \neq 0$ (Claim 1) for all $n \in \mathbb{Z}^*$ and the eigenvectors $(\alpha_1^n, \alpha_2^n, \alpha_3^n)^\dagger$, $(\beta_1^n, \beta_2^n, \beta_3^n)^\dagger$ and $(\gamma_1^n, \gamma_2^n, \gamma_3^n)^\dagger$ of R_n satisfies the following relations:

$$(V_0 - \nu_3^n)\alpha_1^n + Q_0\alpha_2^n = 0, \quad (V_0 - \nu_1^n)\beta_1^n + Q_0\beta_2^n = 0, \quad (V_0 - \nu_2^n)\gamma_1^n + Q_0\gamma_2^n = 0; \tag{3.109}$$

$$\frac{R\psi_0}{Q_0}\alpha_1^n + (\lambda_0 in + V_0 - \nu_3^n)\alpha_2^n + R\alpha_3^n = 0, \quad \frac{R\psi_0}{Q_0}\beta_1^n + (\lambda_0 in + V_0 - \nu_1^n)\beta_2^n + R\beta_3^n = 0, \tag{3.110}$$
$$\frac{R\psi_0}{Q_0}\gamma_1^n + (\lambda_0 in + V_0 - \nu_2^n)\gamma_2^n + R\gamma_3^n = 0;$$

$$\frac{R\psi_0}{c_0}\alpha_2^n + (\kappa_0 in + V_0 - \nu_3^n)\alpha_3^n = 0, \quad \frac{R\psi_0}{c_0}\beta_2^n + (\kappa_0 in + V_0 - \nu_1^n)\beta_3^n = 0, \tag{3.111}$$
$$\frac{R\psi_0}{c_0}\gamma_2^n + (\kappa_0 in + V_0 - \nu_2^n)\gamma_3^n = 0,$$

for all $n \in \mathbb{Z}^*$.

We now consider the following cases:

Case 1. (Control acts in density) We have

$$\begin{aligned}
\mathcal{B}_\rho^*\Phi_n^h &= V_0\xi_n^h(2\pi) + Q_0\eta_n^h(2\pi) = V_0\alpha_1^n + Q_0\alpha_2^n = \nu_3^n\alpha_1^n \neq 0,\\
\mathcal{B}_\rho^*\Phi_n^{p_1} &= V_0\xi_n^{p_1}(2\pi) + Q_0\eta_n^{p_1}(2\pi) = V_0\beta_1^n + Q_0\beta_2^n = \nu_1^n\beta_1^n \neq 0,\\
\mathcal{B}_\rho^*\Phi_n^{p_2} &= V_0\xi_n^{p_2}(2\pi) + Q_0\eta_n^{p_2}(2\pi) = V_0\gamma_1^n + Q_0\gamma_2^n = \nu_2^n\gamma_1^n \neq 0,
\end{aligned}$$

for all $n \in \mathbb{Z}^*$, thanks to the equation (3.109).

Case 2. (Control acts in velocity) We have

$$\begin{aligned}
\mathcal{B}_u^*\Phi_n^h &= R\psi_0\xi_n^h(2\pi) + \lambda_0 Q_0(\eta_n^h)_x(2\pi) + Q_0V_0\eta_n^h(2\pi) + RQ_0\zeta_n^h(2\pi)\\
&= R\psi_0\alpha_1^n + \lambda_0 Q_0 in\alpha_2^n + Q_0V_0\alpha_2^n + RQ_0\alpha_3^n = Q_0\nu_3^n\alpha_2^n \neq 0,\\
\mathcal{B}_u^*\Phi_n^{p_1} &= R\psi_0\xi_n^{p_1}(2\pi) + \lambda_0 Q_0(\eta_n^{p_1})_x(2\pi) + Q_0V_0\eta_n^{p_1}(2\pi) + RQ_0\zeta_n^{p_1}(2\pi)\\
&= R\psi_0\beta_1^n + \lambda_0 Q_0 in\beta_2^n + Q_0V_0\beta_2^n + RQ_0\beta_3^n = Q_0\nu_1^n\beta_2^n \neq 0,\\
\mathcal{B}_u^*\Phi_n^{p_2} &= R\psi_0\xi_n^{p_2}(2\pi) + \lambda_0 Q_0(\eta_n^{p_2})_x(2\pi) + Q_0V_0\eta_n^{p_2}(2\pi) + RQ_0\zeta_n^{p_2}(2\pi)\\
&= R\psi_0\gamma_1^n + \lambda_0 Q_0 in\gamma_2^n + Q_0V_0\gamma_2^n + RQ_0\gamma_3^n = Q_0\nu_2^n\gamma_2^n \neq 0,
\end{aligned}$$

for all $n \in \mathbb{Z}^*$, thanks to the equation (3.110).

Case 3. (Control acts in temperature) We have

$$\begin{aligned}
\mathcal{B}_\theta^*\Phi_n^h &= R\eta_n^h(2\pi) + \frac{c_0V_0}{\psi_0}\zeta_n^h(2\pi) + \frac{c_0\kappa_0}{\psi_0}(\zeta_n^h)_x(2\pi)\\
&= R\alpha_2^n + \frac{c_0}{\psi_0}(V_0 + \kappa_0 in)\alpha_3^n = \frac{c_0}{\psi_0}\nu_3^n\alpha_3^n \neq 0,
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}_\theta^* \Phi_n^{p_1} &= R\eta_n^{p_1}(2\pi) + \frac{c_0 V_0}{\psi_0}\zeta_n^{p_1}(2\pi) + \frac{c_0\kappa_0}{\psi_0}(\zeta_n^{p_1})_x(2\pi) \\
&= R\beta_2^n + \frac{c_0}{\psi_0}(V_0 + \kappa_0 in)\beta_3^n = \frac{c_0}{\psi_0}\nu_1^n \beta_3^n \neq 0, \\
\mathcal{B}_\theta^* \Phi_n^{p_2} &= R\eta_n^{p_2}(2\pi) + \frac{c_0 V_0}{\psi_0}\zeta_n^{p_2}(2\pi) + \frac{c_0\kappa_0}{\psi_0}(\zeta_n^{p_2})_x(2\pi) \\
&= R\gamma_2^n + \frac{c_0}{\psi_0}(V_0 + \kappa_0 in)\gamma_3^n = \frac{c_0}{\psi_0}\nu_2^n \gamma_3^n \neq 0,
\end{aligned}$$

for all $n \in \mathbb{Z}^*$, thanks to the equation (3.111).

The estimates on the observation terms follows directly from the asymptotic expressions (3.91)-(3.92)-(3.93) and Remark 3.3.1. ☐

Remark 3.3.3. *Similar to the barotropic case, we can choose the (finitely many) generalized eigenfunctions $\tilde{\Phi}_{l,j} \in \mathcal{E}(A^*)$ for $1 \leq l < N_j, 1 \leq j \leq j_0$, in such a way that $\mathcal{B}_\rho^*\tilde{\Phi}_{l,j} \neq 0, \mathcal{B}_u^*\tilde{\Phi}_{l,j} \neq 0$ and $\mathcal{B}_\theta^*\tilde{\Phi}_{l,j} \neq 0$. This can be ensured by choosing a suitable multiple of the finitely many generalized eigenfunctions.*

3.3.5 Observability inequalities

As mentioned in the barotropic case, we will write the observability inequalities in this case also, which will help us prove the null controllability results for the system (3.6). The proof is similar and so we skip the details.

Theorem 3.3.2. *Let $T > 0$ be given. Then, the system (3.6)-(3.7)-(3.8) is null controllable at time T in the space $(\dot{L}^2(0, 2\pi))^3$ if and only if the observability inequality*

$$\left\|(\sigma(0), v(0), \varphi(0))^\dagger\right\|^2_{(\dot{L}^2(0,2\pi))^3} \leq C \int_0^T |V_0\sigma(t, 2\pi) + Q_0 v(t, 2\pi)|^2 \, dt \tag{3.112}$$

holds for all solutions $(\sigma, v, \varphi)^\dagger$ of the adjoint system (3.76) with terminal data $(\sigma_T, v_T, \varphi_T)^\dagger \in \mathcal{D}(A^*)$.

Theorem 3.3.3. *Let $T > 0$ be given. Then, the system (3.6)-(3.7)-(3.9) is null controllable at time T in the space $\dot{H}^1_{\text{per}}(0, 2\pi) \times (\dot{L}^2(0, 2\pi))^2$ if and only if the observability inequality*

$$\begin{aligned}
&\left\|(\sigma(0), v(0), \varphi(0))^\dagger\right\|^2_{\dot{H}^{-1}_{\text{per}}(0,2\pi)\times(\dot{L}^2(0,2\pi))^2} \\
&\qquad \leq C \int_0^T |R\psi_0\sigma(t, 2\pi) + \lambda_0 Q_0 v_x(t, 2\pi) + Q_0 V_0 v(t, 2\pi) + RQ_0\varphi(t, 2\pi)|^2 \, dt
\end{aligned} \tag{3.113}$$

holds for all solutions $(\sigma, v, \varphi)^\dagger$ of the adjoint system (3.76) with terminal data $(\sigma_T, v_T, \varphi_T)^\dagger \in \mathcal{D}(A^*)$.

Theorem 3.3.4. *Let $T > 0$ be given. Then, the system (3.6)-(3.7)-(3.10) is null controllable at time T in the space $\dot{H}^1_{\text{per}}(0, 2\pi) \times (\dot{L}^2(0, 2\pi))^2$ if and only if the observability inequality*

$$\left\|(\sigma(0), v(0), \varphi(0))^\dagger\right\|^2_{\dot{H}^{-1}_{\text{per}}(0,2\pi)\times(\dot{L}^2(0,2\pi))^2} \leq C \int_0^T \left| Rv(t, 2\pi) + \frac{c_0 V_0}{\psi_0}\varphi(t, 2\pi) + \frac{c_0\kappa_0}{\psi_0}\varphi_x(t, 2\pi) \right|^2 dt \tag{3.114}$$

holds for all solutions $(\sigma, v, \varphi)^\dagger$ of the adjoint system (3.76) with terminal data $(\sigma_T, v_T, \varphi_T)^\dagger \in \mathcal{D}(A^*)$.

To prove these inequalities, we require lower bound estimates of the corresponding observation terms (given in the right hand side of (3.112),(3.113) and (3.114)) and to obtain these bounds, we will use the Ingham-type inequality (3.14). Similar to the barotropic case, we first prove the null controllability results (Theorem 3.1.3) when all the eigenvalues of A^* are simple. The case when there exist generalized eigenfunctions corresponding to the finitely many multiple eigenvalues will be presented at the end of this section. Throughout the proof of null controllability of the system (3.6), we will assume that all the eigenvalues of A^* have geometric multiplicity 1, as mentioned in the hypothesis of Theorem 3.1.3.

3.3.5.1 The case of simple eigenvalues

Let $(\sigma_T, v_T, \varphi_T)^\dagger \in (\dot{L}^2(0,2\pi))^3$. Since the set of eigenfunctions $\left\{\Phi_n^h, \Phi_n^{p_1}, \Phi_n^{p_2} \ : \ n \in \mathbb{Z}^*\right\}$ of A^* forms a Riesz basis of $(\dot{L}^2(0,2\pi))^3$ (see Proposition 3.3.4), therefore any $(\sigma_T, v_T, \varphi_T)^\dagger \in (\dot{L}^2(0,2\pi))^3$ can be written as

$$(\sigma_T, v_T, \varphi_T)^\dagger = \sum_{n\in\mathbb{Z}^*} a_n^h \Phi_n^h + \sum_{n\in\mathbb{Z}^*} a_n^{p_1} \Phi_n^{p_1} + \sum_{n\in\mathbb{Z}^*} a_n^{p_2} \Phi_n^{p_2},$$

for some $(a_n^h)_{n\in\mathbb{Z}^*}, (a_n^{p_1})_{n\in\mathbb{Z}^*}, (a_n^{p_2})_{n\in\mathbb{Z}^*} \in \ell_2$. Then, the solution to the adjoint system (3.76) is

$$(\sigma(t,x), v(t,x), \varphi(t,x))^\dagger = \sum_{n\in\mathbb{Z}^*} a_n^h e^{\nu_n^h (T-t)} \Phi_n^h(x) + \sum_{n\in\mathbb{Z}^*} a_n^{p_1} e^{\nu_n^{p_1}(T-t)} \Phi_n^{p_1}(x) + \sum_{n\in\mathbb{Z}^*} a_n^{p_2} e^{\nu_n^{p_2}(T-t)} \Phi_n^{p_2}(x),$$

for $(t,x) \in (0,T) \times (0,2\pi)$, that is,

$$\begin{aligned}
\sigma(t,x) &= \sum_{n\in\mathbb{Z}^*} a_n^h e^{\nu_n^h (T-t)} \alpha_1^n e^{inx} + \sum_{n\in\mathbb{Z}^*} a_n^{p_1} e^{\nu_n^{p_1}(T-t)} \beta_1^n e^{inx} + \sum_{n\in\mathbb{Z}^*} a_n^{p_2} e^{\nu_n^{p_2}(T-t)} \gamma_1^n e^{inx}, \\
v(t,x) &= \sum_{n\in\mathbb{Z}^*} a_n^h e^{\nu_n^h (T-t)} \alpha_2^n e^{inx} + \sum_{n\in\mathbb{Z}^*} a_n^{p_1} e^{\nu_n^{p_1}(T-t)} \beta_2^n e^{inx} + \sum_{n\in\mathbb{Z}^*} a_n^{p_2} e^{\nu_n^{p_2}(T-t)} \gamma_2^n e^{inx}, \\
\varphi(t,x) &= \sum_{n\in\mathbb{Z}^*} a_n^h e^{\nu_n^h (T-t)} \alpha_3^n e^{inx} + \sum_{n\in\mathbb{Z}^*} a_n^{p_1} e^{\nu_n^{p_1}(T-t)} \beta_3^n e^{inx} + \sum_{n\in\mathbb{Z}^*} a_n^{p_2} e^{\nu_n^{p_2}(T-t)} \gamma_3^n e^{inx},
\end{aligned}$$

for $(t,x) \in (0,T) \times (0,2\pi)$. We first rewrite the eigenvalues as $\{\nu_n^h, \nu_n^p\}_{n\in\mathbb{Z}^*}$, where

$$\nu_n^p = \begin{cases} \nu_k^{p_1}, & \text{if } n = 2k-1, \ k \in \mathbb{Z} \\ \nu_k^{p_2}, & \text{if } n = 2k, \ k \in \mathbb{Z}^*, \end{cases}$$

for all $n \in \mathbb{Z}^*$ and ν_n^h is as defined earlier (see Lemma 3.3.3). We also denote the eigenfunction

$$\Phi_n^p = \begin{cases} \Phi_k^{p_1}, & \text{if } n = 2k-1, \ k \in \mathbb{Z}, \\ \Phi_k^{p_2}, & \text{if } n = 2k, \ k \in \mathbb{Z}^*, \end{cases}$$

and the observation term

$$\mathcal{B}^*\Phi_n^p = \begin{cases} \mathcal{B}^*\Phi_k^{p_1}, & \text{if } n = 2k-1, \ k \in \mathbb{Z}, \\ \mathcal{B}^*\Phi_k^{p_2}, & \text{if } n = 2k, \ k \in \mathbb{Z}^*, \end{cases}$$

for all $n \in \mathbb{Z}^*$. Also, recall that we have defined the set

$$\mathcal{S} := \left\{ (\lambda_0, \kappa_0) \ : \ \sqrt{\frac{\lambda_0}{\kappa_0}} \notin \mathbb{Q} \right\}.$$

We further denote

$$a_n^p = \begin{cases} a_k^{p_1}, & \text{if } n = 2k-1, \ k \in \mathbb{Z}, \\ a_k^{p_2}, & \text{if } n = 2k, \ k \in \mathbb{Z}^*. \end{cases}$$

Then, we can write

$$(\sigma(t,x), v(t,x), \varphi(t,x))^\dagger = \sum_{n\in\mathbb{Z}^*} a_n^h e^{\nu_n^h (T-t)} \Phi_n^h(x) + \sum_{n\in\mathbb{Z}^*} a_n^p e^{\nu_n^p (T-t)} \Phi_n^p(x),$$

for $(t,x) \in (0,T) \times (0,2\pi)$.

Estimates on the norms of $(\sigma(0), v(0), \varphi(0))^\dagger$: We have

$$\left\|(\sigma(0), v(0), \varphi(0))^\dagger\right\|^2_{(\dot{L}^2(0,2\pi))^3} \leq C \left[\sum_{n\in\mathbb{Z}^*} \left|a_n^h\right|^2 + \sum_{n\in\mathbb{Z}^*} \left|a_n^p\right|^2 e^{2\mathrm{Re}(\nu_n^p)T} \right], \tag{3.115}$$

thanks to the asymptotic expressions (3.84). We also have

$$\left\|(\sigma(0), v(0), \varphi(0))^{\dagger}\right\|^2_{\dot{H}^{-1}_{\rm per}(0,2\pi)\times(\dot{L}^2(0,2\pi))^2} \leq C\left[\sum_{n\in\mathbb{Z}^*}\left|a_n^h\right|^2\frac{1}{|n|^2}+\sum_{n\in\mathbb{Z}^*}\left|a_n^p\right|^2 e^{2\mathrm{Re}(\nu_n^p)T}\right], \tag{3.116}$$

thanks to the asymptotic expressions (3.84).

To prove our null controllability results (Theorem 3.1.3), we will use the Ingham-type inequality (3.14) and for that, we need to prove that the eigenvalues $(\nu_n^h)_{n\in\mathbb{Z}^*}$ and $(\nu_n^p)_{n\in\mathbb{Z}^*}$ satisfy all the hypotheses of Lemma 3.1.1. Recall the asymptotic expressions of the eigenvalues, given by Lemma 3.3.3:

$$\begin{aligned}
\nu_n^h &= V_0 in - \overline{\omega} + O(|n|^{-2}),\\
\nu_n^{p_1} &= -\lambda_0 n^2 + V_0 in + O(1),\\
\nu_n^{p_2} &= -\kappa_0 n^2 + V_0 in + O(1).
\end{aligned}$$

- Due to our assumption on the eigenvalues, we have $\nu_n^h \neq \nu_l^h$, $\nu_n^p \neq \nu_l^p$ for all $n, l \in \mathbb{Z}^*$ with $n \neq l$ and $\{\nu_n^h\ ;\ n\in\mathbb{Z}^*\}\cap\{\nu_n^p\ ;\ n\in\mathbb{Z}^*\} = \emptyset$.
- From the expression of ν_n^h, it is easy to see that the family $(\nu_n^h)_{n\in\mathbb{Z}^*}$ satisfies hypothesis (H2) of Lemma 3.1.1 with $\beta = -\overline{\omega}, \tau = V_0$ and $e_n = O(|n|^{-2})$ for $|n|$ large enough.
- On the other hand, we have

$$\frac{-\mathrm{Re}(\nu_n^p)}{\left|\mathrm{Im}(\nu_n^p)\right|} = \begin{cases} \frac{\lambda_0 k^2 + O(1)}{V_0|k|}, & \text{if } n = 2k-1,\ k\in\mathbb{Z},\\ \frac{\kappa_0 k^2+O(1)}{V_0|k|}, & \text{if } n = 2k,\ k\in\mathbb{Z}^*,\end{cases}$$

  and therefore $\frac{-\mathrm{Re}(\nu_n^p)}{|\mathrm{Im}(\nu_n^p)|} \geq \min(\frac{\lambda_0}{V_0}, \frac{\kappa_0}{V_0})$ for $|n|$ large enough, which verifies hypothesis (P2) of Lemma 3.1.1.
- We also have for $|n|$ large

$$\lambda_0 n^2 \leq \left|\nu_n^{p_1}\right| \leq (\lambda_0+V_0)n^2,\ \text{ and } \kappa_0 n^2 \leq \left|\nu_n^{p_1}\right| \leq (\kappa_0+V_0)n^2,$$

  and therefore (ν_n^p) satisfies hypothesis (P4) of Lemma 3.1.1 for large enough $|n|$.

The family (ν_n^h) satisfy hypotheses (H1)-(H2) of Lemma 3.1.1 for $|n|$ large enough, and therefore one can have the hyperbolic Ingham inequality (3.16). On the other hand, the parabolic branch $(\nu_n^p)_{n\in\mathbb{Z}^*}$ satisfy hypotheses (P1)-(P2) and (P4), but does not necessarily satisfy the gap condition (Hypothesis (P3) of Lemma 3.1.1) when $|n|$ is large enough. However, we can prove the existence of a biorthogonal family to $(e^{\nu_n^p t})_{n\in\mathbb{Z}^*}$ under the stronger assumption (3.13) on the coefficients λ_0 and κ_0; as a consequence we have the parabolic Ingham inequality (3.15) (thanks to Remark 3.1.4).

Lemma 3.3.5. *Let us assume that all eigenvalues $(\nu_n^p)_{n\in\mathbb{Z}^*}$ of A^* are distinct. Then, under the assumption of Theorem 3.1.3 and given $\epsilon > 0$, there exists a sequence $(q_n)_{n\in\mathbb{Z}^*} \subset L^2(0,\infty)$ biorthogonal to the family $(e^{\nu_n^p t})_{n\in\mathbb{Z}^*}$ with the following estimate*

$$\|q_n\|_{L^2(0,\infty)} \leq K(\epsilon)e^{\mathrm{Re}(\nu_n^p)\epsilon} \tag{3.117}$$

for all $n \in \mathbb{Z}^*$.

The proof of this Lemma can be done in a similar way as [FCGBdT10, Lemma 3.1] and [LdT13, Lemma 2], so we omit the details. Indeed, an easy calculation yields that

$$\begin{aligned}
&\left|\nu_n^{p_1} - \nu_j^{p_1}\right| \geq C\left|n^2 - j^2\right|, \quad \left|\nu_n^{p_1} - \nu_j^{p_2}\right| \geq C\left|\lambda_0 n^2 - \kappa_0 j^2\right|,\\
&\left|\nu_n^{p_2} - \nu_j^{p_1}\right| \geq C\left|\kappa_0 n^2 - \lambda_0 j^2\right|, \quad \left|\nu_n^{p_2} - \nu_j^{p_2}\right| \geq C\left|n^2 - j^2\right|,
\end{aligned}$$

for some $C > 0$. With the help of this Lemma and the hyperbolic Ingham inequality (3.16), we can have the combined Ingham-type inequality (3.14) (as mentioned in Remark 3.1.4). With this, we are now ready to prove null controllability results of the system (3.6) in the case of simple eigenvalues.

Proof of Theorem 3.1.3-Part (i). Let $T > \frac{2\pi}{V_0}$. Recall from Theorem 3.3.2 that it is enough to prove the observability inequality (3.112), that is,

$$\int_0^T |V_0\sigma(t,2\pi) + Q_0 v(t,2\pi)|^2\, dt \geq C \left\| (\sigma(0), v(0), \varphi(0))^\dagger \right\|^2_{(\dot{L}^2(0,2\pi))^3},$$

for all $(\sigma_T, v_T, \varphi_T)^\dagger \in \mathcal{D}(A^*)$. Also, recall the observation operator $\mathcal{B}^*_\rho$ given by (3.103). Then, we have the observation term

$$\begin{aligned}
&\int_0^T |V_0\sigma(t,2\pi) + Q_0 v(t,2\pi)|^2\, dt \\
&= \int_0^T \left| \sum_{n\in\mathbb{Z}^*} a_n^h \mathcal{B}^*_\rho \Phi_n^h e^{\nu_n^h (T-t)} + \sum_{n\in\mathbb{Z}^*} a_n^{p_1} \mathcal{B}^*_\rho \Phi_n^{p_1} e^{\nu_n^{p_1}(T-t)} + \sum_{n\in\mathbb{Z}^*} a_n^{p_2} \mathcal{B}^*_\rho \Phi_n^{p_2} e^{\nu_n^{p_2}(T-t)} \right|^2 dt \\
&= \int_0^T \left| \sum_{n\in\mathbb{Z}^*} a_n^h \mathcal{B}^*_\rho \Phi_n^h e^{\nu_n^h (T-t)} + \sum_{n\in\mathbb{Z}^*} a_n^{p} \mathcal{B}^*_\rho \Phi_n^{p} e^{\nu_n^{p}(T-t)} \right|^2 dt.
\end{aligned}$$

Using the combined parabolic-hyperbolic Ingham type inequality (3.14) (Lemma 3.1.1), we have

$$\begin{aligned}
\int_0^T |V_0\sigma(t,2\pi) + Q_0 v(t,2\pi)|^2\, dt &\geq C \left(\sum_{n\in\mathbb{Z}^*} \left|a_n^h\right|^2 \left|\mathcal{B}^*_\rho \Phi_n^h\right|^2 e^{2\mathrm{Re}(\nu_n^h)(T-t)} + \sum_{n\in\mathbb{Z}^*} \left|a_n^p\right|^2 \left|\mathcal{B}^*_\rho \Phi_n^p\right|^2 e^{2\mathrm{Re}(\nu_n^p)T} \right) \\
&\geq C \left(\sum_{n\in\mathbb{Z}^*} \left|a_n^h\right|^2 + \sum_{n\in\mathbb{Z}^*} \left|a_n^p\right|^2 e^{2\mathrm{Re}(\nu_n^p)T} \right),
\end{aligned}$$

thanks to the estimate (3.106). This estimate together with the norm estimate (3.115), the observability inequality (3.112) follows. This completes the proof in the case of simple eigenvalues.

Proof of Theorem 3.1.3-Part (ii). Let $T > \frac{2\pi}{V_0}$. We will consider only the velocity control case. The case when a control acts in temperature (3.10), the proof will be similar (as we have similar lower bounds on the observation term $\mathcal{B}^*_u \Phi_n^h$ and $\mathcal{B}^*_\theta \Phi_n^h$, see Lemma 3.3.4 for instance), and so we omit the details. Thanks to Theorem 3.3.3, it is enough to prove the observability inequality (3.113), that is,

$$\int_0^T |R\psi_0\sigma(t,2\pi) + \lambda_0 Q_0 v_x(t,2\pi) + Q_0 V_0 v(t,2\pi) + RQ_0\varphi(t,2\pi)|^2 dt \geq C \left\| (\sigma(0), v(0), \varphi(0))^\dagger \right\|^2_{\dot{H}^{-1}_{\mathrm{per}} \times (\dot{L}^2)^2}$$

for all $(\sigma_T, v_T, \varphi_T)^\dagger \in \mathcal{D}(A^*)$. We have

$$\begin{aligned}
&\int_0^T |R\psi_0\sigma(t,2\pi) + \lambda_0 Q_0 v_x(t,2\pi) + Q_0 V_0 v(t,2\pi) + RQ_0\varphi(t,2\pi)|^2\, dt \\
&= \int_0^T \left| \sum_{n\in\mathbb{Z}^*} a_n^h \mathcal{B}^*_u \Phi_n^h e^{\nu_n^h (T-t)} + \sum_{n\in\mathbb{Z}^*} a_n^{p_1} \mathcal{B}^*_u \Phi_n^{p_1} e^{\nu_n^{p_1}(T-t)} + \sum_{n\in\mathbb{Z}^*} a_n^{p_2} \mathcal{B}^*_u \Phi_n^{p_2} e^{\nu_n^{p_2}(T-t)} \right|^2 dt \\
&= \int_0^T \left| \sum_{n\in\mathbb{Z}^*} a_n^h \mathcal{B}^*_u \Phi_n^h e^{\nu_n^h (T-t)} + \sum_{n\in\mathbb{Z}^*} a_n^{p} \mathcal{B}^*_u \Phi_n^{p} e^{\nu_n^{p}(T-t)} \right|^2 dt
\end{aligned}$$

where $\mathcal{B}^*_u$ is defined in (3.104). Using the combined parabolic-hyperbolic Ingham type inequality (3.14) (see Lemma 3.1.1) and the observation estimates (3.107), we obtain

$$\int_0^T |R\psi_0\sigma(t,2\pi) + \lambda_0 Q_0 v_x(t,2\pi) + Q_0 V_0 v(t,2\pi) + RQ_0\varphi(t,2\pi)|^2\, dt$$

$$\geq C\left(\sum_{n\in\mathbb{Z}^*}\left|a_n^h\right|^2\left|\mathcal{B}_u^*\Phi_n^h\right|^2 e^{2\mathrm{Re}(\nu_n^h)(T-t)}+\sum_{n\in\mathbb{Z}^*}\left|a_n^p\right|^2\left|\mathcal{B}_u^*\Phi_n^p\right|^2 e^{2\mathrm{Re}(\nu_n^p)T}\right)$$
$$\geq C\left(\sum_{n\in\mathbb{Z}^*}\left|a_n^h\right|^2\frac{1}{|n|^2}+\sum_{n\in\mathbb{Z}^*}\left|a_n^p\right|^2 e^{2\mathrm{Re}(\nu_n^p)T}\right).$$

Thus, we obtain that

$$\int_0^T |R\psi_0\sigma(t,2\pi)+\lambda_0 Q_0 v_x(t,2\pi)+Q_0V_0v(t,2\pi)+RQ_0\varphi(t,2\pi)|^2\,dt$$
$$\geq C\left\|(\sigma(0),v(0),\varphi(0))^\dagger\right\|^2_{\dot{H}^{-1}_{\mathrm{per}}(0,2\pi)\times(\dot{L}^2(0,2\pi))^2},$$

thanks to the estimate (3.115). This proves the observability inequality (3.113) and the proof is complete for simple eigenvalues.

3.3.5.2 The case of multiple eigenvalues

Throughout the proof, we assume that all the eigenvalues of A^* have geometric multiplicity 1. Recall that ν_j is the eigenvalues of A^* with multiplicity N_j for $j=1,2,\ldots,j_0$, and for all $|n|>n_0$, the eigenvalues ν_n of A^* are algebraically simple. Also, recall the set of (generalized) eigenfunctions corresponding to ν_j (for $1\leq j\leq j_0$) as

$$\left\{\Phi_j=(\xi_j,\eta_j,\zeta_j)\ ;\ \tilde{\Phi}_{l,j}=(\tilde{\xi}_{l,j},\tilde{\eta}_{l,j},\tilde{\zeta}_{l,j})\ :\ l=1,\ldots,N_j-1,\ j=1,\ldots,j_0\right\}.$$

The proof of null controllability of the system (3.6) in the presence of multiple eigenvalue will be similar to the barotropic case, so we give a brief proof of Theorem 3.1.3 in each cases (control acting in density, velocity and temperature).

Control in density. Let $(\sigma_T,v_T,\varphi_T)^\dagger\in(\dot{L}^2(0,2\pi))^3$. We decompose it as follows:

$$(\sigma_T,v_T,\varphi_T)^\dagger=(\sigma_{T,1},v_{T,1},\varphi_{T,1})^\dagger+(\sigma_{T,2},v_{T,2},\varphi_{T,2})^\dagger, \tag{3.118}$$

with

$$(\sigma_{T,1},v_{T,1},\varphi_{T,1})^\dagger=\sum_{j=1}^{j_0}\left(a_j\Phi_j+\sum_{l=1}^{N_j-1}\tilde{a}_{l,j}\tilde{\Phi}_{l,j}\right)$$

and

$$(\sigma_{T,2},v_{T,2},\varphi_{T,2})^\dagger=\sum_{|n|\geq n_0}\left(a_n^h\Phi_n^h+a_n^{p_1}\Phi_n^{p_1}+a_n^{p_2}\Phi_n^{p_2}\right).$$

Let (σ_1,v_1,φ_1) and (σ_2,v_2,φ_2) denote the solutions of the adjoint system (3.76) associated to the terminal data $(\sigma_{T,1},v_{T,1},\varphi_{T,1})$ and $(\sigma_{T,2},v_{T,2},\varphi_{T,2})$ respectively. Then, we can write these solutions as

$$(\sigma_1(t),v_1(t),\varphi_1(t))^\dagger=\sum_{j=1}^{j_0}e^{\nu_j(T-t)}\left(a_j\Phi_j+\sum_{l=1}^{N_j-1}(T-t)^l\tilde{a}_{l,j}\tilde{\Phi}_{l,j}\right), \tag{3.119}$$
$$(\sigma_2(t),v_2(t),\varphi_2(t))^\dagger=\sum_{|n|\geq n_0}\left(a_n^h e^{\nu_n^h(T-t)}\Phi_n^h+a_n^{p_1}e^{\nu_n^{p_1}(T-t)}\Phi_n^{p_1}+a_n^{p_2}e^{\nu_n^{p_2}(T-t)}\Phi_n^{p_2}\right), \tag{3.120}$$

for $t\in[0,T]$. Using the observability inequality (3.112) in the case of simple eigenvalues, we get

$$\int_0^T|V_0\sigma_2(t,2\pi)+Q_0v_2(t,2\pi)|^2\,dt\geq C\left\|(\sigma_2(0),v_2(0),\varphi_2(0))^\dagger\right\|_{(\dot{L}^2(0,2\pi))^3}. \tag{3.121}$$

Note that

$$V_0\sigma_1(t,2\pi)+Q_0v_1(t,2\pi)=\sum_{j=1}^{j_0}e^{\nu_j(T-t)}\left(a_j\mathcal{B}_\rho^*\Phi_j+\sum_{l=1}^{N_j-1}\tilde{a}_{l,j}(T-t)^l\mathcal{B}_\rho^*\tilde{\Phi}_{l,j}\right) \tag{3.122}$$

for $t \in (0,T)$. Proceeding similarly as in the barotropic case and using the well-posedness result (Lemma 3.3.2)

$$\int_0^{\epsilon} |V_0\sigma_2(t,2\pi) + Q_0 v_2(t,2\pi)|^2\,dt \le C\left\|(\sigma_2(\epsilon), v_2(\epsilon), \varphi_2(\epsilon))^\dagger\right\|^2_{(\dot{L}^2(0,2\pi))^3},$$

(for $\epsilon > 0$ small enough) and finite dimensional norm equivalence (thanks to Lemma 3.3.4-Remark 3.3.3), we can add these finitely many terms in the above observability inequality to obtain

$$\int_0^{T} |V_0\sigma(t,2\pi) + Q_0 v(t,2\pi)|^2\,dt \ge C\left\|(\sigma_2(0), v_2(0), \varphi_2(0))^\dagger\right\|^2_{(\dot{L}^2(0,2\pi))^3}. \tag{3.123}$$

and

$$\int_0^{T} |V_0\sigma(t,2\pi) + Q_0 v(t,2\pi)|^2\,dt \ge C\left\|(\sigma_1(0), v_1(0), \varphi_1(0))^\dagger\right\|^2_{(\dot{L}^2(0,2\pi))^3}. \tag{3.124}$$

Combining these two inequalities (3.123) and (3.124), we obtain the desired observability inequality (3.112), proving Theorem 3.1.3-Part (i) in the case of multiple eigenvalues.

Control in velocity. As mentioned in the barotropic case, it is enough to prove the following inequality:

$$\begin{aligned}\int_0^{\frac{\epsilon}{2}} |R\psi_0\sigma(t,2\pi) + \lambda_0 Q_0 v_x(t,2\pi) + Q_0V_0 v(t,2\pi) + RQ_0\varphi(t,2\pi)|^2\,dt \\ \le C\left\|(\sigma_2(\epsilon), v_2(\epsilon), \varphi_2(\epsilon))^\dagger\right\|^2_{\dot{H}^{-1}_{\text{per}}(0,2\pi)\times(\dot{L}^2(0,2\pi))^2}.\end{aligned} \tag{3.125}$$

Recall that (equation (3.120))

$$(\sigma_2(t), v_2(t), \varphi_2(t))^\dagger = \sum_{|n|\ge n_0}\left(a_n^h e^{\nu_n^h(T-t)}\Phi_n^h + a_n^{p_1} e^{\nu_n^{p_1}(T-t)}\Phi_n^{p_1} + a_n^{p_2} e^{\nu_n^{p_2}(T-t)}\Phi_n^{p_2}\right)$$

for $t \in (0,T)$. Since the observation term $\mathcal{B}_u^*\Phi_n^h$ have similar upper bound (of order $\frac{1}{n}$), proceeding similarly as in the barotropic case, we can obtain

$$\begin{aligned}\int_0^{\frac{\epsilon}{2}} |R\psi_0\sigma(t,2\pi) + \lambda_0 Q_0 v_x(t,2\pi) + Q_0V_0 v(t,2\pi) + RQ_0\varphi(t,2\pi)|^2\,dt \\ \le C\sum_{|n|\ge n_0}\frac{|a_n^h|^2}{|n|^2} + C\sum_{|n|\ge n_0}|a_n^p|^2 e^{2\text{Re}(\nu_n^p)(T-\epsilon)},\end{aligned} \tag{3.126}$$

see for instance the inequality (3.59). On the other hand, we compute

$$\begin{aligned}&\|(\sigma_2(\epsilon), v_2(\epsilon), \varphi_2(\epsilon))\|_{\dot{H}^{-1}_{\text{per}}(0,2\pi)\times(\dot{L}^2(0,2\pi))^2} \\ &= \sum_{|n|\ge n_0}\left(\frac{R\psi_0}{|n|^2}\left|a_n^h e^{\nu_n^h(T-\epsilon)}\alpha_1^n + a_n^{p_1} e^{\nu_n^{p_1}(T-\epsilon)}\beta_1^n + a_n^{p_2} e^{\nu_n^{p_2}(T-\epsilon)}\gamma_1^n\right|^2\right. \\ &\qquad + Q_0^2\left|a_n^h e^{\nu_n^h(T-\epsilon)}\alpha_2^n + a_n^{p_1} e^{\nu_n^{p_1}(T-\epsilon)}\beta_2^n + a_n^{p_2} e^{\nu_n^{p_2}(T-\epsilon)}\gamma_2^n\right|^2 \\ &\qquad\left. + \frac{Q_0^2 c_0}{\psi_0}\left|a_n^h e^{\nu_n^h(T-\epsilon)}\alpha_3^n + a_n^{p_1} e^{\nu_n^{p_1}(T-\epsilon)}\beta_3^n + a_n^{p_2} e^{\nu_n^{p_2}(T-\epsilon)}\gamma_3^n\right|^2\right).\end{aligned}$$

Thanks to Remark 3.3.1, we can write

$$\begin{aligned}&\|(\sigma_2(\epsilon), v_2(\epsilon), \varphi_2(\epsilon))\|_{\dot{H}^{-1}_{\text{per}}(0,2\pi)\times(\dot{L}^2(0,2\pi))^2} \\ &\ge C\left(\sum_{|n|\ge n_0}\frac{|a_n^h|^2}{|n|^2} + \sum_{|n|\ge n_0}|a_n^{p_1}|^2 e^{2\text{Re}(\nu_n^{p_1})(T-\epsilon)} + \sum_{|n|\ge n_0}|a_n^{p_2}|^2 e^{2\text{Re}(\nu_n^{p_2})(T-\epsilon)}\right)\end{aligned} \tag{3.127}$$

$$= C\left(\sum_{|n|\geq n_0} \frac{\left|a_n^h\right|^2}{|n|^2} + \sum_{|n|\geq n_0} \left|a_n^p\right|^2 e^{2\mathrm{Re}(\nu_n^p)(T-\epsilon)} \right).$$

Comparing this inequality with (3.126), we deduce that

$$\int_0^{\frac{\epsilon}{2}} |R\psi_0\sigma(t,2\pi) + \lambda_0 Q_0 v_x(t,2\pi) + Q_0 V_0 v(t,2\pi) + RQ_0\varphi(t,2\pi)|^2\,dt \\ \leq C\,\|(\sigma_2(\epsilon), v_2(\epsilon), \varphi_2(\epsilon))\|_{\dot{H}^{-1}_{\mathrm{per}}(0,2\pi)\times(\dot{L}^2(0,2\pi))^2},$$

proving the required inequality (3.125).

<u>**Control in Temperature.**</u> The proof will be similar to the velocity case (due to the similar bounds on the observation terms) and so we skip the details.

This concludes the proof of Theorem 3.1.3 in the case of multiple eigenvalues. □

3.3.6 Lack of null controllability for less regular initial states

Similar to the barotropic case, we first write the following result:

Proposition 3.3.5. *Let $0 \leq s < 1$ and $T > 0$ be given. Then,*

- *the system (3.6)-(3.7)-(3.9) is null controllable at time T in the space $\dot{H}^s_{\mathrm{per}}(0,2\pi) \times (\dot{L}^2(0,2\pi))^2$ if and only if the inequality*

$$\left\|(\sigma(0), v(0), \varphi(0))^\dagger\right\|^2_{\dot{H}^{-s}_{\mathrm{per}}(0,2\pi)\times(\dot{L}^2(0,2\pi))^2} \\ \leq C\int_0^T |R\psi_0\sigma(t,2\pi) + \lambda_0 Q_0 v_x(t,2\pi) + Q_0 V_0 v(t,2\pi) + RQ_0\varphi(t,2\pi)|^2\,dt \tag{3.128}$$

  holds for all $(\sigma, v, \varphi)^\dagger$ of the adjoint system (3.76) with $(\sigma_T, v_T, \varphi_T)^\dagger \in \mathcal{D}(A^*)$.

- *the system (3.6)-(3.7)-(3.10) is null controllable at time T in the space $\dot{H}^s_{\mathrm{per}}(0,2\pi) \times (\dot{L}^2(0,2\pi))^2$ if and only if the inequality*

$$\left\|(\sigma(0), v(0), \varphi(0))^\dagger\right\|^2_{\dot{H}^{-s}_{\mathrm{per}}(0,2\pi)\times(\dot{L}^2(0,2\pi))^2} \leq C\int_0^T \left| Rv(t,2\pi) + \frac{c_0 V_0}{\psi_0}\varphi(t,2\pi) + \frac{c_0\kappa_0}{\psi_0}\varphi_x(t,2\pi)\right|^2 dt \tag{3.129}$$

  holds for all solutions $(\sigma, v, \varphi)^\dagger$ of (3.76) with $(\sigma_T, v_T, \varphi_T)^\dagger \in \mathcal{D}(A^*)$.

3.3.6.1 Proof of Proposition 3.1.2- Part (ii)

We will present the proof for velocity case only; the temperature case will be exactly similar, because the observation terms $B^*_\theta\Phi^h_n$ and $B^*_u\Phi^h_n$ have same upper bounds (see Lemma 3.3.4). For $(\sigma^n_T, v^n_T, \varphi^n_T)^\dagger = \Phi^h_n$, the solution to the adjoint system is

$$(\sigma^n(t,x), v^n(t,x), \varphi^n(t,x))^\dagger = e^{\nu^h_n(T-t)}\Phi^h_n(x),$$

for $(t,x) \in (0,T)\times(0,2\pi)$ and $n \in \mathbb{Z}^*$. For large $|n|$, we have the following estimate

$$\left\|\Phi^h_n\right\|_{H^{-s}_{\mathrm{per}}(0,2\pi)\times(L^2(0,2\pi))^2} \geq \frac{C}{|n|^s},$$

and therefore

$$\left\|(\sigma^n(0), v^n(0), \varphi^n(0))^\dagger\right\|^2_{H^{-s}_{\mathrm{per}}(0,2\pi)\times(L^2(0,2\pi))^2} \geq \frac{C}{|n|^{2s}}$$

for all $|n|$ large. We also have

$$\int_0^T \left|R\psi_0\sigma^n(t,2\pi) + \lambda_0 Q_0 v^n_x(t,2\pi) + Q_0 V_0 v^n(t,2\pi) + RQ_0\varphi^n(t,2\pi)\right|^2 dt \leq \frac{C}{|n|^2},$$

for all $n \in \mathbb{Z}^*$ (see equation (3.126) for instance). Thus, if the observability inequality (3.128) holds, then we get

$$\frac{C}{|n|^{2s}} \leq \frac{C}{|n|^2} \implies |n|^{2-2s} \leq C,$$

which is not possible since $0 \leq s < 1$. This completes the proof. □

3.3.7 Lack of controllability at small time

The proof will be similar to the barotropic case, that is, the proof of Theorem 3.1.1- Part (ii). For the sake of completeness, we give the proof below.

3.3.7.1 Proof of Proposition 3.1.2-Part (i)

Let $0 < T < \frac{2\pi}{V_0}$. Following the notations in the proof of Theorem 3.1.1- Part (ii) (Section 3.2.7), we consider the system

$$\begin{cases} \tilde{\sigma}_t + V_0 \tilde{\sigma}_x = \overline{\omega}\tilde{\sigma}, & \text{in } (0,T)\times(0,2\pi), \\ \tilde{\sigma}(t,0) = \tilde{\sigma}(t,2\pi), & \text{for } t \in (0,T), \\ \tilde{\sigma}(T,x) = \tilde{\sigma}_T^N(x), & \text{in } (0,2\pi). \end{cases} \tag{3.130}$$

Since $\text{supp}(\tilde{\sigma}_T^N) \subset \text{supp}(\tilde{\sigma}_T) \subset (T, 2\pi)$, the solution satisfies $\tilde{\sigma}^N(t,0) = \tilde{\sigma}^N(t,2\pi) = 0$ for all $t \in (0,T)$. We now consider the adjoint to our main system

$$\begin{cases} -\sigma_t - V_0\sigma_x - Q_0 v_x = 0, & \text{in } (0,T)\times(0,2\pi), \\ -v_t - \lambda_0 v_{xx} - \dfrac{R\psi_0}{Q_0}\sigma_x - V_0 v_x - R\varphi_x = 0, & \text{in } (0,T)\times(0,2\pi), \\ -\varphi_t - \kappa_0\varphi_{xx} - \dfrac{R\psi_0}{c_0} v_x - V_0\varphi_x = 0, & \text{in } (0,T)\times(0,2\pi), \\ \sigma(t,0) = \sigma(t,2\pi), & \text{for } t \in (0,T), \\ v(t,0) = v(t,2\pi), \quad v_x(t,0) = v_x(t,2\pi), & \text{for } t \in (0,T), \\ \varphi(t,0) = \varphi(t,2\pi), \quad \varphi_x(t,0) = \varphi_x(t,2\pi), & \text{for } t \in (0,T), \\ \sigma(T,x) = \tilde{\sigma}_T^N(x), \quad v(T,x) = v_T^N(x), \quad \varphi(T,x) = \varphi_T^N(x), & \text{in } (0,2\pi), \end{cases} \tag{3.131}$$

where we choose v_T^N and φ_T^N such that

$$(\tilde{\sigma}_T^N, v_T^N, \varphi_T^N)^\dagger = \sum_{|n|\geq N+1} \tilde{a}_n^h \Phi_n^h$$

with $\tilde{a}_n^h \alpha_1^n := a_n P^N(n)$ for all $|n| \geq N+1$. We write the solutions to the systems (3.130) and (3.131) respectively as

$$\tilde{\sigma}^N(t,x) = \sum_{|n|\geq N+1} a_n P^N(n) e^{(V_0 in - \overline{\omega})(T-t)} e^{inx}, \tag{3.132}$$

$$\sigma^N(t,x) = \sum_{|n|\geq N+1} a_n P^N(n) e^{\nu_n^h (T-t)} e^{inx}, \tag{3.133}$$

$$v^N(t,x) = \sum_{|n|\geq N+1} a_n P^N(n) \frac{\beta_1^n}{\alpha_1^n} e^{\nu_n^h (T-t)} e^{inx}, \tag{3.134}$$

$$\varphi^N(t,x) = \sum_{|n|\geq N+1} a_n P^N(n) \frac{\gamma_1^n}{\alpha_1^n} e^{\nu_n^h (T-t)} e^{inx}, \tag{3.135}$$

for all $(t,x) \in [0,T]\times[0,2\pi]$. Similar to the barotropic case, we prove that the solution component σ^N approximates the solution $\tilde{\sigma}^N$. Indeed, we have

$$\left\|\sigma^N(\cdot,x) - \tilde{\sigma}^N(\cdot,x)\right\|^2_{L^2(0,T)} \leq \sum_{|n|\geq N+1} |a_n|^2 \left|P^N(n)\right|^2 \left\| e^{\nu_n^h (T-t)} - e^{(V_0 in - \overline{\omega})(T-t)} \right\|^2_{L^2(0,T)}$$

$$\leq \sum_{|n|\geq N+1} |a_n|^2 \left|P^N(n)\right|^2 \left\| e^{(V_0 in - \overline{\omega})(T-t)} e^{O(|n|^{-1})(T-t)} - 1 \right\|^2_{L^2(0,T)}$$
$$\leq \frac{C}{|n|^2} \sum_{|n|\geq N+1} |a_n|^2 \left|P^N(n)\right|^2,$$

for all $x \in [0, 2\pi]$. We also have for all $x \in [0, 2\pi]$

$$\begin{aligned}\left\|v^N(\cdot, x)\right\|^2_{L^2(0,T)} &\leq \sum_{|n|\geq N+1} |a_n|^2 \left|P^N(n)\right|^2 \frac{|\beta_1^n|^2}{|\alpha_1^n|^2} \left\| e^{\nu_n^h(T-\cdot)} \right\|^2_{L^2(0,T)} \\ &\leq C \sum_{|n|\geq N+1} |a_n|^2 \left|P^N(n)\right|^2 \frac{1}{|n|^2} \\ &\leq \frac{C}{|N|^2} \sum_{|n|\geq N+1} |a_n|^2 \left|P^N(n)\right|^2\end{aligned}$$

We suppose that the following observability inequality holds

$$\int_0^T \left|V_0 \sigma^N(t, 2\pi) + Q_0 v^N(t, 2\pi)\right|^2 dt \geq C \left\|(\sigma^N(0), v^N(0), \varphi^N(0))^\dagger\right\|^2_{(\dot{L}^2(0,2\pi))^3}.$$

Then, we have

$$\begin{aligned}&\left\|(\sigma^N(0), v^N(0), \varphi^N(0))^\dagger\right\|^2_{(\dot{L}^2(0,2\pi))^3} \\ &\leq C \int_0^T \left|V_0 \sigma^N(t, 2\pi) + Q_0 v^N(t, 2\pi)\right|^2 dt \\ &\leq C \int_0^T \left(V_0^2 \left|(\sigma^N(t, 2\pi) - \tilde{\sigma}^N(t, 2\pi))\right|^2 + V_0^2 \left|\tilde{\sigma}^N(t, 2\pi)\right|^2 + Q_0^2 \left|v^N(t, 2\pi)\right|^2\right) dt \\ &\leq \frac{C}{N^2} \sum_{|n|\geq N+1} |a_n|^2 \left|P^N(n)\right|^2,\end{aligned}$$

since $\tilde{\sigma}^N(t, 0) = 0 = \tilde{\sigma}^N(t, 2\pi)$ for all $t \in (0, T)$. Thus, we get

$$\begin{aligned}\left\|\sigma^N(0)\right\|^2_{\dot{L}^2(0,2\pi)} &\leq \left\|(\sigma^N(0), v^N(0), \varphi^N(0))^\dagger\right\|^2_{(\dot{L}^2(0,2\pi))^3} \\ &\leq \frac{C}{N^2} \sum_{|n|\geq N+1} |a_n|^2 \left|P^N(n)\right|^2 \leq \frac{C}{N^2} \left\|\sigma^N(0)\right\|^2_{\dot{L}^2(0,2\pi)},\end{aligned}$$

since $\mathrm{Re}(\nu_n^h)$ is bounded. Therefore, $1 \leq \frac{C}{N^2}$ for all N and hence the above inequality cannot hold. This is a contradiction and the proof is complete. □

3.3.8 Lack of approximate controllability

In this section, we find the existence of certain coefficients $Q_0, V_0, \psi_0, \lambda_0, \kappa_0, R, c_0$ such that the system (3.6) is not approximately controllable at any time $T > 0$ in $(L^2(0, 2\pi))^2$ (that is, Proposition 3.1.3). Full characterization of these coefficients is very difficult due to the cubic polynomial (3.87). We present the proof of Proposition 3.1.3 in the case when there is a boundary control acting in density component. The proof will be similar in other cases (that is, when the control is acting in the velocity or temperature components) and so we omit the details.

3.3.8.1 Proof of Proposition 3.1.3

Let $T > 0$ be given and let us choose the coefficients

$$Q_0 = V_0 = \lambda_0 = 1 = R\psi_0 = \frac{R^2\psi_0}{c_0} = 1, \ \kappa_0 = 2.$$

To prove this result (in the density case), it is enough to find a terminal data $(\sigma_T, v_T, \varphi_T) \in \mathcal{D}(A^*)$ such that the associated solution (σ, v, φ) of (3.76) fails to satisfy the following unique continuation property:

$$V_0\sigma(t,2\pi) + Q_0 v(t,2\pi) = 0 \quad \text{for all } t \in (0,T) \text{ implies } \ (\sigma, v, \varphi) = (0,0,0).$$

Thanks to Remark 3.3.2, A^* has an eigenvalue $\nu_1 = \nu_{-1} = -1$ for $n = 1$ and $n = -1$ respectively. Let $\Phi_1 := \begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix} e^{ix}$ and $\Phi_{-1} := \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix} e^{-ix}$ (for some $\alpha_i, \beta_i \in \mathbb{C}, i = 1,2,3$) denote the independent eigenfunctions of A^* corresponding to this multiple eigenvalue -1. We now choose the terminal data as

$$(\sigma_T, v_T, \varphi_T)^\dagger = C\Phi_1 + D\Phi_{-1},$$

where C, D are complex constants that will be chosen later. The solution of (3.76) is then given by

$$(\sigma(t), v(t), \varphi(t))^\dagger = e^{-(T-t)}\left(C\Phi_1 + D\Phi_{-1}\right)$$

for all $t \in (0,T)$. Therefore

$$V_0\sigma(t,2\pi) + Q_0 v(t,2\pi) = e^{-(T-t)}\left(C\mathcal{B}_\rho^*\Phi_1 + D\mathcal{B}_\rho^*\Phi_{-1}\right)$$

for all $t \in (0,T)$. If we take $C = -\mathcal{B}_\rho^*\Phi_{-1}$ and $D = \mathcal{B}_\rho^*\Phi_1$, then $C, D \neq 0$ (thanks to Lemma 3.3.4) and for these choice of C, D, we have $V_0\sigma(t,2\pi) + Q_0 v(t,2\pi) = 0$ for all $t \in (0,T)$ but $(\sigma, v, \varphi) \neq (0,0,0)$. This completes the proof. □

3.4 Further comments and conclusions

3.4.1 Controllability results using Neumann boundary conditions

We consider the system (3.1) with the initial state (3.2) and the boundary conditions

$$\rho(t,0) = \rho(t,2\pi), \quad u(t,0) = u(t,2\pi), \quad u_x(t,0) = u_x(t,2\pi) + q_1(t), \quad t \in (0,T), \tag{3.136}$$

where q_1 is a boundary control that acts on the velocity through Neumann conditions. Since the observation terms satisfies similar estimates as in (3.36), following the proof of Theorem 3.1.2, we can obtain the null controllability of the system (3.1)-(3.2)-(3.136) at time $T > \frac{2\pi}{V_0}$ in the space $\dot{H}^1_{\text{per}}(0,2\pi) \times \dot{L}^2(0,2\pi)$, and the null controllability fails in the space $\dot{H}^s_{\text{per}}(0,2\pi) \times \dot{L}^2(0,2\pi)$ for $0 \leq s < 1$. In this case also, null controllability of the system (3.1)-(3.2)-(3.136) is inconclusive when the time is small $(0 < T \leq \frac{2\pi}{V_0})$.

Similar to the barotropic case, we consider the system (3.6) with the initial state (3.7) and the boundary conditions

$$\begin{aligned} \rho(t,0) = \rho(t,2\pi), \quad u(t,0) = u(t,2\pi), \quad u_x(t,0) = u_x(t,2\pi) + q_2(t), \\ \theta(t,0) = \theta(t,2\pi), \quad \theta_x(t,0) = \theta_x(t,2\pi), \quad t \in (0,T). \end{aligned} \tag{3.137}$$

In this case also, following the proof of Theorem 3.1.3-Part (ii) and Proposition 3.1.2-Part (ii), we get null controllability of the system (3.6)-(3.7)-(3.137) at time $T > \frac{2\pi}{V_0}$ in the space $\dot{H}^1_{\text{per}}(0,2\pi) \times (\dot{L}^2(0,2\pi))^2$, and null controllability fails in the space $\dot{H}^s_{\text{per}}(0,2\pi) \times (\dot{L}^2(0,2\pi))^2$ for $0 \leq s < 1$.
We next consider the system (3.6) with the initial state (3.7) and the boundary conditions

$$\begin{aligned} \rho(t,0) = \rho(t,2\pi), \quad u(t,0) = u(t,2\pi), \quad u_x(t,0) = u_x(t,2\pi), \\ \theta(t,0) = \theta(t,2\pi), \quad \theta_x(t,0) = \theta_x(t,2\pi) + q_3(t), \quad t \in (0,T). \end{aligned} \tag{3.138}$$

Similar to the previous case, following the proof of Theorem 3.1.3-Part (ii) and Proposition 3.1.2-Part (ii), we get null controllability of the system (3.6)-(3.7)-(3.138) at time $T > \frac{2\pi}{V_0}$ in the space $\dot{H}^1_{\text{per}}(0,2\pi) \times (\dot{L}^2(0,2\pi))^2$, and null controllability fails in the space $\dot{H}^s_{\text{per}}(0,2\pi) \times (\dot{L}^2(0,2\pi))^2$ for $0 \leq s < 1$. For both systems (3.6)-(3.7)-(3.137) and (3.6)-(3.7)-(3.138), null controllability is inconclusive for a small time $0 < T \leq \frac{2\pi}{V_0}$.

3.4.2 Backward uniqueness property

The backward uniqueness property of the system (3.1) or (3.6) is itself an interesting question from the mathematical point of view. It says that, when there is no control acting on the system and the solution vanishes at time $T > 0$, then the solution must vanish identically at all time $t \in [0, T]$. Our system (3.1) with the initial condition (3.2) and boundary condition (3.3) (with $p = 0$) or (3.4) (with $q = 0$) satisfies the backward uniqueness property, more precisely, $(\rho(T), u(T)) = (0, 0)$ implies $(\rho(t), u(t)) = 0$ for all $t \in [0, T]$. This can be seen easily as the eigenfunctions of A^*, and hence of A, form a complete set in $(L^2(0, 2\pi))^2$. We can similarly conclude that the non-barotropic system (3.6) with the initial condition (3.7) and boundary condition (3.8) (with $p = 0$) or (3.9) (with $q = 0$) or (3.10) (with $r = 0$) satisfies the backward uniqueness property.

If a system has backward uniqueness property, then null controllability of the system at some time $T > 0$ will give approximate controllability at that time T. This can be seen easily, because the observability inequality (for null controllability) and the backward uniqueness implies the unique continuation property for the corresponding adjoint system. Thus, using a boundary control in density, our systems (3.1) and (3.6) are approximately controllable at time $T > \frac{2\pi}{V_0}$ in the spaces $(\dot{L}^2(0, 2\pi))^2$ and $(\dot{L}^2(0, 2\pi))^3$ respectively (thanks to Theorem 3.1.1 and Theorem 3.1.3). Similarly, when a boundary control is acting in the velocity or in temperature (for the non-barotropic case), the systems (3.1) and (3.6) are approximately controllable at time $T > \frac{2\pi}{V_0}$ in the spaces $\dot{H}^1_{\text{per}}(0, 2\pi) \times \dot{L}^2(0, 2\pi)$ and $\dot{H}^1_{\text{per}}(0, 2\pi) \times (\dot{L}^2(0, 2\pi))^2$ respectively (thanks to Theorem 3.1.2 and Theorem 3.1.3).

In this context, we must mention that proving the backward uniqueness property might be difficult (in general) when the associated operator do not have complete set of eigenfunctions; see for instance [Ren15], where the author proved backward uniqueness of the linearized compressible Navier-Stokes system (3.1) under Dirichlet boundary conditions $\rho(t, 0) = u(t, 0) = u(t, 1) = 0$ $(t \in (0, T))$, by proving injectivity of the associated semigroup.

3.4.3 More number of controls

Adding controls in both velocity and temperature components does not improve the null controllability result of the system (3.6) with respect to the regularity of the initial states. Estimates of the observation terms remain the same as in the control acts in velocity or temperature.

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Chapter **4**

Linearized compressible Navier-Stokes system (barotropic fluids)

This chapter is taken from the article [BCDK22]:

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Abstract

In this paper, we prove the boundary null-controllability of the compressible Navier-Stokes equations linearized around a positive constant steady state in a bounded interval when the time is sufficiently large. The novelty of this work is that we consider only one Dirichlet boundary control at one end of the interval acting either on the velocity or density part of the concerned system, where the first-order couplings between transport and heat-type equations arise. Moreover, we establish that the null-controllability results are optimal/sharp concerning the regularity of initial states for the velocity case and with respect to time for the density case.

The proofs of controllability results rely on a parabolic-hyperbolic joint Ingham-type inequality, which is derived in this work, and a mixed parabolic-hyperbolic moments method. In light of the requirement, we need to use some complex analytic arguments to check the Fattorini-Hautus criterion. To this end, a careful spectral analysis of the associated non-self-adjoint operator is performed, where the spectrum consists of parabolic and hyperbolic branches of eigenvalues. It is one of the involved parts of this article because we analyze more general boundary conditions in contrast to the periodic case appearing in [CMRR14].

4.1 Introduction and main results

4.1.1 The system under study

The Navier-Stokes (NS) system for a viscous compressible isentropic fluid in $(0,L)$ is

$$\begin{cases} \rho_t + (\rho u)_x = 0, & \text{in } (0,+\infty)\times(0,L), \\ \rho\big(u_t + uu_x\big) + (p(\rho))_x - \nu u_{xx} = 0, & \text{in } (0,+\infty)\times(0,L), \end{cases} \tag{4.1}$$

where $L > 0$ denotes the finite length of the interval, ρ is the fluid density and u is the velocity. The viscosity of the fluid is denoted by $\nu > 0$ and we assume that the pressure p satisfies the constitutive law $p(\rho) = a\rho^\gamma$ for $a > 0$ and $\gamma \geq 1$. Upon linearization of (4.1) around some constant steady state (Q_0, V_0) (with $Q_0 > 0, V_0 > 0$), we have

$$\begin{cases} \rho_t + V_0\rho_x + Q_0 u_x = 0, & \text{in } (0,+\infty)\times(0,L), \\ u_t - \dfrac{\nu}{Q_0}u_{xx} + V_0 u_x + a\gamma Q_0^{\gamma-2}\rho_x = 0, & \text{in } (0,+\infty)\times(0,L). \end{cases} \tag{4.2}$$

Now, if we consider the change of variables:

$$\rho(t,x) \longrightarrow \alpha\rho(\beta t, \delta x), \quad u(t,x) \longrightarrow u(\beta t, \delta x), \quad \forall (t,x) \in (0,+\infty)\times(0,L),$$

with the choices of $\alpha, \beta, \delta > 0$ as

$$\alpha := \left(a\gamma Q_0^{\gamma-3}\right)^{-1/2}, \quad \beta := \frac{Q_0 V_0^2}{\nu}, \quad \delta := \frac{Q_0 V_0}{\nu},$$

then the system (4.2) reduces to

$$\begin{cases} \rho_t + \rho_x + c u_x = 0, & \text{in } (0,+\infty)\times(0,\delta L), \\ u_t - u_{xx} + u_x + c\rho_x = 0, & \text{in } (0,+\infty)\times(0,\delta L), \end{cases} \tag{4.3}$$

with $c = \frac{Q_0}{V_0}\left(a\gamma Q_0^{\gamma-3}\right)^{1/2}$. Let us describe the problems on which we are going to work in the present article. Our goal is to study the boundary controllability properties of the linearized Navier-Stokes system (4.3) at time $T > 0$ with a single control force acting either on the velocity or density component. Here, we must mention that the whole analysis of this paper will be performed in the space domain $(0,1)$, which is mainly for the simplicity of spectral computations. The same can be done in the interval $(0,\delta L)$.

I. Control on velocity: The first problem under consideration is

$$\begin{cases} \rho_t + \rho_x + c u_x = 0, & \text{in } (0,T) \times (0,1), \\ u_t - u_{xx} + u_x + c\rho_x = 0, & \text{in } (0,T) \times (0,1), \\ \rho(t,0) = \rho(t,1), & \text{for } t \in (0,T), \\ u(t,0) = 0, \quad u(t,1) = q(t), & \text{for } t \in (0,T), \\ \rho(0,x) = \rho_0(x), \quad u(0,x) = u_0(x), & \text{in } (0,1), \end{cases} \tag{4.4}$$

with a Dirichlet control q acting at the right boundary point only through the velocity component u, and (ρ_0, u_0) is the given initial state from some suitable Hilbert space.

II. Control on density: Next, we consider the case when a boundary control p acts on the density part instead of velocity. More precisely, the system under consideration is

$$\begin{cases} \rho_t + \rho_x + c u_x = 0, & \text{in } (0,T) \times (0,1), \\ u_t - u_{xx} + u_x + c\rho_x = 0, & \text{in } (0,T) \times (0,1), \\ \rho(t,0) = \rho(t,1) + p(t), & \text{for } t \in (0,T), \\ u(t,0) = 0, \quad u(t,1) = 0, & \text{for } t \in (0,T), \\ \rho(0,x) = \rho_0(x), \quad u(0,x) = u_0(x), & \text{in } (0,1). \end{cases} \tag{4.5}$$

The aim is to study the null-controllability of the systems (4.4) and (4.5) at a given time $T > 0$. Moreover, as a consequence of the null controllability result for the system (4.5), we can also achieve the null-controllability for the following full Dirichlet system when a control is exerted on the density part, that is

$$\begin{cases} \rho_t + \rho_x + c u_x = 0, & \text{in } (0,T) \times (0,1), \\ u_t - u_{xx} + u_x + c\rho_x = 0, & \text{in } (0,T) \times (0,1), \\ \rho(t,0) = h(t), & \text{for } t \in (0,T), \\ u(t,0) = 0, \quad u(t,1) = 0, & \text{for } t \in (0,T), \\ \rho(0,x) = \rho_0(x), \quad u(0,x) = u_0(x), & \text{in } (0,1). \end{cases} \tag{4.6}$$

Let us prescribe the notions of null and approximate controllability for the concerned systems (4.4)–(4.6).

Definition 4.1.1. *Let H be a Hilbert space. We say the system (4.4) (resp. (4.5) and (4.6)) is*

- ***null controllable*** *at a finite time $T > 0$ in H if, for any given initial state $(\rho_0, u_0) \in H$, there exists a control $q \in L^2(0,T)$ (resp. $p, h \in L^2(0,T)$) such that the solution (ρ, u) to (4.4) (resp. (4.5) and (4.6)) can be driven to 0 at the time T, that is,*

$$(\rho(T,x), u(T,x)) = (0,0), \quad \textit{for all } x \in (0,1).$$

- ***approximately controllable*** *at a finite time $T > 0$ in H if, for any given initial state $(\rho_0, u_0) \in H$, final state $(\rho_T, u_T) \in H$ and given $\epsilon > 0$, there exists a control $q \in L^2(0,T)$ (resp. $p, h \in L^2(0,T)$) such that the solution (ρ, u) to (4.4) (resp. (4.5) and (4.6)) satisfies*

$$\|(\rho(T), u(T)) - (\rho_T, u_T)\|_H \le \epsilon.$$

If the system (4.4) is null controllable at some time $T > 0$ by using a control $q \in L^2(0,T)$ acting only on the velocity part, then we have the following compatibility condition (obtained by integrating the first equation of (4.4)):

$$\int_0^1 \rho_0(x)dx = c\int_0^T q(t)dt.$$

We also get a similar compatibility condition for the density case (that is, for system (4.5)), given by

$$\int_0^1 \rho_0(x)dx = -\int_0^T p(t)dt.$$

To avoid these constraints, we shall work on the Hilbert space $\dot{L}^2(0,1) \times L^2(0,1)$, where

$$\dot{L}^2(0,1) := \left\{ f \in L^2(0,1) \ : \ \int_0^1 f dx = 0 \right\}.$$

4.1.2 Functional setting

For any $s > 0$, we introduce the following Sobolev space

$$H^s_\sharp(0,L) := \{\varphi \in H^s(0,L) \ : \ \varphi(0) = \varphi(L)\}$$

and denote $(H^s_\sharp(0,L))'$ as the dual space of $H^s_\sharp(0,L)$ with respect to the pivot space $L^2(0,L)$. We also denote, for any $s > 0$, $H^{-s}(0,L)$ and $(\dot{H}^s_\sharp(0,L))'$ as the dual spaces of $H^s_0(0,L)$ and $\dot{H}^s_\sharp(0,L)$ with respect to the pivot spaces $L^2(0,L)$ and $\dot{L}^2(0,L)$ respectively. We note here that, although the trace $\varphi(0)$ or $\varphi(L)$ is meaningful only for $s > \frac{1}{2}$, we still keep the same notation for $s \leq \frac{1}{2}$ to simplify the presentation. Let us now write the underlying operator associated with the control systems (4.4) or (4.5), given by

$$A = \begin{pmatrix} -\partial_x & -c\partial_x \\ -c\partial_x & \partial_{xx} - \partial_x \end{pmatrix}, \tag{4.7}$$

with its domain

$$D(A) = \left\{ \Phi = (\xi, \eta) \in H^1(0,1) \times H^2(0,1) \ : \ \xi(0) = \xi(1), \quad \eta(0) = \eta(1) = 0 \right\}. \tag{4.8}$$

The adjoint of the operator A has the following formal expression

$$A^* = \begin{pmatrix} \partial_x & c\partial_x \\ c\partial_x & \partial_{xx} + \partial_x \end{pmatrix}, \tag{4.9}$$

also with the same domain $D(A^*) = D(A)$, given by (4.8). Note that, the operator A is non-self-adjoint in nature.

Notations: Throughout the chapter, $C, C_i > 0$ for $i \in \mathbb{N}^*$, denote the generic constants that may vary from line to line and may depend on T.

4.1.3 Main results

This section is devoted to announce the main results of this chapter.

Theorem 4.1.1 (Control on velocity). *Let $T > 1$ and $c > 0$ such that $c^4 + 8c^2 + 5 < 4\pi^2$. Then, there exists a countable set $\mathcal{N}$ such that for chosen $c \notin \mathcal{N}$ and any given $(\rho_0, u_0) \in \dot{H}^{\frac{1}{2}}_\sharp(0,1) \times L^2(0,1)$, there exists a Dirichlet boundary control $q \in L^2(0,T)$ acting on the velocity component such that the system (4.4) is null-controllable at time T, that is*

$$\rho(T,x) = u(T,x) = 0, \quad \forall x \in (0,1). \tag{4.10}$$

Moreover, if $0 \leq s < \frac{1}{2}$, the system (4.4) fails to satisfy the null-controllability criterion (4.10) in the space $\dot{H}^s_\sharp(0,1) \times L^2(0,1)$ for any given time $T > 0$ and $c > 0$.

Theorem 4.1.2 (Control on density)**.** *Let $T > 1$ and $c > 0$ such that $c^4 + 8c^2 + 5 < 4\pi^2$. Then, for any given initial state $(\rho_0, u_0) \in \dot{L}^2(0,1) \times L^2(0,1)$, there exists a boundary control $p \in L^2(0,T)$ acting through the density component such that the system (4.5) is null-controllable at time T, that is*

$$\rho(T,x) = u(T,x) = 0, \quad \forall x \in (0,1). \tag{4.11}$$

Remark 4.1.1. *We must mention here that the restrictions on c appear in the above results because of the difficulty in proving that roots of the auxiliary equation (which comes from the differential equation satisfied by the eigenfunctions of A^*) are distinct, see Lemma 4.4.1 for instance. In particular, this property ensures that all the eigenvalues of A^* has geometric multiplicity 1 (see Proposition 4.3.1-Part (iv)), which is very crucial to obtain null controllability of the systems (4.4) and (4.5). Moreover, the set $\mathcal{N}$ appears while proving that all the observation terms are non-zero in the case when a control acts only on the velocity part; see Proposition 4.4.1–Part 2 for details. Note that, in particular, the set $\{c > 0 \ : \ c^4 + 8c^2 + 5 < 4\pi^2\}$ contains the interval $(0,1]$.*

Moreover, we have the lack of null-controllability result for the system (4.5) when $T < 1$. Precisely, we prove the following proposition.

Proposition 4.1.1. *Let $0 < T < 1$. The system (4.5) is not null-controllable at time T in the space $L^2(0,1) \times L^2(0,1)$.*

As a consequence of Theorem 4.1.2, we also achieve the null-controllability for the system (4.6) with a Dirichlet control on the density part. More precisely, we have the following result.

Theorem 4.1.3 (Dirichlet control on density)**.** *Let $T > 1$ and $c > 0$ such that $c^4+8c^2+5 < 4\pi^2$. Then, for any given initial state $(\rho_0, u_0) \in \dot{L}^2(0,1) \times L^2(0,1)$, there exists a boundary control $h \in L^2(0,T)$ acting through the density component such that the system (4.6) is null-controllable at time T, that is*

$$\rho(T,x) = u(T,x) = 0, \quad \forall x \in (0,1). \tag{4.12}$$

Moreover, if $0 < T < 1$, the system (4.6) is not null controllable at time T in $L^2(0,1) \times L^2(0,1)$.

Indeed, by Theorem 4.1.2, there exists a control $p \in L^2(0,T)$ which drives the solution (ρ, u) of the system (4.5) to $(0,0)$ with initial state $(\rho_0, u_0) \in \dot{L}^2(0,1) \times L^2(0,1)$. Then, by showing $\rho(\cdot, 1) \in L^2(0,T)$, one can consider $h(t) := \rho(t,1) + p(t)$ for $t \in (0,T)$, which acts as a null-control for the system (4.6). Similarly, we can prove null controllability of (4.5) by assuming null controllability of the system (4.6). As a consequence, null controllability of the system (4.5) is equivalent to that for the system (4.6). This kind of technique has been applied for instance in [CC09a, CHO16].

To prove the main results of this paper, we notably use an Ingham-type inequality and the moments technique. In fact, we establish the following Ingham-type inequality which is of independent interest.

Proposition 4.1.2 (A combined Ingham-type inequality)**.** *Let $\{\lambda_k\}_{k\in\mathbb{N}^*}$ and $\{\gamma_k\}_{k\in\mathbb{Z}}$ be two sequences in $\mathbb{C}$ with the following properties: there is $N \in \mathbb{N}^*$ such that*

(i) for all $k, j \in \mathbb{Z}$, $\gamma_k \neq \gamma_j$ unless $j = k$;

(ii) $\gamma_k = \beta + 2k\pi i + \nu_k$ for all $|k| \geq N$;

where $\beta \in \mathbb{C}$ and $\{\nu_k\}_{|k|\geq N} \in \ell_2$.

Also, there exist constants $A_0 \geq 0$, $B_0 \geq \delta$ with $\delta > 0$ and some $\epsilon > 0$ for which $\{\lambda_k\}_{k\in\mathbb{N}^*}$ satisfies

(i) for all $k, j \in \mathbb{N}^*$, $\lambda_k \neq \lambda_j$ unless $j = k$;

(ii) $\frac{-\mathrm{Re}(\lambda_k)}{|\mathrm{Im}(\lambda_k)|} \geq \widehat{c}$ for some $\widehat{c} > 0$ and $k \geq N$;

(iii) there exists some $r > 1$ such that $\left|\lambda_k - \lambda_j\right| \geq \delta\,|k^r - j^r|$ for all $k \neq j$ with $k, j \geq N$ and

(iv) $\epsilon(A_0 + B_0 k^r) \leq |\lambda_k| \leq A_0 + B_0 k^r$ for all $k \geq N$.

We also assume that the families are disjoint, i.e.,

$$\{\gamma_k,\ k\in\mathbb{Z}\}\cap\{\lambda_k,\ k\in\mathbb{N}^*\}=\emptyset.$$

Then, for any time $T>1$, there exists a positive constant C depending only on T such that

$$\int_0^T\left|\sum_{k\in\mathbb{N}^*}a_k e^{\lambda_k t}+\sum_{k\in\mathbb{Z}}b_k e^{\gamma_k t}\right|^2 dt\geq C\left(\sum_{k\in\mathbb{N}^*}|a_k|^2 e^{2\mathrm{Re}(\lambda_k)T}+\sum_{k\in\mathbb{Z}}|b_k|^2\right),\tag{4.13}$$

for all sequences $\{a_k\}_{k\in\mathbb{N}^*}$ and $\{b_k\}_{k\in\mathbb{Z}}$ in ℓ_2.

Remark 4.1.2. *The first Ingham inequality was proved in 1936 by Ingham [Ing36]. He considered a hyperbolic family of the form $(i\gamma_k)_{k\in\mathbb{N}^*}$, where $(\gamma_k)_{k\in\mathbb{N}^*}$ is a sequence of real numbers satisfying the gap condition $\inf_{k\in\mathbb{N}}|\gamma_{k+1}-\gamma_k|>0$. Since then, there are many variations of this inequality available in the literature including the parabolic Ingham inequality (commonly known as the Müntz-Szász theorem). We refer to the works [AI95, JTZ97, You01, FCGBdT10, Edw06, LZ02, Lóp99, KL05, MZ04] for proofs of these variations of Ingham-type inequality.*

Zhang and Zuazua [ZZ03a, ZZ03b, ZZ04] proved a joint parabolic-hyperbolic Ingham-type inequality with a parabolic branch of the form $-k^2\pi^2+2+O(k^{-1})$ and a hyperbolic branch of the form $(\frac{1}{2}+k)\pi i+O(|k|^{-1})$ (Lemma 4.1 in [ZZ03a] or [ZZ04] and Lemma 4.5 in [ZZ03b]). This result has been generalized by Komornik and Tenenbaum [KT15]. In this article, we prove a joint parabolic-hyperbolic Ingham-type inequality under more general assumptions on the parabolic and hyperbolic branches compare to the assumptions in [KT15, Theorem 1.1]. Our proof is based on a decoupling idea as mentioned in [Zua16, Section 2.4] by Zuazua and [CMRR14, Theorem 4.2] by Chowdhury, Mitra, Ramaswamy and Renardy. In fact, our proof works with more general assumptions on the sequences $(\lambda_k)_{k\in\mathbb{N}^*}$ and $(\gamma_k)_{k\in\mathbb{Z}}$ for which each of the individual parabolic and hyperbolic Ingham inequalities hold.

4.1.4 Literature on the controllability results related to the compressible Navier-Stokes equations

In the past few years, the controllability of the compressible and incompressible fluids has turned into a very significant topic to the control community. Fernández-Cara et al. [FCGIP04b] proved the local exact distributed controllability of the incompressible Navier-Stokes system when a control is supported in a small open set; see also the references therein. A local null-controllability result of 3D Navier-Stokes system with distributed control for incompressible fluids having two vanishing components has been addressed in [CL14] by Coron and Lissy. Badra, Ervedoza and Guerrero in [BEG16] proved the local exact controllability to the trajectories for non-homogeneous (variable density) incompressible 2D Navier-Stokes equations using boundary controls for both density and velocity.

In the case of compressible Navier-Stokes equations, we first mention the work by E. V. Amosova [Amo11] where she considered a compressible viscous fluid in 1D w.r.t. the Lagrangian coordinates with zero boundary condition on the velocity and an interior control acting on the velocity equation. She proved a local exact controllability result when the initial density is already on the targeted trajectory. Ervedoza, Glass, Guerrero and Puel in [EGGP12] proved a local exact controllability result for the 1D compressible Navier-Stokes system in a bounded domain $(0,L)$ for regular initial data in $H^3(0,L)\times H^3(0,L)$ with two boundary controls, when time is large enough. This result has been improved by Ervedoza and Savel in [ES18] by choosing the initial data from $H^1(0,L)\times H^1(0,L)$; see also a generalized result [EGG16] by Ervedoza, Glass and Guerrero for dimensions 2 and 3.

We also refer that Chowdhury, Ramaswamy and Raymond in [CRR12] established a null controllability and stabilizability result of a linearized (around a constant steady-state $(Q_0,0)$, $Q_0>0$) 1D compressible Navier-Stokes equations. The authors proved that their system is null-controllable in $H^1_0\times L^2$ by a distributed control acting everywhere in the velocity equation. Their result is proved to be sharp in the following sense: the null-controllability cannot be achieved by a localized interior control (or by a boundary control) acting on the velocity part.

Martin, Rosier and Rouchon in [MRR13] considered the wave equation with structural damping in 1D; using the spectral analysis and method of moments, they obtained that their equation is null-controllable with a moving distributed control for regular initial conditions in $H^{s+2} \times H^s$ for $s > 15/2$ at sufficiently large time. See also the work of Chaves-Silva, Rosier and Zuazua [CSRZ14a] for the higher dimensional case.

The 1D compressible Navier–Stokes equations linearized around a constant steady state with periodic boundary conditions is closely related to the structurally damped wave equation studied in [MRR13]. Chowdhury and Mitra [CM15] studied the interior null-controllability of the linearized (around constant steady state (Q_0, V_0), $Q_0 > 0, V_0 > 0$) 1D compressible Navier–Stokes system with periodic boundary conditions. Following the approach of [MRR13], the authors in [CM15] established that their system is null-controllable by a localized interior control when the time is large enough, and for regular initial data in $\dot{H}^{s+1}_{per} \times H^s_{per}$ with $s > 13/2$. They also achieved that, for any $T > \frac{2\pi}{V_0}$, the system is approximately controllable at time $T > \frac{2\pi}{V_0}$ in $\dot{L}^2 \times L^2$ using a localized interior control (of the form $f(t,x) = h(t)g(x)$ for $(t,x) \in (0,T) \times (0, 2\pi)$) and, it is null-controllable at time T using periodic boundary control with regular initial data $\dot{H}^{s+1}_{per} \times \dot{H}^s_{per}$ for $s > 9/2$.

In [CMRR14], Chowdhury, Mitra, Ramaswamy and Renardy considered the one-dimensional compressible Navier–Stokes equations linearized around a constant steady state $(Q_0, V_0), Q_0 > 0, V_0 > 0$, with homogeneous periodic boundary conditions in the interval $(0, 2\pi)$. They proved that the linearized system with homogeneous periodic boundary conditions is null controllable in $\dot{H}^1_{per} \times L^2$ by a localized interior control when the time $T > \frac{2\pi}{V_0}$. Moreover, in their work the distributed null-controllability result in $\dot{H}^1_{per} \times L^2$ is sharp in the sense that the controllability fails in $\dot{H}^s_{per} \times L^2$ for any $0 \le s < 1$. As usual, the large time for controllability is needed due to the presence of transport part and indeed, the null-controllability fails for small time, see [Mai15] by Maity and [AMM22] by Ahamed, Maity and Mitra.

Chowdhury in [Cho15] considered the same linearized Navier–Stokes system around (Q_0, V_0) with $Q_0 > 0, V_0 > 0$ in $(0, L)$ with homogeneous Dirichlet boundary conditions and an interior control acting only on the velocity equation on a open subset $(0, l) \subset (0, L)$. He proved the approximate controllability of the linearized system in $L^2(0, L) \times L^2(0, L)$ with a localized control in $L^2(0, T; L^2(0, l))$ when $T > \frac{L-l}{V_0}$.

In the context of the controllability of coupled transport-parabolic system (which is the main feature of linearized compressible Navier-Stokes equations), we must mention the work [LZ98] by Lebeau and Zuazua where the distributed null-controllability of Thermoelasticity system has been studied. More recently, Beauchard, Koenig and Le Balc'h [BKLB20] considered the linear parabolic-transport system with constant coefficients and coupling of order zero and one with locally distributed controls posed on the one-dimensional torus $\mathbb{T}$. Following the approach of [LZ98], they proved the null-controllability at sufficiently large time when there are as many controls as equations. On the other hand, when the control acts only on the transport (resp. parabolic) component, they obtained an algebraic necessary and sufficient condition on the coupling term for the null-controllability, and their controllability studies based on a detailed spectral analysis. According to the more general result established in [BKLB20], we can say that for a 2×2 coupled parabolic-transport system (with periodic boundary conditions), the null-controllability with one localized interior control holds true in $L^2(\mathbb{T}) \times \dot{L}^2(\mathbb{T})$ (resp. in $\dot{H}^2(\mathbb{T})\times H^2(\mathbb{T})$) when the control acts only on the transport (resp. parabolic) component. More recently, the distributed null-controllability of underactuated linear parabolic-transport systems with constant coefficients in one-dimensional torus has been established in [KL23] by Koenig and Lissy for regular enough initial data and large time.

Finally, one may find few stabilization results for linearized compressible Navier-Stokes system available in [ABBEFR11, CRR12, CMRR15, CDM21, MRR15, MRR17].

4.1.5 Our approach and achievement of the present work

As mentioned earlier, in compressible Navier-Stokes system, the interesting feature is the first order coupling between transport equation and the momentum equation of parabolic type. It was shown in [CMRR14, CM15] that the linearized compressible Navier-Stokes system with Periodic-Periodic boundary conditions, there is a sequence of generalized eigenfunctions of the associated adjoint operator that forms a Riesz Basis for the state Hilbert space. The success in obtaining this result lies in the simplicity of the corresponding characteristic equations as well as the explicit structure of all eigenfunctions in terms of Fourier basis.

But for the operator $(A^*, D(A^*))$ defined in (4.9), the characteristic equation is a third order ODE and the eigenvalue equation is a non-standard transcendental equation, which is quite challenging to handle. In fact, the method (invariant subspace idea) used in [CMRR14, CM15] is not practically applicable to our case. However, we manage to characterize the set of eigenvalues and eigenfunctions for the operator A^*. More precisely, the spectrum of A^* consists of: a parabolic part containing the eigenvalues λ_k^p such that $\mathrm{Re}(\lambda_k^p)$ behaves like $-k^2\pi^2$ for large enough $k \in \mathbb{N}^*$ while $\mathrm{Im}(\lambda_k^p)$ is bounded; a hyperbolic part made up of the eigenvalues λ_k^h such that $\mathrm{Im}(\lambda_k^h)$ behaves like $2k\pi$ for large enough $k \in \mathbb{Z}$ while $\mathrm{Re}(\lambda_k^h)$ is bounded; and a finite set of lower frequencies. The Riesz basis property of the set of (generalized) eigenfunctions has been then established by using an abstract result of B.-Z. Guo [Guo01].

To study the boundary null controllability, we mention that the usual extension method is not really convenient for the Navier-Stokes system. This is because, when we put one interior control in the system, then upon extending the domain and restricting the solution on the boundary will give rise to two boundary controls for the system. In this regard, we refer some earlier null-controllability results [MRR17, EGGP12, ES18] with one interior control acting in the velocity equation or two boundary controls both for density and velocity.

The main novelty of the present work is that we directly handle the boundary null controllability with only one control acting on the density or velocity part where the boundary conditions are of mixed type (In this regard, we mention the work [CMZ20] by Cerpa, Montoya and Zhang, where some mixed boundary conditions has been appeared in the context of KdV-Burgers equation). More precisely, when a control acts in velocity, we use the Ingham-type inequality given by Proposition 4.1.2 to prove an observability inequality for the adjoint to the system (4.4) in $(\dot{H}^{\frac{1}{2}}_{\sharp}(0,1))' \times L^2(0,1)$, leading to the null-controllability of (4.4) at time $T > 1$ with initial data in $\dot{H}^{\frac{1}{2}}_{\sharp}(0,1) \times L^2(0,1)$. On the other hand, when a boundary control acts on the density part, we proceed in the following way: first, using the Ingham-type inequality (4.13) we obtain the null-controllability of the system (4.5) at time $T > 1$ in the space $\dot{L}^2(0,1) \times H^1_0(0,1)$; secondly, we apply a parabolic-hyperbolic joint moments technique as developed in [Han94] by Hansen to conclude the null-controllability of the same system (4.5) in the space $\dot{H}^s_{\sharp}(0,1) \times L^2(0,1)$ for $s > \frac{1}{2}$ at $T > 1$. Then, due to the linearity of the solution map of the system (4.5), these two results provide the null-controllability of that system in the space $\dot{L}^2(0,1) \times L^2(0,1)$ when $T > 1$. And, consequently, we deduce the null-controllability of the system (4.6) at time $T > 1$ in $\dot{L}^2(0,1) \times L^2(0,1)$. Finally, we obtain that null controllability of the systems (4.5) and (4.6) fails in $L^2(0,1) \times L^2(0,1)$ when the time is small, that is, when $0 < T < 1$.

4.1.6 Chapter organization

The chapter is organized as follows.

- In Section 4.2, we discuss the well-posedness results of the main systems and some associated results have been proved in the Appendix.
- We split the spectral analysis for the associated adjoint operator into two sections for the ease of reading. Section 4.3 contains a short description of the spectral properties whereas the detailed analysis is prescribed in Section 4.8.
- In Section 4.4, we obtain the lower bounds for the observation terms which are crucial to determine the null-controllability for the system (4.4) or (4.5).
- Section 4.6 is devoted to prove the null-controllability of the system (4.4), that is Theorem 4.1.1. An Ingham-type inequality (Proposition 4.1.2), proved in Section 4.5, is the main ingredient for the required null-controllability proof.

– Then, in Section 4.7, we prove the null-controllability of the system (4.5), that is Theorem 4.1.2 by using both the method of moments and the Ingham-type inequality obtained in Section 4.5. As a consequence, we conclude the result in Theorem 4.1.3. Further, a lack of null controllability result (Proposition 4.1.1) for this system (4.5) is also included in this section.

– Finally, we conclude our paper by providing some open question and remarks in Section 4.9.

4.2 Well-posedness of the system

Let us first recall the operator A^* defined by (4.9). Then, we write the adjoint system associated to the control problems (4.4) and (4.5): let (σ, v) be the adjoint state and the system reads as

$$\begin{cases} -\sigma_t - \sigma_x - cv_x = f, & \text{in } (0,T)\times(0,1), \\ -v_t - v_{xx} - v_x - c\sigma_x = g, & \text{in } (0,T)\times(0,1), \\ \sigma(t,0) = \sigma(t,1), & \text{for } t \in (0,T), \\ v(t,0) = v(t,1) = 0, & \text{for } t \in (0,T), \\ \sigma(T,x) = \sigma_T(x), \quad v(T,x) = v_T(x), & \text{in } (0,1). \end{cases} \tag{4.14}$$

Shortly, one may express it by

$$-V'(t) = A^*V(t) + F(t), \quad \forall t \in (0,T), \qquad V(T) = V_T, \tag{4.15}$$

where the state is $V := (\sigma, v)$, given final data is $V_T := (\sigma_T, v_T)$ and source term is $F := (f, g)$.

To show the well-posedness of the solutions to (4.4) and (4.5), let us first write the following lemma.

Lemma 4.2.1. *The operator A (resp. A^*) is maximal dissipative in $L^2(0,1)\times L^2(0,1)$, that is, $(A, D(A))$ (resp. $(A^*, D(A^*))$) generates a strongly continuous semigroup of contractions in $L^2(0,1)\times L^2(0,1)$.*

The proof of Lemma 4.2.1 can be done in a standard fashion. For the sake of completeness, we give the proof in Appendix A.0.1. As a consequence of this result, we now guarantee the existence of a strong solution of the Navier-Stokes equation (4.4) (resp. (4.5)) when there is no control input acting on the system.

Lemma 4.2.2. *For any given $(\rho_0, u_0) \in \mathcal{D}(A)$, the system (4.4) with $q = 0$ (or the system (4.5) with $p = 0$) admits a unique strong solution $(\rho, u) \in C^1([0,T]; L^2(0,1)\times L^2(0,1)) \cap C^0([0,T]; \mathcal{D}(A))$.*

Once we have the existence of semigroup generated by the operator A^*, we can write the following result:

Proposition 4.2.1. *For any given $F := (f, g) \in L^2(0,T; L^2(0,1)\times L^2(0,1))$ and $V_T = (\sigma_T, v_T) \in L^2(0,1)\times L^2(0,1)$, there exists a unique weak solution $V := (\sigma, v)$ to the system (4.15) in the space $C([0,T]; L^2(0,1)) \times [C([0,T]; L^2(0,1)) \cap L^2(0,T; H^1_0(0,1))]$ with the estimate*

$$\|(\sigma, v)\|_{C^0([0,T];L^2(0,1)\times L^2(0,1))} + \|v\|_{L^2(0,T;H^1_0(0,1))} \le C\left(\|F\|_{L^2(0,T;L^2(0,1)\times L^2(0,1))} + \|V_T\|_{L^2(0,1)\times L^2(0,1)}\right).$$

Moreover, we have the hidden regularity property $\sigma(\cdot, 1) \in L^2(0,T)$.

In particular, if $F \in L^2(0,T; H^1(0,1)\times L^2(0,1))$ and $V_T = (0,0)$, the solution (σ, v) to (4.15) belongs to $C^0([0,T]; H^1_\sharp(0,1)) \times [C^0([0,T]; H^1_0(0,1)) \cap L^2(0,T; H^2(0,1))]$.

The proof of this result can be adapted from the work [Gir08, Chapter IV, Sec. 4.3] and so we omit the details. For the hidden regularity property, we give a detailed proof in Appendix A.1.

Now, we can define the notion of solutions to the control systems (4.4) and (4.5) in the sense of transposition (see for instance [Cor07]) where a non-trivial boundary source term is appearing.

Definition 4.2.1. *We write the following definitions bases on the act of the control.*

- *For given initial state* $U_0 := (\rho_0, u_0) \in L^2(0,1) \times L^2(0,1)$ *and boundary data* $q \in L^2(0,T)$*, a function* $U := (\rho, u) \in L^2(0,T; (H^1_\sharp(0,1))') \times L^2(0,T; L^2(0,1))$ *is a solution to the system* (4.4) *if for any given* $F := (f,g) \in L^2(0,T; H^1(0,1)) \times L^2(0,T; L^2(0,1))$*, the following identity holds true:*

$$\int_0^T \langle \rho(t,\cdot), f(t,\cdot)\rangle_{(H^1)',H^1}\, dt + \int_0^T \int_0^1 u(t,x)g(t,x)dxdt$$
$$= \langle U_0(\cdot), V(0,\cdot)\rangle_{L^2\times L^2} + \int_0^T \left[c\sigma(t,1) + v_x(t,1)\right] q(t)dt,$$

  where $V := (\sigma, v)$ *is the unique weak solution to the adjoint system* (4.15) *with* $V_T = (0,0)$.

- *For given initial state* $U_0 := (\rho_0, u_0) \in L^2(0,1) \times L^2(0,1)$ *and boundary data* $p \in L^2(0,T)$*, a function* $U := (\rho, u) \in L^2(0,T; L^2(0,1)) \times L^2(0,T; L^2(0,1))$ *is a solution to the system* (4.5) *if for any given* $F := (f,g) \in L^2(0,T; L^2(0,1)) \times L^2(0,T; L^2(0,1))$*, the following identity holds true:*

$$\int_0^T \int_0^1 \rho(t,x)f(t,x)dxdt + \int_0^T \int_0^1 u(t,x)g(t,x)dxdt = \langle U_0(\cdot), V(0,\cdot)\rangle_{L^2\times L^2} + \int_0^T \sigma(t,1)p(t)dt,$$

  where $V := (\sigma, v)$ *is the unique weak solution to the adjoint system* (4.15) *with* $V_T = (0,0)$.

Let us state the following theorems that concern the existence and uniqueness of solutions to the control problems (4.4) and (4.5).

Theorem 4.2.1. *For every* $q \in L^2(0,T)$ *and* $U_0 := (\rho_0, u_0) \in L^2(0,1) \times L^2(0,1)$*, the system* (4.4) *has a unique solution* $U := (\rho, u)$ *belonging to the space* $C^0([0,T]; (H^1_\sharp(0,1))') \times [C^0([0,T]; H^{-1}(0,1)) \cap L^2(0,T; L^2(0,1))]$ *in the sense of transposition.*

Moreover, this solution (ρ, u) *satisfies the following estimate*

$$\|\rho\|_{C^0([0,T];(H^1_\sharp(0,1))')} + \|u\|_{C^0([0,T];H^{-1}(0,1))\cap L^2(0,T;L^2(0,1))} \leq C\left(\|(\rho_0,u_0)\|_{L^2(0,1)\times L^2(0,1)} + \|q\|_{L^2(0,T)}\right)$$

for some constant $C > 0$.

The proof for Theorem 4.2.1 will be followed from [CR13, Section 3]. In fact, if $(\rho_0, u_0) \in L^2(0,1) \times L^2(0,1)$ and $q \in L^2(0,T)$, the solution (ρ, u) of (4.4) belong to $L^2(0,T; (H^1_\sharp(0,1))') \times L^2(0,T; L^2(0,L))$. Using the continuity estimate for the transport equation and properties of the parabolic equation, we can deduce that $\rho \in C^0([0,T]; (H^1_\sharp(0,1))')$ and $u \in C^0([0,T]; H^{-1}(0,1))$.

Theorem 4.2.2. *For every* $p \in L^2(0,T)$ *and* $U_0 := (\rho_0, u_0) \in L^2(0,1) \times L^2(0,1)$*, the system* (4.5) *has a unique solution* $U := (\rho, u)$ *belonging to the space* $L^2(0,T; L^2(0,1)) \times L^2(0,T; L^2(0,1))$ *in the sense of transposition and the operator defined by*

$$(U_0, p) \mapsto U(U_0, p),$$

is linear and continuous from $(L^2(0,1) \times L^2(0,1)) \times L^2(0,T)$ *into* $L^2(0,T; L^2(0,1)) \times L^2(0,T; L^2(0,1))$.

Moreover, the solution satisfies the following regularity result,

$$(\rho, u) \in C^0([0,T]; L^2(0,1)) \times [C^0([0,T]; L^2(0,1)) \cap L^2(0,T; H^1_0(0,1))] \tag{4.16}$$

with the estimate

$$\|\rho\|_{C^0([0,T];L^2(0,1))} + \|u\|_{C^0([0,T];L^2(0,1))\cap L^2(0,T;H^1_0(0,1))}$$
$$\leq C\left(\|\rho_0\|_{L^2(0,1)} + \|u_0\|_{L^2(0,1)} + \|p\|_{L^2(0,T)}\right), \tag{4.17}$$

for some constant $C > 0$.

Further, we have the hidden regularity property $\rho(\cdot, 1) \in L^2(0,T)$.

We give a sketch of the proof for Theorem 4.2.2 in Appendix A.0.2-A.1.

4.3 A short description of the spectral properties of the adjoint operator

In this section, we briefly describe the spectral properties of the adjoint operator A^* associated to our control system (4.4) or (4.5). This part is crucial in our analysis but it is the most technical part, and thus a detailed study will be presented in Section 4.8.

4.3.1 The eigenvalue problem

Let us denote $\Phi := (\xi, \eta)$ and consider the following eigenvalue problem:

$$A^*\Phi = \lambda\Phi, \quad \text{for } \lambda \in \mathbb{C},$$

which is explicitly given by

$$\begin{aligned} \xi'(x) + c\eta'(x) &= \lambda\xi(x), \quad x \in (0,1), \\ \eta''(x) + \eta'(x) + c\xi'(x) &= \lambda\eta(x), \quad x \in (0,1), \\ \xi(0) &= \xi(1), \\ \eta(0) = \eta(1) &= 0. \end{aligned} \tag{4.18}$$

We prove the following proposition.

Proposition 4.3.1. *The following results are true.*

(i) $\ker A^* = \operatorname{span}\{(1,0)\}$.

(ii) All non-zero eigenvalues of A^* have negative real parts.

(iii) The resolvent operator associated with A^* is compact and hence the spectrum of A^* is discrete.

(iv) Let $c > 0$ be such that $c^4 + 8c^2 + 5 < 4\pi^2$. Then, the eigenvalues of A^* are geometrically simple.

A quick observation tells that: when $\lambda = 0$, then $\alpha(1,0)$ with $\alpha \neq 0$ are the only eigenfunctions of the operator A^*, which is nothing but the part (i) of the above proposition. Proofs of the other parts are given in Section 4.8.

4.3.2 The set of eigenvalues

Let us write the properties of the eigenvalues of the operator A^*. More precisely, we have the following lemma.

Lemma 4.3.1. *Let $(A^*, D(A^*))$ be the operator given by (4.9). Then, there exist $k_0, n_0 \in \mathbb{N}^*$ such that A^* has three sets of eigenvalues: the parabolic part $\{\lambda_k^p\}_{k\geq k_0}$, the hyperbolic part $\{\lambda_k^h\}_{|k|\geq k_0}$ and a finite family $\{0\} \cup \{\widehat{\lambda}_n\}_{n=1}^{n_0}$ of lower frequencies. Moreover, the parabolic and hyperbolic branches satisfy the following asymptotic properties:*

$$\lambda_k^p = -k^2\pi^2 + O(1), \quad \text{for all } k \geq k_0 \text{ large}, \tag{4.19a}$$

$$\lambda_k^h = -c^2 - 2ik\pi + O(|k|^{-1}), \quad \text{for all } |k| \geq k_0 \text{ large}. \tag{4.19b}$$

The proof of the above lemma is one of the crucial part of our work and it is heavy; the details have been provided in Sections 4.8.1 and 4.8.3.

For simplicity, we set $\lambda_0 = 0$ and the associated eigenfunction by $\Phi_{\lambda_0} = (1,0)$. We further denote the set of eigenvalues associated to the parabolic and hyperbolic parts respectively by

$$\Lambda_p := \{\lambda_k^p,\ k \geq k_0\}, \quad \Lambda_h := \{\lambda_k^h,\ |k| \geq k_0\}, \tag{4.20}$$

and for the lower frequencies by

$$\Lambda_0 := \left\{\widehat{\lambda}_n,\ 1 \le n \le n_0\right\}. \tag{4.21}$$

Finally, the set of all eigenvalues are denoted by $\sigma(A^*)$, where

$$\sigma(A^*) := \left\{\lambda_0\right\} \cup \Lambda_0 \cup \Lambda_p \cup \Lambda_h. \tag{4.22}$$

4.3.3 The set of eigenfunctions

We start by writing the following proposition.

Proposition 4.3.2. *Let k_0 be as given by Lemma 4.3.1. Then, the operator A^* has the following sets of (generalized) eigenfunctions: the parabolic part $\{\Phi_{\lambda_k^p}\}_{k\ge k_0}$, the hyperbolic part $\{\Phi_{\lambda_k^h}\}_{|k|\ge k_0}$, the singleton set $\{\Phi_{\lambda_0}\}$ and a finite set $\{\Phi^i_\lambda;\ \lambda \in \Lambda_0,\ i = 0,\dots,m_\lambda - 1\}$, where $m_\lambda \ge 1$ is the length of Jordan chain associated to each of the eigenvalues $\lambda \in \Lambda_0$.*

Furthermore, we have the following:

1. *The parabolic part of the eigenfunctions*

$$\Phi_{\lambda_k^p} := (\xi_{\lambda_k^p}, \eta_{\lambda_k^p}) \tag{4.23}$$

have asymptotic expressions for large $k \ge k_0$, given by

$$\xi_{\lambda_k^p}(x) = \frac{ib}{k\pi} e^{-\frac{1}{2}(1+x)} \cos(k\pi(1-x)) + e^{x\left(-k^2\pi^2 + O(1)\right)} \times O\left(\frac{1}{k}\right) + O\left(\frac{1}{k^2}\right), \tag{4.24}$$

$$\eta_{\lambda_k^p}(x) = e^{-\frac{1}{2}(1+x)} \sin(k\pi(1-x)) + O\left(\frac{1}{k}\right), \tag{4.25}$$

for all $x \in (0,1)$ and the hyperbolic part of the eigenfunctions

$$\Phi_{\lambda_k^h} := (\xi_{\lambda_k^h}, \eta_{\lambda_k^h}) \tag{4.26}$$

have asymptotic expressions for large $|k| \ge k_0$, given by

$$\xi_{\lambda_k^h}(x) = \frac{2i}{b e^{\frac{1}{\sqrt{|k|}}}} \operatorname{sgn}(k) e^{-\frac{1}{2} - i\operatorname{sgn}(k)\sqrt{|k\pi|}} e^{-2ik\pi x} + O\left(|k|^{-1}\right), \tag{4.27}$$

$$\begin{aligned}\eta_{\lambda_k^h}(x) = {} & \frac{1}{k\pi e^{\frac{1}{\sqrt{|k|}}}} \operatorname{sgn}(k) e^{-\frac{1}{2} - i\operatorname{sgn}(k)\sqrt{|k\pi|}} e^{-2ik\pi x} \\ & + \frac{1}{k\pi e^{\frac{1}{\sqrt{|k|}}}} \operatorname{sgn}(k) e^{-(1-x)\left(\sqrt{|k\pi|} - \frac{1}{2} - i\operatorname{sgn}(k)\sqrt{|k\pi|}\right)} + O\left(|k|^{-1}\right),\end{aligned} \tag{4.28}$$

for all $x \in (0,1)$, where the sgn *function is defined as*

$$\operatorname{sgn}(k) = \begin{cases} 1 & \text{when } k \ge 0, \\ -1 & \text{when } k < 0, \end{cases} \tag{4.29}$$

2. *The eigenfamily, denoted by*

$$\mathcal{E}(A^*) := \left\{\Phi_{\lambda_k^p},\ k \ge k_0\right\} \cup \left\{\Phi_{\lambda_k^h},\ |k| \ge k_0\right\} \cup \left\{\Phi_{\lambda_0}\right\} \cup \left\{\Phi^i_\lambda;\ \lambda \in \Lambda_0,\ i = 0,\dots,m_\lambda - 1\right\}, \tag{4.30}$$

forms a Riesz basis in $L^2(0,1) \times L^2(0,1)$.

The last property (Riesz basis) can also be proved in the space $(H^{s_1}_\sharp(0,1))' \times H^{-s_2}(0,1)$ $(s_1, s_2 \ge 0)$ by normalize the eigenfunctions suitably, as written below.

Corollary 4.3.1. *Let $s_1, s_2 \geq 0$ be given. The family of (generalized) eigenfunctions*

$$\mathcal{E}(A^*) := \left\{k^{s_2}\Phi_{\lambda_k^p},\ k \geq k_0\right\} \cup \left\{k^{s_1}\Phi_{\lambda_k^h},\ |k| \geq k_0\right\} \cup \left\{\Phi_{\lambda_0}\right\} \cup \left\{\Phi_\lambda^j;\ \lambda \in \Lambda_0,\ j = 0, \ldots, m_\lambda - 1\right\},$$

forms a Riesz basis in $(H_\sharp^{s_1}(0,1))' \times H^{-s_2}(0,1)$.

We have taken the same finitely many eigenfunctions as before, which can be ensured by choosing suitable multiples of the generalized eigenfunctions. We will use this Riesz basis property (with appropriate s_1 and s_2) to prove the required observability inequalities, see the proof of our main results in Sections 4.6–4.7.

The existence of parabolic and hyperbolic parts of the family of eigenfunctions are proved in Sections 4.8.2–4.8.4. Then, using a result from [Guo01], we shall prove the existence of lower frequencies of eigenvalues $\{\widehat{\lambda}_n\}_{n=1}^{n_0}$ and the associated (generalized) eigenfunctions. Moreover, we will prove that the set of eigenfunctions $\mathcal{E}(A^*)$ forms a Riesz basis for $L^2(0,1) \times L^2(0,1)$.

Lemma 4.3.2 (Bounds of the eigenfunctions)**.** *Recall the eigenfunctions $\Phi_{\lambda_k^p} = (\xi_{\lambda_k^p}, \eta_{\lambda_k^p})$, $\forall k \geq k_0$ and $\Phi_{\lambda_k^h} = (\xi_{\lambda_k^h}, \eta_{\lambda_k^h})$, $\forall |k| \geq k_0$ given by (4.24)–(4.25) and (4.27)–(4.28) respectively. Then there exist constants $C_1, C_2 > 0$ independent in k, such that we have the following.*

1. *For any $s \geq 0$ and $k \geq k_0$, we have*

$$\begin{cases} C_1 k^{-s-1} \leq \|\xi_{\lambda_k^p}\|_{(H_\sharp^s(0,1))'} \leq C_2 k^{-s-1}, \\ C_1 k^{-s} \leq \|\eta_{\lambda_k^p}\|_{H^{-s}(0,1)} \leq C_2 k^{-s}. \end{cases} \tag{4.31}$$

2. *On the other hand, for any $|k| \geq k_0$ and $s \geq 0$, we have*

$$\begin{cases} C_1\,|k|^{-s} \leq \|\xi_{\lambda_k^h}\|_{(H_\sharp^s(0,1))'} \leq C_2 |k|^{-s}, \\ C_1\,|k|^{-s-1} \leq \|\eta_{\lambda_k^h}\|_{H^{-s}(0,1)} \leq C_2 |k|^{-s-1}. \end{cases} \tag{4.32}$$

Again, the proofs can be found in Section 4.8.5.

Riesz basis property of the (generalized) eigenfunctions. Let us first recall the following result.

Theorem 4.3.1 (B.-Z. GUO [Guo01])**.** *Let A be a densely defined discrete operator (i.e., the resolvent of A is compact) in a Hilbert space H. Let $\{\phi_n\}_1^\infty$ be a Riesz basis of H. If there are an integer $N \geq 0$ and a sequence of generalized eigenvectors $\{\psi_n\}_{N+1}^\infty$ of A such that*

$$\sum_{N+1}^{\infty} \|\phi_n - \psi_n\|^2 < +\infty,$$

then the following results hold.

(i) There are a constant $M > N$ and generalized eigenvectors $\{\psi_{n0}\}_1^M$ of A such that $\{\psi_{n0}\}_1^M \cup \{\psi_n\}_{M+1}^\infty$ forms a Riesz Basis for H.

(ii) Let $\{\psi_{n0}\}_1^M \cup \{\psi_n\}_{M+1}^\infty$ correspond to the eigenvalues $\{\lambda_n\}_1^\infty$ of A. Then the spectrum $\sigma(A) = \{\lambda_n\}_1^\infty$, where λ_n is counted according to its algebraic multiplicity.

(iii) If there is an $M_0 > 0$ such that $\lambda_n \neq \lambda_m$ for all $m, n > M_0$, then there is an $N_0 > M_0$ such that all λ_n are algebraically simple if $n > N_0$.

The first assumption of Theorem 4.3.1 is true in our case since we know that the resolvent operator of A^* is compact, thanks to the Proposition 4.3.1–part (iii). So, the next duty is to find a known family $\{\Psi_k,\ k \in \mathbb{N}^*;\ \widetilde{\Psi}_k,\ k \in \mathbb{Z}\}$ that defines a Riesz basis for $L^2(0,1) \times L^2(0,1)$ and that is quadratically close to the countable family $\left\{\Phi_{\lambda_k^p},\ k \geq k_0\right\} \cup \left\{\Phi_{\lambda_k^h},\ |k| \geq k_0\right\}$. Precisely, our goal is to show the following:

$$\sum_{k \geq k_0} \left\| \Phi_{\lambda_k^p} - \Psi_k \right\|^2_{L^2 \times L^2} + \sum_{|k| \geq k_0} \left\| \Phi_{\lambda_k^h} - \widetilde{\Psi}_k \right\|^2_{L^2 \times L^2} < +\infty. \tag{4.33}$$

To this end, let us consider the following functions on $(0,1)$:

$$\Psi_k(x) := \begin{pmatrix} \phi_k(x) \\ \psi_k(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 2ie^{-\frac{1}{2}(1+x)} \sin(k\pi(1-x)) \end{pmatrix}, \quad \forall k \in \mathbb{N}^*, \tag{4.34a}$$

$$\widetilde{\Psi}_k(x) := \begin{pmatrix} \widetilde{\phi}_k(x) \\ \widetilde{\psi}_k(x) \end{pmatrix} = \begin{pmatrix} \frac{2i}{ce^{\frac{1}{\sqrt{|k|}}}} \operatorname{sgn}(k) e^{-\frac{1}{2} - i \operatorname{sgn}(k)\sqrt{|k\pi|}} e^{-2ik\pi x} \\ 0 \end{pmatrix}, \quad \forall k \in \mathbb{Z}. \tag{4.34b}$$

It can be shown that the family $\{\Psi_k,\ k \in \mathbb{N}^*;\ \widetilde{\Psi}_k,\ k \in \mathbb{Z}\}$ of above functions forms a Riesz basis for $L^2(0,1) \times L^2(0,1)$ and we have the following result.

Lemma 4.3.3. *The family $\{\Psi_k, k \in \mathbb{N}^*;\ \widetilde{\Psi}_k, k \in \mathbb{Z}\}$ given by (4.34) is quadratically close to the family of eigenfunctions $\left\{\Phi_{\lambda_k^p},\ k \geq k_0\right\} \cup \left\{\Phi_{\lambda_k^h},\ |k| \geq k_0\right\}$.*

Proof. Looking at the expressions of the eigenfunctions $\Phi_{\lambda_k^p}$, $\Phi_{\lambda_k^h}$ for large modulus of k, given by (4.23)–(4.24)–(4.25) and (4.26)–(4.27)–(4.28) (resp.) and the known functions Ψ_k, $\widetilde{\Psi}_k$ given by (4.34), it is straightforward to compute that

$$\left\| \Phi_{\lambda_k^p} - \Psi_k \right\|^2_{L^2 \times L^2} \leq \frac{C}{k^2}, \quad \forall k \geq k_0 \text{ large enough},$$

and

$$\left\| \Phi_{\lambda_k^h} - \widetilde{\Psi}_k \right\|^2_{L^2 \times L^2} \leq \frac{C}{k^2}, \quad \forall |k| \geq k_0 \text{ large enough},$$

which implies the required property (4.33). □

Sketch of the proof for Proposition 4.3.2. First, recall that the countable (infinite) number of eigenfunctions $\{\Phi_{\lambda_k^p}\}_{k \geq k_0}$ and $\{\Phi_{\lambda_k^h}\}_{|k| \geq k_0}$, with their asymptotic expressions are already given by (4.24)–(4.25), (4.27)–(4.28).

Now, thanks to Lemma 4.3.3, we can apply the point (i) of Theorem 4.3.1 to ensure the existence of eigenmodes for lower frequencies. Accordingly, there exist an $n_0 \in \mathbb{N}^*$ and a finite set eigenvalues

$$\Lambda_0 := \{\widehat{\lambda}_n\}_1^{n_0}$$

of the operator A^*. But there may exist some generalized eigenfunctions corresponding to the eigenvalues of the finite set Λ_0. Thus, for each $\lambda \in \Lambda_0$, we associate a Jordan chain of length $m_\lambda \geq 1$, denoted by $\Phi_\lambda^0, \ldots, \Phi_\lambda^{m_\lambda - 1}$ which verify

$$(A^* - \lambda I)\Phi_\lambda^j = \Phi_\lambda^{j-1}, \quad \forall j \in \{1, \ldots, m_\lambda - 1\}, \quad \lambda \in \Lambda_0,$$

where in particular $\Phi_\lambda^0 := \Phi_\lambda$, the eigenfunction corresponding to λ. Moreover, by virtue of Theorem 4.3.1, we can guarantee that the family, given by

$$\mathcal{E}(A^*) := \left\{\Phi_{\lambda_k^p},\ k \geq k_0\right\} \cup \left\{\Phi_{\lambda_k^h},\ |k| \geq k_0\right\} \cup \left\{\Phi_{\lambda_0}\right\} \cup \left\{\Phi_\lambda^j;\ \lambda \in \Lambda_0,\ j = 0, \ldots, m_\lambda - 1\right\},$$

forms a Riesz basis in $L^2(0,1) \times L^2(0,1)$.

The proof ends. □

Remark 4.3.1. *In the same way, one can prove that the set of eigenvalues and (generalized) eigenfunctions of A (denoted by $\sigma(A)$ and $\mathcal{E}(A)$ respectively) have similar properties as of the eigenpairs of A^*. In this case, we can find some $\widetilde{k}_0 \in \mathbb{N}^*$ (large enough) such that A has the eigenvalues of parabolic and hyperbolic nature for $k \geq \widetilde{k}_0$ and $|k| \geq \widetilde{k}_0$ respectively. For later use, we denote the eigenfunctions of A, respectively by $\widetilde{\Phi}_k^p$, $k \geq \widetilde{k}_0$ and $\widetilde{\Phi}_k^h$, $|k| \geq \widetilde{k}_0$ corresponding to the parabolic and hyperbolic branches of eigenvalues. Moreover, using the result of Theorem 4.3.1, we can show that the set $\mathcal{E}(A)$ forms a Riesz basis for the space $L^2(0,1) \times L^2(0,1)$.*

4.4 Estimations of the observation terms

In this section, we are going to find some lower bounds of the observation terms associated to our control systems. In this regard, we use the notations $\mathcal{B}_\rho^*$ and $\mathcal{B}_u^*$ which represent the observation operators for the density and velocity case respectively, and their formal expressions are given below.

- The observation operator corresponding to (4.5) (control in density) is defined by

$$\mathcal{B}_\rho^* = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbb{1}_{\{x=1\}} : D(A^*) \to \mathbb{R}, \tag{4.35}$$

  such that

$$\mathcal{B}_\rho^* \Phi = \xi(1), \quad \forall\, \Phi = (\xi, \eta) \in D(A^*). \tag{4.36}$$

- The observation operator corresponding to (4.4) (control in velocity) is defined by

$$\mathcal{B}_u^* = c\mathbb{1}_{\{x=1\}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathbb{1}_{\{x=1\}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \frac{\partial}{\partial x} : D(A^*) \to \mathbb{R}, \tag{4.37}$$

  such that

$$\mathcal{B}_u^* \Phi = c\xi(1) + \eta'(1), \quad \forall\, \Phi = (\xi, \eta) \in D(A^*). \tag{4.38}$$

4.4.1 Characteristics of the observation terms

Let us pick any

$$\Phi := (\xi, \eta) \in \left\{\Phi_\lambda;\ \lambda \in \Lambda_p \cup \Lambda_h \cup \Lambda_0\right\},$$

and recall the eigenvalue problem (4.18). Substituting the first equation of (4.18) in the second one, we get

$$\eta''(x) - (c^2 - 1)\eta'(x) + c\lambda\xi(x) - \lambda\eta(x) = 0, \quad \forall x \in (0,1). \tag{4.39}$$

Differentiating, we have

$$\eta'''(x) - (c^2 - 1)\eta''(x) + c\lambda\xi'(x) - \lambda\eta'(x) = 0, \quad \forall x \in (0,1).$$

By substituting $c\xi' = \lambda\eta - \eta'' - \eta'$ in above, we get a third order ode satisfied only by η as follows

$$\begin{cases} \eta'''(x) - (\lambda + c^2 - 1)\eta''(x) - 2\lambda\eta'(x) + \lambda^2\eta(x) = 0, \quad \forall x \in (0,1), \\ \eta(0) = 0,\ \eta(1) = 0, \\ \eta''(0) - (c^2 - 1)\eta'(0) = \eta''(1) - (c^2 - 1)\eta'(1). \end{cases} \tag{4.40}$$

Let m_1, m_2 and m_3 be roots of the cubic auxiliary equation (associated to (4.40))

$$m^3 - (\lambda + c^2 - 1)m^2 - 2\lambda m + \lambda^2 = 0. \tag{4.41}$$

Then, we have the following result which states some properties of the roots m_1, m_2 and m_3.

Lemma 4.4.1. *The following statements hold.*

- *Roots of the cubic equation (4.41) has multiplicity less than 3.*
- *If $c > 0$ is such that $c^4 + 8c^2 + 5 < 4\pi^2$, the relation $e^{m_1} = e^{m_2} = e^{m_3}$ cannot hold.*

Proof. From the relation between roots and the coefficients, we have

$$\begin{cases} m_1 + m_2 + m_3 = \lambda + c^2 - 1, \\ m_1 m_2 + m_2 m_3 + m_3 m_1 = -2\lambda, \\ m_1 m_2 m_3 = -\lambda^2. \end{cases} \tag{4.42}$$

We prove all the statements separately.

- Let $m_1 = m_2 = m_3 = m$. Then, we have from the first equation of (4.42)

  $$m = \frac{1}{3}(\lambda + c^2 - 1).$$

  Next, from the second and third equations of (4.42), we have $3m^2 = -2\lambda$ and $m^3 = -\lambda^2$ which further yields

  $$(\lambda + c^2 - 1)^2 = -6\lambda, \quad (\lambda + c^2 - 1)^3 = -27\lambda^2, \tag{4.43}$$

  so that $\lambda + c^2 - 1 = \frac{9}{2}\lambda$. By means of the first equality in (4.43), we then have $\lambda = -\frac{8}{27}$ which eventually gives

  $$c^2 = 1 + \frac{7}{2}\lambda = 1 - \frac{28}{27} = -\frac{1}{27} < 0,$$

  and this is not possible. Hence m_1, m_2 and m_3 cannot be equal together.

- Let us now assume

  $$e^{m_1} = e^{m_2} = e^{m_3},$$

  that is,

  $$m_2 = m_1 + 2il\pi, \quad m_3 = m_1 + 2in\pi,$$

  for some $l, n \in \mathbb{Z}$. From the first equation of (4.42), we have that

  $$3m_1 + 2il\pi + 2in\pi = \lambda + c^2 - 1, \quad \text{i.e.,} \quad m_1 = \frac{1}{3}(\lambda + c^2 - 1 - 2il\pi - 2in\pi), \tag{4.44}$$

  and so,

  $$m_2 = \frac{1}{3}(\lambda + c^2 - 1 + 4il\pi - 2in\pi), \quad m_3 = \frac{1}{3}(\lambda + c^2 - 1 - 2il\pi + 4in\pi). \tag{4.45}$$

  Substituting the above m_1, m_2, m_3 in the second equation of (4.42), we deduce (upon simplifications)

  $$\lambda^2 + 2(c^2 + 2)\lambda + 4(l^2 - ln + n^2)\pi^2 + (c^2 - 1)^2 = 0.$$

  Solving the above equation, we get some particular values of λ, namely

  $$\begin{aligned} \lambda &= \frac{-2(c^2 + 2) \pm \sqrt{4(c^2 + 2)^2 - 16\pi^2(l^2 - ln + n^2) - 4(c^2 - 1)^2}}{2} \\ &= -c^2 - 2 \pm \sqrt{3(2c^2 + 1) - 4\pi^2(l^2 - ln + n^2)}. \end{aligned}$$

  Since $l, n \in \mathbb{Z}$, one has $l^2 - ln + n^2 \geq 0$[1] and $l^2 - ln + n^2 = 0$ if and only if $l = n = 0$[2]. Thus for $(l, n) \neq (0, 0)$ the values of λ are

  $$\lambda = -c^2 - 2 \pm i\sqrt{4\pi^2(l^2 - ln + n^2) - 3(2c^2 + 1)}. \tag{4.46}$$

[1] For $ln = 0$, $l^2 - ln + n^2 = l^2 + n^2 \geq 0$, for $ln < 0, l^2 - ln + n^2 > 0$ and for $ln > 0, l^2 - ln + n^2 = (l - n)^2 + ln > 0$.

[2] If $l^2 - ln + n^2 = 0$ and $n \neq 0$ then $(\frac{l}{n})^2 - (\frac{l}{n}) + 1 = 0$ has no real solutions. Therefore $n = 0$ and hence $l = 0$.

Note that $4\pi^2(l^2-ln+n^2)-3(2c^2+1)$ is always non-negative under the assumption $c^4+8c^2+5<4\pi^2$ and for all $(l,n)\neq(0,0)$.

On the other hand, putting the values of m_1,m_2,m_3 (given by (4.44)–(4.45)) in the third equation of (4.42), we get

$$(\lambda+c^2-1-2il\pi-2in\pi)(\lambda+c^2-1+4il\pi-2in\pi)(\lambda+c^2-1-2il\pi+4in\pi)=-27\lambda^2,$$

which further yields

$$\begin{aligned}&\lambda^3+3(c^2+8)\lambda^2+(3(c^2-1)^2+12l^2\pi^2-12ln\pi^2+12n^2\pi^2)\lambda+(c^2-1)^3\\&+12\pi^2(c^2-1)(l^2-ln+n^2)-16il^3\pi^3+24il^2n\pi^3+24iln^2\pi^3-16in^3\pi^3=0.\end{aligned}$$

The real part of above equality satisfies

$$\begin{aligned}\mathrm{Re}(\lambda^3)+3(c^2+8)\mathrm{Re}(\lambda^2)+[3(c^2-1)^2+12\pi^2(l^2-ln+n^2)]\mathrm{Re}(\lambda)\\+(c^2-1)^3+12\pi^2(c^2-1)(l^2-ln+n^2)=0.\end{aligned}\tag{4.47}$$

Now, from (4.46), one may find that

$$\begin{aligned}&\mathrm{Re}(\lambda)=-(c^2+2),\\&\mathrm{Re}(\lambda^2)=c^4+10c^2+7-4\pi^2(l^2-ln+n^2),\\&\mathrm{Re}(\lambda^3)=-c^6-24c^4-57c^2-26+12\pi^2(c^2+2)(l^2-ln+n^2).\end{aligned}$$

Replacing the above values in (4.47), we obtain

$$\begin{aligned}&-c^6-24c^4-57c^2-26+12\pi^2(c^2+2)(l^2-ln+n^2)\\&+3(c^2+8)[c^4+10c^2+7-4\pi^2(l^2-ln+n^2)]\\&-[3(c^2-1)^2+12\pi^2(l^2-ln+n^2)](c^2+2)+(c^2-1)^3+12\pi^2(c^2-1)(l^2-ln+n^2)=0\end{aligned}$$

Simplifying, we eventually have

$$27c^4+216c^2+135-108\pi^2(l^2-ln+n^2)=0,$$

so that

$$l^2-ln+n^2=\frac{27c^4+216c^2+135}{108\pi^2}=\frac{c^4+8c^2+5}{4\pi^2}<1,$$

by our assumption $c^4+8c^2+5<4\pi^2$, which is a contradiction as $l^2-ln+n^2\geq 1$ for any $(l,n)\neq(0,0)$.

Therefore, the only possibility could be $l=n=0$, but in that case, the expressions (4.44) and (4.45) provides us $m_1=m_2=m_3$, which is again a contradiction to the first part of the lemma.

Hence, the results of this lemma are true. □

We are now ready to prove that all the observation terms are non-zero for both density and velocity control cases. For $\lambda=0$, the eigenfunction is $(1,0)$, and thus from the expressions of observation terms (4.36) and (4.38), we immediately get

$$\mathcal{B}^*_\rho(1,0)=1,\quad \mathcal{B}^*_u(1,0)=c,$$

which are non-zero.

We thus focus only on the case when $\lambda\neq 0$. In such a situation, for any eigenfunction Φ of A^*, the observation terms can be rewritten as

$$\mathcal{B}^*_\rho\Phi=-\frac{1}{c\lambda}\left(\eta''(1)-(c^2-1)\eta'(1)\right),\tag{4.48}$$

$$\mathcal{B}^*_u\Phi=-\frac{1}{\lambda}\left(\eta''(1)-(\lambda+c^2-1)\eta'(1)\right),\tag{4.49}$$

where we have used the equation (4.39).

We now prove the following proposition.

Proposition 4.4.1. *We have the following results for any non-zero eigenvalue λ of A^*.*

1. *Let $c > 0$ be such that $c^4 + 8c^2 + 5 < 4\pi^2$. Then, the solution η of (4.40) satisfies $\eta''(1) \neq (c^2 - 1)\eta'(1)$.*
2. *There exists a countable set $\mathcal{N} \subset (0, \infty)$ such that for all $c \in (0, \infty) \setminus \mathcal{N}$ with $c^4 + 8c^2 + 5 < 4\pi^2$, the solution η of (4.40) satisfies $\eta''(1) \neq (\lambda + c^2 - 1)\eta'(1)$.*

Proof. 1. To prove the first part, we suppose on contrary that $\eta''(1) = (c^2 - 1)\eta'(1)$. This will also give us $\eta''(0) = (c^2 - 1)\eta'(0)$ since $\xi(0) = \xi(1)$ and consequently, $\eta''(1) - (c^2 - 1)\eta'(1) = \eta''(0) - (c^2 - 1)\eta'(0)$. We will use the Fourier transform technique together with some complex analytical arguments to prove that $\eta = 0$ on $(0, 1)$. This kind of technique is applied in many works, see for instance [Ros97] for KdV the equation.

Let us define an extension map $\vartheta : \mathbb{R} \to \mathbb{R}$ by

$$\vartheta(x) = \begin{cases} \eta(x), & x \in (0,1), \\ 0, & x \in \mathbb{R} \setminus (0,1). \end{cases} \tag{4.50}$$

Then the transformed equation for (4.40) is

$$\begin{aligned} &\vartheta'''(x) - (\lambda + c^2 - 1)\vartheta''(x) - 2\lambda\vartheta'(x) + \lambda^2\vartheta(x) \\ &= -\eta''(1)\delta_{x=1} + \eta''(0)\delta_{x=0} - \eta'(1)\left[\delta'_{x=1} - (\lambda + c^2 - 1)\delta_{x=1}\right] + \eta'(0)\left[\delta'_{x=0} - (\lambda + c^2 - 1)\delta_{x=0}\right] \end{aligned} \tag{4.51}$$

for all $x \in \mathbb{R}$.

Let us use the conditions $\eta''(1) = (c^2 - 1)\eta'(1)$ and $\eta''(0) = (c^2 - 1)\eta'(0)$ in (4.51), which gives

$$\begin{aligned} &\vartheta'''(x) - (\lambda + c^2 - 1)\vartheta''(x) - 2\lambda\vartheta'(x) + \lambda^2\vartheta(x) \\ &= -\eta'(1)\left[\delta'_{x=1} - \lambda\delta_{x=1}\right] + \eta'(0)\left[\delta'_{x=0} - \lambda\delta_{x=0}\right], \quad \forall x \in \mathbb{R}. \end{aligned} \tag{4.52}$$

Observe that, the existence of an η satisfying (4.40) is equivalent to the existence of α, β, λ with $(\alpha, \beta) \neq (0, 0)$, such that

$$\begin{aligned} &\vartheta'''(x) - (\lambda + c^2 - 1)\vartheta''(x) - 2\lambda\vartheta'(x) + \lambda^2\vartheta(x) \\ &= -\alpha\left[\delta'_{x=1} - \lambda\delta_{x=1}\right] + \beta\left[\delta'_{x=0} - \lambda\delta_{x=0}\right], \quad \forall x \in \mathbb{R}. \end{aligned} \tag{4.53}$$

Without loss of generality, we can assume $\alpha \neq 0$. Indeed, $\alpha = \eta'(1) = 0$ implies $\eta''(1) = 0$ from our assumption and thus from the equation (4.40), one has $\eta = 0$.

Taking Fourier transform on both sides of (4.53), we get

$$\begin{aligned} &\left((iz)^3 - (\lambda + c^2 - 1)(iz)^2 - 2\lambda(iz) + \lambda^2\right)\hat{\vartheta}(z) \\ &= -\alpha(ize^{-iz} - \lambda e^{-iz}) + \beta(iz - \lambda), \quad \text{for } z \in \mathbb{C}, \end{aligned}$$

which yields

$$\hat{\vartheta}(z) = \frac{(-\alpha e^{-iz} + \beta)(iz - \lambda)}{(iz)^3 - (\lambda + c^2 - 1)(iz)^2 - 2\lambda(iz) + \lambda^2}, \quad \text{for } z \in \mathbb{C}.$$

Since $\hat{\vartheta}$ is the Fourier transform of a function $\eta \in H_0^1(0, 1)$, by the Paley-Wiener theorem, the function $\hat{\vartheta}$ is entire. Thus, the roots of $(iz)^3 - (\lambda + c^2 - 1)(iz)^2 - 2\lambda(iz) + \lambda^2$ are also the roots of $(-\alpha e^{-iz} - \beta)(\lambda - iz)$ with the same multiplicity. So, the main work is to find the roots of

$$(-\alpha e^{-iz} + \beta)(iz - \lambda) = 0, \quad \text{for } z \in \mathbb{C}. \tag{4.54}$$

In fact, rewriting $\hat{\vartheta}$ as a function $iz \in \mathbb{C}$, we have

$$\hat{\vartheta}(iz) = \frac{(-\alpha e^{z} + \beta)(-z - \lambda)}{-z^3 - (\lambda + c^2 - 1)z^2 + 2\lambda z + \lambda^2}, \quad \text{for } z \in \mathbb{C}. \tag{4.55}$$

In (4.55), the roots of $(-\alpha e^z + \beta)(-z - \lambda)$ are $z = -\lambda$ and the zeros of $e^z = \frac{\beta}{\alpha}$ (as we have $\alpha \neq 0$). We also note that $-\lambda$ is not a root of the polynomial equation

$$-z^3 - (\lambda + c^2 - 1)z^2 + 2\lambda z + \lambda^2 = 0, \tag{4.56}$$

since $\lambda c \neq 0$.

Let r_1, r_2, r_3 be the roots of the equation (4.56). Then one must have

$$e^{r_1} = e^{r_2} = e^{r_3} = \frac{\beta}{\alpha},$$

which is not possible, due to Lemma 4.4.1.

Therefore, the only possibility is $\alpha = \beta = 0$, which gives (comparing (4.52) and (4.53)) that $\eta'(0) = \eta'(1) = 0$. But, we have the boundary condition $\eta(0) = \eta(1) = 0$ and by assumption $\eta''(1) - (c^2 - 1)\eta'(1) = \eta''(0) - (c^2 - 1)\eta'(0)$, i.e., $\eta''(1) = \eta''(0) = 0$. Consequently, $\eta = 0$ in $(0, 1)$ and thus $\xi = 0$ in $(0, 1)$.

So our assumption was false, and that the assertion of first part holds true.

2. To prove the second statement, we assume on contrary that

$$\eta''(1) = (\lambda + c^2 - 1)\eta'(1). \tag{4.57}$$

Now, our claim is to show that $\eta = 0$ in $(0, 1)$. We note here that the Fourier transform technique used earlier will not work here due to the difficulty of the boundary condition $\eta''(1) = (\lambda + b^2 - 1)\eta'(1)$. However, we use a different complex analytic method, addressed for instance in [LB20b], to conclude the proof.

Consider the following adjoint system of (4.40) as

$$\begin{cases} -\theta'''(x) - (\lambda + c^2 - 1)\theta''(x) + 2\lambda\theta'(x) + \lambda^2\theta(x) = 0, \\ \theta(0) = 0, \quad \theta'(0) = 0, \quad \theta'(1) \neq 0. \end{cases} \tag{4.58}$$

Multiplying the equation (4.40) by θ and then integrating by parts, we obtain

$$\eta''(1)\theta(1) - \eta'(1)\theta'(1) - (\lambda + c^2 - 1)\eta'(1)\theta(1) = 0.$$

Then, due to our assumption (4.57), we get

$$\eta'(1)\theta'(1) = 0. \tag{4.59}$$

Let us make the following claim.

Claim. There exists a countable set $\mathcal{N}$ such that for any $c \in (0, \infty) \setminus \mathcal{N}$ with $c^4 + 8c^2 + 5 < 4\pi^2$, the equation (4.58) has a non-trivial solution.

Proof of the Claim. Let m_1^*, m_2^*, m_3^* be roots of the following auxiliary equation

$$-m^3 - (\lambda + c^2 - 1)m^2 + 2\lambda m + \lambda^2 = 0. \tag{4.60}$$

Since c satisfies $c^4 + 8c^2 + 5 < 4\pi^2$, the roots of (4.60) does not satisfy $e^{m_1^*} = e^{m_2^*} = e^{m_3^*}$, thanks to Lemma 4.4.1. Note also that the map $c \mapsto m(c)$ is injective. In fact, $m(c_1) = m(c_2)$ implies $(c_1^2 - c_2^2)m(c_1) = 0$ and hence $c_1 = c_2$ (since $m(c_1) \neq 0$ for any $\lambda \neq 0$). We then write the solution θ of (4.58) as

$$\theta(x) = C_1 e^{m_1^* x} + C_2 e^{m_2^* x} + C_3 e^{m_3^* x}, \quad x \in (0, 1). \tag{4.61}$$

Consider the following system of equations

$$C_1 + C_2 + C_3 = 0$$

$$C_1 m_1^* + C_2 m_2^* + C_3 m_3^* = 0$$
$$C_1 m_1^* e^{m_1^*} + C_2 m_2^* e^{m_2^*} + C_3 m_3^* e^{m_3^*} = \theta'(1),$$

which has a solution if and only if the matrix

$$\mathcal{R}_c := \begin{pmatrix} 1 & 1 & 1 \\ m_1^* & m_2^* & m_3^* \\ m_1^* e^{m_1^*} & m_2^* e^{m_2^*} & m_3^* e^{m_3^*} \end{pmatrix} \tag{4.62}$$

is invertible. The determinant of $\mathcal{R}_c$ is given by

$$\det(\mathcal{R}_c) = m_2^* m_3^* (e^{m_2^*} - e^{m_3^*}) + m_3^* m_1^* (e^{m_3^*} - e^{m_1^*}) + m_1^* m_2^* (e^{m_1^*} - e^{m_2^*}). \tag{4.63}$$

We now characterize all $c \in (0, \infty)$ such that $\det(\mathcal{R}_c) \neq 0$. Let us define three entire functions $F_i : \mathbb{C} \to \mathbb{C}$ $(i = 1, 2, 3)$ by

$$F_1(z) : = z \left[(m_2^* - m_3^*) e^z - m_2^* e^{m_2^*} + m_3^* e^{m_3^*} \right] + m_2^* m_3^* \left(e^{m_2^*} - e^{m_3^*} \right) \tag{4.64}$$
$$F_2(z) : = z \left[(m_3^* - m_1^*) e^z + m_1^* e^{m_1^*} - m_3^* e^{m_3^*} \right] + m_3^* m_1^* \left(e^{m_3^*} - e^{m_1^*} \right) \tag{4.65}$$
$$F_3(z) : = z \left[(m_1^* - m_2^*) e^z - m_1^* e^{m_1^*} + m_2^* e^{m_2^*} \right] + m_1^* m_2^* \left(e^{m_1^*} - e^{m_2^*} \right). \tag{4.66}$$

We first consider the function F_1. Note that if $F_1(0) = 0$, then $e^{m_2^*} = e^{m_3^*}$, which implies $F_1(z) = (m_2^* - m_3^*) z (e^z - e^{m_3^*})$ and hence $F_1(m_1^*) \neq 0$, else $e^{m_1^*} = e^{m_2^*} = e^{m_3^*}$ which is not possible due to Lemma 4.4.1. Therefore, the function F_1 does not vanish identically. This implies that the zero set of F_1, defined as

$$Z_{F_1} := \{ z \in \mathbb{C} \ : \ F_1(z) = 0 \} \tag{4.67}$$

is at most countable. In a similar manner, we can say that the zero sets of F_2 and F_3, defined as

$$Z_{F_2} : = \{ z \in \mathbb{C} \ : \ F_2(z) = 0 \}, \tag{4.68}$$
$$Z_{F_3} : = \{ z \in \mathbb{C} \ : \ F_3(z) = 0 \} \tag{4.69}$$

are at most countable. Since the map $c \mapsto m(c)$ is injective, the set

$$\mathcal{N}_j := \left\{ c \in (0, \infty) \ : \ F_j(m_j(c)) = 0 \right\} \tag{4.70}$$

for $j = 1, 2, 3$, is also at most countable. Let us then define the set

$$\mathcal{N} := \bigcup_{j=1}^{3} \mathcal{N}_j. \tag{4.71}$$

From the construction of the set $\mathcal{N}$, it is clear that for all $c \in (0, \infty) \setminus \mathcal{N}$ with $c^4 + 8c^2 + 5 < 4\pi^2$, $\det(\mathcal{R}_c)$ is non-zero. This proves our claim.

From the previous fact, we can see that for $c \in (0, \infty) \setminus \mathcal{N}$ with $c^4 + 8c^2 + 5 < 4\pi^2$, solution of the adjoint equation (4.58) verifies $\theta'(1) \neq 0$, which implies from (4.59) that $\eta'(1) = 0$. Hence $\eta \equiv 0$ on $(0, 1)$.

This completes the proof of the Lemma. □

4.4.2 Lower bounds of the observation terms

The next lemmas show that the observation terms satisfy some lower bounds which are not exponentially small. In fact, these lower bounds are crucial to conclude the null-controllability of the concerned systems (4.4) and (4.5).

Lemma 4.4.2 (Observation estimates: control on density)**.** *There exist constants $C_1, C_2 > 0$, independent in k, such that we have the following observation estimates for the parabolic and hyperbolic parts of the set of eigenfunctions of A^*, namely*

$$\frac{C_1}{k\pi} \leq |\mathcal{B}_\rho^* \Phi_{\lambda_k^p}| \leq \frac{C_2}{k\pi}, \quad \textit{for large } k \geq k_0, \tag{4.72a}$$

$$C_1 \leq |\mathcal{B}_\rho^* \Phi_{\lambda_k^h}| \leq C_2, \quad \textit{for large } k \geq k_0, \tag{4.72b}$$

where the number k_0 is introduced by Lemma 4.3.1.

Proof. Using the definition of $\mathcal{B}_\rho^*$ introduced by (4.36), we have

$$\mathcal{B}_\rho^* \Phi_{\lambda_k^p} = \xi_{\lambda_k^p}(1), \quad \forall k \geq k_0,$$
$$\mathcal{B}_\rho^* \Phi_{\lambda_k^h} = \xi_{\lambda_k^h}(1), \quad \forall |k| \geq k_0.$$

(i) Let us recall the expressions of $\xi_{\lambda_k^p}$ from (4.24), so that we have

$$\xi_{\lambda_k^p}(1) = \frac{ic}{k\pi} e^{-1} + e^{-k^2\pi^2 + O(1)} \times O\left(\frac{1}{k}\right) + O\left(\frac{1}{k^2}\right)$$

From the above expression, it is easy to observe that

$$k\pi \left| \xi_{\lambda_k^p}(1) \right| \to c e^{-1} \quad \text{as } k \to +\infty,$$

and thus the result (4.72a) holds for large enough k.

(ii) On the other hand, from the expression of $\xi_{\lambda_k^h}$ given by (4.27), we have

$$\xi_{\lambda_k^h}(1) = \frac{2i}{c} \operatorname{sgn}(k) e^{-\frac{1}{2} - i \operatorname{sgn}(k) \sqrt{|k\pi|}} + O\left(|k|^{-1}\right),$$

and so,

$$\left| \xi_{\lambda_k^h}(1) \right| \to \frac{2}{c} e^{-\frac{1}{2}} \quad \text{as } k \to +\infty.$$

As a consequence, the estimate (4.72b) follows.

The proof is completed. □

Lemma 4.4.3 (Observation estimates: control in velocity)**.** *There exist some constants $C_1, C_2 > 0$, independent in k, such that we have the following observation estimates:*

$$C_1 k\pi \leq |\mathcal{B}_u^* \Phi_{\lambda_k^p}| \leq C_2 k\pi, \quad \textit{for large } k, \tag{4.73a}$$

$$\frac{C_1}{\sqrt{|k\pi|}} \leq |\mathcal{B}_u^* \Phi_{\lambda_k^h}| \leq \frac{C_2}{\sqrt{|k\pi|}}, \quad \textit{for large } k, \tag{4.73b}$$

Proof. Using the definition of $\mathcal{B}_u^*$ given by (4.37)–(4.38), we have

$$\mathcal{B}_u^* \Phi_{\lambda_k^p} = c\xi_{\lambda_k^p}(1) + \eta'_{\lambda_k^p}(1), \quad \forall k \geq k_0,$$
$$\mathcal{B}_u^* \Phi_{\lambda_k^h} = c\xi_{\lambda_k^h}(1) + \eta'_{\lambda_k^h}(1), \quad \forall |k| \geq k_0.$$

(i) Recall the expressions of $\xi_{\lambda_k^p}$ and $\eta_{\lambda_k^p}$, given by (4.24) and (4.25) respectively, so that we have

$$C\xi_{\lambda_k^p}(1) + \eta'_{\lambda_k^p}(1) = \frac{ic^2}{k\pi}e^{-1} + be^{-k^2\pi^2+O(1)} \times O\left(\frac{1}{k}\right) + k\pi e^{-1} + O\left(\frac{1}{k}\right).$$

Observe that,

$$\frac{1}{k\pi}\left|c\xi_{\lambda_k^p}(1) + \eta'_{\lambda_k^p}(1)\right| \to e^{-1} \quad \text{as } k \to +\infty,$$

and hence the estimate (4.73a) holds.

(ii) For the set of eigenfunctions (4.27)–(4.28) associated to λ_k^h, the observation terms are

$$c\xi_{\lambda_k^h}(1) + \eta'_{\lambda_k^h}(1) = \text{sgn}(k)\frac{\sqrt{|k\pi|} - \frac{1}{2} - i\,\text{sgn}(k)\sqrt{|k\pi|}}{k\pi e^{\frac{1}{\sqrt{|k|}}}} + O\left(|k|^{-1}\right)$$

Here, one can show that

$$\sqrt{|k\pi|}\left|c\xi_{\lambda_k^h}(1) + \eta'_{\lambda_k^h}(1)\right| \to \sqrt{2} \quad \text{as } k \to +\infty,$$

which concludes the required observation estimate (4.73b).

The proof ends. □

4.5 A combined parabolic-hyperbolic Ingham-type inequality

This section is devoted to prove the Ingham-type inequality stated in Proposition 4.1.2 which will be intensively used to prove the controllability results of this paper. We will closely follow the decoupling idea given by [CMRR14, Theorem 4.2] [Zua16, Section 2.4].

Proof of Proposition 4.1.2. Recall the sequences $\{\lambda_k\}_{k\in\mathbb{N}^*}$ and $\{\gamma_k\}_{k\in\mathbb{Z}}$ and the hypothesis of Proposition 4.1.2. We denote $\tilde{\lambda}_k = \lambda_k - \beta$, $\forall k \in \mathbb{N}^*$ and $\tilde{\gamma}_k = \gamma_k - \beta$, $\forall k \in \mathbb{Z}$. Let $N \in \mathbb{N}^*$ be as given in the hypothesis. Then, we have the following known parabolic and hyperbolic Ingham inequalities

$$\int_0^T \left|\sum_{k\geq N} a_k e^{\tilde{\lambda}_k(T-t)}\right|^2 dt \geq C \sum_{k\geq N} |a_k|^2 e^{2\text{Re}(\tilde{\lambda}_k)T} \quad \text{for any } T > 0, \tag{4.74}$$

$$C_1 \sum_{|k|\geq N} |b_k|^2 \leq \int_0^T \left|\sum_{|k|\geq N} b_k e^{\tilde{\gamma}_k(T-t)}\right|^2 dt \leq C_2 \sum_{|k|\geq N} |b_k|^2 \quad \text{for any } T > 1, \tag{4.75}$$

see for instance, [Han91, Lóp99, Edw06, Ing36, LZ02, FCGBdT10, KL05, MZ04].

Let us denote

$$U^p(t) = \sum_{k\geq N} a_k e^{\tilde{\lambda}_k(T-t)}, \quad U^h(t) = \sum_{|k|\geq N} b_k e^{\tilde{\gamma}_k(T-t)}, \quad t \geq 0, \tag{4.76}$$

and

$$U(t) = U^p(t) + U^h(t), \quad t \geq 0. \tag{4.77}$$

Motivating from [Zua16], we define for $t > 1$

$$\widetilde{U}^p(t) = U^p(t) - U^p(t-1) = \sum_{k\geq N} a_k \left(1 - e^{\tilde{\lambda}_k}\right) e^{\tilde{\lambda}_k(T-t)}, \tag{4.78a}$$

$$\widetilde{U}^h(t) = U^h(t) - U^h(t-1) = \sum_{|k|\geq N} b_k \left(1 - e^{\tilde{\gamma}_k}\right) e^{\tilde{\gamma}_k(T-t)}, \tag{4.78b}$$

and
$$\widetilde{U}(t) = \widetilde{U}^p(t) + \widetilde{U}^h(t) = U(t) - U(t-1). \tag{4.79}$$

Then, we have

$$\begin{aligned}\int_1^T \left|\widetilde{U}(t)\right|^2 dt &\le \int_1^T |U(t)|^2\, dt + \int_1^T |U(t-1)|^2\, dt \\ &\le C \int_0^T |U(t)|^2\, dt.\end{aligned}$$

We now compute the L^2-norms of the functions $\widetilde{U}^p$ and $\widetilde{U}^h$ separately. Applying the hyperbolic Ingham inequality given by (4.75), we get

$$\int_1^T \left|\widetilde{U}^h(t)\right|^2 dt \le C \sum_{|k|\ge N} |b_k|^2 \left|1 - e^{\tilde{\gamma}_k}\right|^2.$$

Since $1 - e^{\tilde{\gamma}_k} = 1 - e^{\nu_k}$ and $\{\nu_k\}_{|k|\ge N} \in \ell_2$, we can choose N large enough such that $\left|1 - e^{\tilde{\gamma}_k}\right|^2 < \epsilon$ for all $|k| \ge N$. Thus, it follows that,

$$\int_1^T \left|\widetilde{U}^h(t)\right|^2 dt \le C\epsilon \sum_{|k|\ge N} |b_k|^2. \tag{4.80}$$

Now, recall (4.79) so that one has $\widetilde{U}^p(t) = \widetilde{U}(t) - \widetilde{U}^h(t)$. Using the triangle inequality, we get

$$\begin{aligned}\int_1^T \left|\widetilde{U}^p(t)\right|^2 dt &\le C \int_1^T \left|\widetilde{U}(t)\right|^2 dt + C \int_1^T \left|\widetilde{U}^h(t)\right|^2 dt \\ &\le C \int_0^T |U(t)|^2\, dt + C\epsilon \sum_{|k|\ge N} |b_k|^2.\end{aligned} \tag{4.81}$$

Let be $0 < \tau < T$. Applying the parabolic Ingham inequality (4.74) to the quantity $\widetilde{U}^p(t)$ (given by (4.78a)), we obtain

$$\begin{aligned}\int_{T-\tau}^T \left|\widetilde{U}^p(t)\right|^2 dt = \int_0^\tau \left|\widetilde{U}^p(T-t)\right|^2 dt &\ge C \sum_{k\ge N} |a_k|^2 \, |1 - e^{\tilde{\lambda}_k}|^2 e^{2\mathrm{Re}(\tilde{\lambda}_k)\tau} \\ &\ge C \sum_{k\ge N} |a_k|^2\, e^{2\mathrm{Re}(\tilde{\lambda}_k)\tau},\end{aligned}$$

thanks to the properties of $\tilde{\lambda}_k$. Note that the above constant C depends on τ. Let us now choose $\tau > 0$ small enough such that $T - \tau > 1$. Thus, we get

$$\int_1^T \left|\widetilde{U}^p(t)\right|^2 dt \ge \int_{T-\tau}^T \left|\widetilde{U}^p(t)\right|^2 dt \ge C \sum_{k\ge N} |a_k|^2\, e^{2\mathrm{Re}(\tilde{\lambda}_k)\tau}. \tag{4.82}$$

Recall the function $U^p(t)$ given by (4.76), we deduce that

$$\begin{aligned}\int_0^{T-\tau} \left|U^p(t)\right|^2 dt &\le \sum_{k\ge N} |a_k|^2 \int_0^{T-\tau} e^{2\mathrm{Re}(\tilde{\lambda}_k)(T-t)} dt \\ &\le \sum_{k\ge N} |a_k|^2 \left|\frac{e^{\mathrm{Re}(\tilde{\lambda}_k)\tau} - e^{2\mathrm{Re}(\tilde{\lambda}_k)T}}{2\mathrm{Re}(\tilde{\lambda}_k)}\right| \\ &\le C \sum_{k\ge N} |a_k|^2\, e^{2\mathrm{Re}(\tilde{\lambda}_k)\tau},\end{aligned} \tag{4.83}$$

thanks to fact that $\left|\mathrm{Re}(\tilde{\lambda}_k)\right|^2 \ge C$ for $k \ge N$ large enough (combining the hypothesis (ii) and (iv) in Proposition 4.1.2 satisfied by $\{\lambda_k\}_{k\in\mathbb{N}^*}$).

Now, using the facts (4.82) and (4.81) in (4.83), we have

$$\int_0^{T-\tau} \left|U^p(t)\right|^2 dt \leq C\left(\int_0^T |U(t)|^2\, dt + \epsilon \sum_{|k|\geq N} |b_k|^2\right). \tag{4.84}$$

Since $T-\tau > 1$, applying the hyperbolic Ingham inequality (4.75) to $U^h(t)$ and then following a triangle inequality, we have

$$\begin{aligned}\sum_{|k|\geq N} |b_k|^2 \leq C\int_0^{T-\tau} \left|U^h(t)\right|^2 dt &\leq C\left(\int_0^{T-\tau} |U(t)|^2\, dt + \int_0^{T-\tau} \left|U^p(t)\right|^2 dt\right)\\ &\leq C\left(\int_0^T |U(t)|^2\, dt + \epsilon \sum_{|k|\geq N} |b_k|^2\right),\end{aligned}$$

thanks to the estimate (4.84).

Now, fix $\epsilon > 0$ small enough such that $1 - C\epsilon > 0$. As a consequence, there is some constant $C > 0$ depending only on T such that, we have

$$\sum_{|k|\geq N} |b_k|^2\, dt \leq C\int_0^T |U(t)|^2\, dt. \tag{4.85}$$

On the other hand, using the parabolic Ingham inequality to $U^p(t)$, followed by a triangle inequality, hyperbolic Ingham inequality to $U_h(t)$ and the result (4.85), we obtain

$$\begin{aligned}\sum_{k\geq N} |a_k|^2\, e^{2\mathrm{Re}(\tilde\lambda_k)T} \leq C\int_0^T \left|U^p(t)\right|^2 dt &\leq C\left(\int_0^T |U(t)|^2\, dt + \int_0^T \left|U^h(t)\right|^2 dt\right)\\ &\leq C\left(\int_0^T |U(t)|^2\, dt + \sum_{|k|\geq N} |b_k|^2\, dt\right)\\ &\leq C\int_0^T |U(t)|^2\, dt.\end{aligned}$$

Thus, eventually we have

$$\sum_{k\geq N} |a_k|^2\, e^{2\mathrm{Re}(\tilde\lambda_k)T} + \sum_{|k|\geq N} |b_k|^2 \leq C\int_0^T |U(t)|^2\, dt. \tag{4.86}$$

Recall that $\tilde\lambda_k = \lambda_k - \beta$, $\tilde\gamma_k = \gamma_k - \beta$, and that

$$\begin{aligned}\int_0^T |U(t)|^2\, dt &= \int_0^T \left|\sum_{k\geq N} a_k e^{(\lambda_k-\beta)(T-t)} + \sum_{|k|\geq N} b_k e^{(\gamma_k-\beta)(T-t)}\right|^2 dt \\ &\leq C\int_0^T \left|\sum_{k\geq N} a_k e^{\lambda_k(T-t)} + \sum_{|k|\geq N} b_k e^{\gamma_k(T-t)}\right|^2 dt.\end{aligned} \tag{4.87}$$

Moreover, it is easy to see that

$$e^{2\mathrm{Re}(\tilde\lambda_k)T} = e^{2\mathrm{Re}(\lambda_k)T - 2\mathrm{Re}(\beta)T} \geq Ce^{2\mathrm{Re}(\lambda_k)T}$$

for some $C > 0$ and thus combining (4.86) and (4.87), we obtain

$$\sum_{k\geq N} |a_k|^2\, e^{2\mathrm{Re}(\lambda_k)T} + \sum_{|k|\geq N} |b_k|^2 \leq C\int_0^T \left|\sum_{k\geq N} a_k e^{\lambda_k(T-t)} + \sum_{|k|\geq N} b_k e^{\gamma_k(T-t)}\right|^2 dt.$$

Finally, adding the finitely many terms in the above summation using a similar idea as in [MZ04, Theorem 4.3, Chapter 4] (since $\{\gamma_k\}_{k\in\mathbb{Z}}$ and $\{\lambda_k\}_{k\in\mathbb{N}^*}$ are disjoint), we can conclude that

$$\sum_{k\in\mathbb{N}^*} |a_k|^2 e^{2\mathrm{Re}(\lambda_k)T} + \sum_{k\in\mathbb{Z}} |b_k|^2 \leq C \int_0^T \left| \sum_{k\in\mathbb{N}^*} a_k e^{\lambda_k(T-t)} + \sum_{k\in\mathbb{Z}} b_k e^{\gamma_k(T-t)} \right|^2 dt. \tag{4.88}$$

This completes the proof. □

Remark 4.5.1. *Note that, in the proof we have only used the individual (parabolic and hyperbolic) Ingham inequalities. Thus, the hypotheses on the sequences $(\lambda_k)_{k\in\mathbb{N}^*}$ and $(\gamma_k)_{k\in\mathbb{Z}}$ can be relaxed so that each of the inequalities (4.74) and (4.75) holds. In this context, we refer to the works [FCGBdT10, LZ02, LdT13] for a proof of the parabolic Ingham's inequality (4.74) under different hypotheses on the sequence $(\lambda_k)_{k\in\mathbb{N}^*}$.*

4.6 Null-controllability for the velocity case

In this section, we prove the null-controllability of the system (4.4) (that is, Theorem 4.1.1) by establishing a proper observability inequality. The parabolic-hyperbolic joint Ingham-type inequality as obtained in Section 4.5, is the main ingredient to conclude this result.

Let (ρ, u) be the solution to the system (4.4) with a boundary control q acting on the velocity part. The following lemma gives an equivalent criterion for the null-controllability of the concerned model (4.4).

Lemma 4.6.1. *The system (4.4) is null-controllable at time $T > 0$ in $\dot{H}^{\frac{1}{2}}_{\sharp}(0,1) \times L^2(0,1)$ if and only if there exists a control $q \in L^2(0,T)$ such that*

$$\left\langle \begin{pmatrix} \sigma(0) \\ v(0) \end{pmatrix}, \begin{pmatrix} \rho_0 \\ u_0 \end{pmatrix} \right\rangle_{(\dot{H}^{\frac{1}{2}}_{\sharp})' \times L^2, \dot{H}^{\frac{1}{2}}_{\sharp} \times L^2} = \int_0^T \left(\overline{c\sigma(t,1)} + \overline{v_x(t,1)} \right) q(t) dt, \tag{4.89}$$

where (σ, v) is the solution to the adjoint system (4.14) with $(f,g) = (0,0)$ and any given final data $(\sigma_T, v_T) \in D(A^*)$.

With this result, we can now write the observability inequality that is required to prove null controllability of the system (4.4). Recall the observation operator $\mathcal{B}_u^*$ defined by (4.37)–(4.38).

Theorem 4.6.1. *The system (4.4) is null-controllable at time $T > 0$ in the space $\dot{H}^{\frac{1}{2}}_{\sharp}(0,1) \times L^2(0,1)$ if and only if the following observability inequality*

$$\int_0^T \left| \mathcal{B}_u^*(\sigma(t), v(t)) \right|^2 dt \geq C \, \|(\sigma(0), v(0))\|^2_{(\dot{H}^{\frac{1}{2}}_{\sharp}(0,1))' \times L^2(0,1)} \tag{4.90}$$

hold for every $(\sigma_T, v_T) \in D(A^*)$ and $(f,g) = (0,0)$.

Proof. We only proof the null controllability by assuming the observability inequality (4.90), and for the other part we refer to the article [MZ04]. To prove null controllability of the system (4.4), it is enough to prove the existence of a minimizer of certain quadratic functional, see for instance [MZ04, Zua07]. For this, we define the following set

$$\mathcal{H} := \Big\{ (\sigma_T, v_T) \in (\dot{H}^{\frac{1}{2}}_{\sharp}(0,1))' \times L^2(0,1) \ : \ \text{the solution } (\sigma, v) \text{ of (4.15) with } (f,g) = (0,0)$$
$$\text{satisfies } \int_0^T \left| \mathcal{B}_u^*(\sigma(t), v(t)) \right|^2 dt < \infty \Big\}$$

and define a quadratic functional $J_u : \mathcal{H} \to \mathbb{R}$ by

$$J_u(\sigma_T, v_T) := \frac{1}{2} \int_0^T \left| \mathcal{B}_u^*(\sigma(t), v(t)) \right|^2 dt + \left\langle \begin{pmatrix} \sigma(0) \\ v(0) \end{pmatrix}, \begin{pmatrix} \rho_0 \\ u_0 \end{pmatrix} \right\rangle_{(\dot{H}^{\frac{1}{2}}_{\sharp})' \times L^2, \dot{H}^{\frac{1}{2}}_{\sharp} \times L^2}, \quad (\sigma_T, v_T) \in \mathcal{H}. \tag{4.91}$$

The map J_u may not be coercive in H with respect to the usual $(\dot{H}^{\frac{1}{2}}_{\sharp})' \times L^2$-norm. Thus, we define a new norm on $\mathcal{H}$ by

$$\|(\sigma_T, v_T)\|_{\mathcal{H}} := \left(\int_0^T |\mathcal{B}^*(\sigma(t), v(t))|^2 \, dt\right)^{\frac{1}{2}}.$$

Indeed, if $\|(\sigma_T, v_T)\|_{\mathcal{H}} = 0$ then $\mathcal{B}^*_u(\sigma(t), v(t)) = 0$ for all $t \in (0,T)$. The observability inequality (4.90) is then yields $(\sigma(0), v(0)) = (0,0)$ and as a consequence of the backward uniqueness property of the adjoint system (4.15) (see Section 4.9), it follows that $(\sigma, v) \equiv (0,0)$.

With this new norm on $\mathcal{H}$, the operator J_u is continuous and coercive in $\mathcal{H}$. Thus, it has a unique minimizer $(\hat{\sigma}_T, \hat{v}_T) \in \mathcal{H}$. Let $(\hat{\sigma}, \hat{v})$ be the solution of (4.15) with respect to this terminal data $(\hat{\sigma}_T, \hat{v}_T)$. Then the function $q = \mathcal{B}^*_u(\hat{\sigma}, \hat{v}) \in L^2(0,T)$ will be a null control of the system (4.4). □

We are now ready to prove our first main result, i.e., Theorem 4.1.1 of our work.

Proof of Theorem 4.1.1. We prove each part separately.

Null-controllability in $\dot{H}^{\frac{1}{2}}_{\sharp}(0,1) \times L^2(0,1)$. Recall that the set of (generalized) eigenfunctions

$$\left\{\Phi_{\lambda_k^p},\, k \geq k_0\right\} \cup \left\{k^{\frac{1}{2}} \Phi_{\lambda_k^h},\, |k| \geq k_0\right\} \cup \left\{\Phi^i_\lambda;\, \lambda \in \Lambda_0,\, i = 0, \ldots, m_\lambda - 1\right\}$$

forms a Riesz basis in $(\dot{H}^{\frac{1}{2}}_{\sharp}(0,1))' \times L^2(0,1)$, due to Proposition 4.3.2 and Corollary 4.3.1, and thus one can consider any given final data $(\sigma_T, v_T) \in (\dot{H}^{\frac{1}{2}}_{\sharp}(0,1))' \times L^2(0,1)$ as follows:

$$(\sigma_T, v_T) = \sum_{k \geq k_0} a_k \Phi_{\lambda_k^p} + \sum_{|k| \geq k_0} b_k k^{\frac{1}{2}} \Phi_{\lambda_k^h} + \sum_{\lambda \in \Lambda_0} \sum_{j=0}^{m_\lambda - 1} c_{\lambda,j} \Phi^j_\lambda, \tag{4.92}$$

where $\sum_{k \geq k_0} |a_k|^2 + \sum_{|k| \geq k_0} |b_k|^2 < +\infty$, and $c_{\lambda,j}$ for $\lambda \in \Lambda_0$ and $j \in \{0, \cdots, m_\lambda - 1\}$ are constants.

Therefore, the solution to the adjoint system (4.14) with the above terminal data and $(f,g) = (0,0)$ can be written as

$$(\sigma(t), v(t)) = \sum_{k \geq k_0} a_k e^{\lambda_k^p (T-t)} \Phi_{\lambda_k^p} + \sum_{|k| \geq k_0} b_k k^{\frac{1}{2}} e^{\lambda_k^h (T-t)} \Phi_{\lambda_k^h} + \sum_{\lambda \in \Lambda_0} \sum_{j=0}^{m_\lambda - 1} c_{\lambda,j} (T-t)^j e^{\lambda(T-t)} \Phi^j_\lambda, \tag{4.93}$$

for $t \in [0,T]$. Now, we find that

$$\begin{aligned}
\mathcal{B}^*_u(\sigma(t), v(t)) &= c\sigma(t,1) + v_x(t,1) \\
&= \sum_{k \geq k_0} a_k \, e^{\lambda_k^p (T-t)} \mathcal{B}^*_u \Phi_{\lambda_k^p} + \sum_{|k| \geq k_0} b_k k^{\frac{1}{2}} \, e^{\lambda_k^h (T-t)} \mathcal{B}^*_u \Phi_{\lambda_k^h} + \sum_{\lambda \in \Lambda_0} \sum_{j=0}^{m_\lambda - 1} c_{\lambda,j} (T-t)^j e^{\lambda(T-t)} \mathcal{B}^*_u \Phi^j_\lambda,
\end{aligned}$$

for $t \in (0,T)$. At this point, we may assume that

$$\mathcal{B}^*_u \Phi^j_\lambda \neq 0, \quad \forall \lambda \in \Lambda_0, \; j = 1, \cdots, m_\lambda - 1,$$

which can be ensured as one can add any multiple of the eigenfunction to each (finitely many) generalized eigenfunction and adjust accordingly.

We start with $T > 1$. Then, in one hand, using the Ingham-type inequality (4.13) for $|k| \geq k_0$, we have

$$\begin{aligned}
&\int_0^T \left| \sum_{k \geq k_0} a_k \, e^{\lambda_k^p (T-t)} \mathcal{B}^*_u \Phi_{\lambda_k^p} + \sum_{|k| \geq k_0} b_k k^{\frac{1}{2}} \, e^{\lambda_k^h (T-t)} \mathcal{B}^*_u \Phi_{\lambda_k^h} \right|^2 dt \\
&\geq C_1 \left(\sum_{k \geq k_0} \left| a_k \mathcal{B}^*_u \Phi_{\lambda_k^p} \right|^2 e^{2\mathrm{Re}(\lambda_k^p) T} + \sum_{|k| \geq k_0} \left| b_k k^{\frac{1}{2}} \mathcal{B}^*_u \Phi_{\lambda_k^h} \right|^2 \right) \\
&\geq C_1 \left(\sum_{k \geq k_0} |a_k|^2 k^2 e^{2\mathrm{Re}(\lambda_k^p) T} + \sum_{|k| \geq k_0} |b_k|^2 \right),
\end{aligned} \tag{4.94}$$

for some $C_1 > 0$, where we have also used the observation estimates given by Lemma 4.4.3.

On the other hand, thanks to the Riesz basis property (see Corollary 4.3.1), we have

$$\left\| \sum_{k\geq k_0} a_k e^{\lambda_k^p T} \Phi_{\lambda_k^p} + \sum_{|k|\geq k_0} b_k k^{\frac{1}{2}} e^{\lambda_k^h T} \Phi_{\lambda_k^h} \right\|^2_{(\dot{H}^{\frac{1}{2}}_\sharp)'\times L^2} \leq C_2 \Bigg(\sum_{k\geq k_0} |a_k|^2 e^{2\mathrm{Re}(\lambda_k^p)T} + \sum_{|k|\geq k_0} |b_k|^2 e^{2\mathrm{Re}(\lambda_k^h)T} \Bigg),$$

for some $C_2 > 0$. Thus, we deduce that

$$\begin{aligned} \int_0^T \left| \sum_{k\geq k_0} a_k\, e^{\lambda_k^p (T-t)} \mathcal{B}_u^* \Phi_{\lambda_k^p} + \sum_{|k|\geq k_0} b_k k^{\frac{1}{2}}\, e^{\lambda_k^h (T-t)} \mathcal{B}_u^* \Phi_{\lambda_k^h} \right|^2 dt \\ \geq C \left\| \sum_{k\geq k_0} a_k e^{\lambda_k^p T} \Phi_{\lambda_k^p} + \sum_{|k|\geq k_0} b_k k^{\frac{1}{2}} e^{\lambda_k^h T} \Phi_{\lambda_k^h} \right\|^2_{(\dot{H}^{\frac{1}{2}}_\sharp)'\times L^2} \end{aligned} \tag{4.95}$$

for some $C > 0$. But the solution (σ, v) also contains some finitely many terms as written in (4.93). Thus, to conclude the required observability inequality (4.90), we need to consider those finite number of terms in the inequality (4.95). Indeed, this can be done by using the strategy developed in [KL05] and [CMRR14, Section 4.2] since all the observation terms $\mathcal{B}_u^*\Phi \neq 0$ for any (generalized) eigenfunction Φ of A^* as long as we consider $c \notin \mathcal{N}$ with $c^4 + 8c^2 + 5 < 4\pi^2$ (see Proposition 4.4.1– Part 2). However, we give a detailed proof here for the sake of completeness.

Let $(\sigma_T, v_T) \in (\dot{H}^{\frac{1}{2}}_\sharp(0,1))' \times L^2(0,1)$ be given. We write $(\sigma_T, v_T) = (\sigma_{T,1}, v_{T,1}) + (\sigma_{T,2}, v_{T,2})$ with

$$(\sigma_{T,1}, v_{T,1}) = \sum_{\lambda\in\Lambda_0} \sum_{j=0}^{m_\lambda -1} c_{\lambda,j} \Phi_\lambda^j, \text{ and } (\sigma_{T,2}, v_{T,2}) = \sum_{k\geq k_0} a_k \Phi_{\lambda_k^p} + \sum_{|k|\geq k_0} b_k k^{\frac{1}{2}} \Phi_{\lambda_k^h}.$$

The corresponding solutions of the adjoint system (4.15) with these $(\sigma_{T,1}, v_{T,1})$ and $(\sigma_{T,2}, v_{T,2})$ are respectively

$$\begin{aligned} (\sigma_1(t), v_1(t)) &= \sum_{\lambda\in\Lambda_0} \sum_{j=0}^{m_\lambda -1} c_{\lambda,j} e^{\lambda(T-t)} (T-t)^j \Phi_\lambda^j, \\ (\sigma_2(t), v_2(t)) &= \sum_{k\geq k_0} a_k e^{\lambda_k^p (T-t)} \Phi_{\lambda_k^p} + \sum_{|k|\geq k_0} b_k k^{\frac{1}{2}} e^{\lambda_k^h (T-t)} \Phi_{\lambda_k^h}. \end{aligned}$$

From the previous computations (the case of high frequencies), we have the following inequality

$$\int_0^T \left| \mathcal{B}_u^* (\sigma_2(t), v_2(t)) \right|^2 dt \geq C \left\| (\sigma_2(0), v_2(0)) \right\|^2_{(\dot{H}^{\frac{1}{2}}_\sharp)'\times L^2}. \tag{4.96}$$

To prove the observability inequality (4.90), we have to include the observation term $\mathcal{B}_u^*(\sigma_1(t), v_1(t)) = \sum_{\lambda\in\Lambda_0} \sum_{j=0}^{m_\lambda -1} c_{\lambda,j} e^{\lambda(T-t)} (T-t)^j \mathcal{B}_u^* \Phi_\lambda^j$ in the above inequality. We give a detailed proof below by adding only one term, say for instance $e^{\lambda_{j_0}(T-t)} \left(c_{j_0} \mathcal{B}_u^* \Phi_{j_0} + (T-t) \tilde{c}_{j_0} \mathcal{B}_u^* \tilde{\Phi}_{j_0} \right)$ corresponding to the eigenvalue $\lambda = \lambda_{j_0} \in \Lambda_0$, where Φ_{j_0} and $\tilde{\Phi}_{j_0}$ denote the (generalized) eigenfunctions corresponding to λ_{j_0}. All the remaining finitely many terms can be added one by one using the same argument. We denote

$$\mathcal{F}(t) := \mathcal{B}_u^*(\sigma_2(t), v_2(t)) + e^{\lambda_{j_0}(T-t)} \left(c_{j_0} \mathcal{B}_u^* \Phi_{j_0} + (T-t) \tilde{c}_{j_0} \mathcal{B}_u^* \tilde{\Phi}_{j_0} \right), \quad \text{for } t \in (0,T), \tag{4.97}$$

and define

$$\mathcal{G}(t) := \mathcal{F}(t) - \frac{1}{2\delta} \int_{-\delta}^{\delta} e^{\lambda_{j_0} s} \mathcal{F}(t+s) ds, \quad t \in (\delta, T-\delta),$$

where we will choose $\delta > 0$ later accordingly. Then, one can obtain the following estimate (see for instance [KL05, Section 4.4]):

$$\int_\delta^{T-\delta} |\mathcal{G}(t)|^2 dt \leq C \int_0^T |\mathcal{F}(t)|^2 dt \tag{4.98}$$

for some constant $C > 0$.

On the other hand, we have

$$\mathcal{G}(t) = \sum_{k \geq k_0} a_k e^{\lambda_k^p (T-t)} \mathcal{B}_u^* \Phi_{\lambda_k^p} \left(1 - \frac{\sinh((\lambda_k^p - \lambda_{j_0})\delta)}{(\lambda_k^p - \lambda_{j_0})\delta}\right) + \sum_{|k| \geq k_0} b_k k^{\frac{1}{2}} e^{\lambda_k^h (T-t)} \mathcal{B}_u^* \Phi_{\lambda_k^h} \left(1 - \frac{\sinh((\lambda_k^h - \lambda_{j_0})\delta)}{(\lambda_k^h - \lambda_{j_0})\delta}\right)$$

for $t \in (\delta, T-\delta)$. Since $T > 1$, choosing $\delta > 0$ small enough so that $T - 2\delta > 1$, we obtain by using the Ingham-type inequality (4.13)

$$\int_\delta^{T-\delta} |\mathcal{G}(t)|^2 \, dt \geq C \left(\sum_{k \geq k_0} \left| a_k \mathcal{B}_u^* \Phi_{\lambda_k^p} \right|^2 e^{2\mathrm{Re}(\lambda_k^p) T} + \sum_{|k| \geq k_0} \left| b_k k^{\frac{1}{2}} \mathcal{B}_u^* \Phi_{\lambda_k^h} \right|^2 \right).$$

This can be ensured from the fact that $\inf_{k \geq k_0} \left|\lambda_k^p - \lambda_{j_0}\right|, \inf_{k \geq k_0} \left|\lambda_k^h - \lambda_{j_0}\right| > 0$, which then gives (by taking $\delta > 0$ suitably) that

$$\inf_{k \geq k_0} \left| 1 - \frac{\sinh((\lambda_k^p - \lambda_{j_0})\delta)}{(\lambda_k^p - \lambda_{j_0})\delta} \right|, \ \inf_{k \geq k_0} \left| 1 - \frac{\sinh((\lambda_k^h - \lambda_{j_0})\delta)}{(\lambda_k^h - \lambda_{j_0})\delta} \right| > 0.$$

Using this inequality, we readily have (see eq. (4.94)-(4.95))

$$\int_\delta^{T-\delta} |\mathcal{G}(t)|^2 \, dt \geq C \, \|(\sigma_2(0), v_2(0))\|^2_{(\dot{H}_\sharp^{\frac{1}{2}})' \times L^2}.$$

Combining this with the estimate (4.98), we deduce that

$$\int_0^T |\mathcal{F}(t)|^2 \, dt \geq C \, \|(\sigma_2(0), v_2(0))\|^2_{(\dot{H}_\sharp^{\frac{1}{2}})' \times L^2}. \tag{4.99}$$

Since $T > 1$, we can choose $\varepsilon > 0$ small enough so that $T - \varepsilon > 1$. Then, we obtain from the above inequality

$$\int_0^T |\mathcal{F}(t)|^2 \, dt \geq \int_\varepsilon^T |\mathcal{F}(t)|^2 \, dt \geq C \, \|(\sigma_2(\varepsilon), v_2(\varepsilon))\|^2_{(\dot{H}_\sharp^{\frac{1}{2}})' \times L^2}. \tag{4.100}$$

We now prove a weak admissibility inequality

$$\int_0^{\frac{\varepsilon}{2}} \left| \mathcal{B}_u^* (\sigma_2(t), v_2(t)) \right|^2 dt \leq C \, \|(\sigma_2(\varepsilon), v_2(\varepsilon))\|^2_{(\dot{H}_\sharp^{\frac{1}{2}})' \times L^2}. \tag{4.101}$$

In fact, applying Hölder's inequality and the hyperbolic Ingham inequality (4.75) (right side), we deduce that

$$\begin{aligned}
\int_0^{\frac{\varepsilon}{2}} \left| \mathcal{B}_u^* (\sigma_2(t), v_2(t)) \right|^2 dt &\leq 2 \int_0^{\frac{\varepsilon}{2}} \left| \sum_{k \geq k_0} a_k e^{\lambda_k^p (T-t)} \mathcal{B}_u^* \Phi_{\lambda_k^p} \right|^2 dt + 2 \int_0^{\frac{\varepsilon}{2}} \left| \sum_{|k| \geq k_0} b_k k^{\frac{1}{2}} e^{\lambda_k^h (T-t)} \mathcal{B}_u^* \Phi_{\lambda_k^h} \right|^2 dt \\
&\leq C \sum_{k \geq k_0} |a_k|^2 e^{2\mathrm{Re}(\lambda_k^p)(T-\varepsilon)} \sum_{k \geq k_0} \left| \mathcal{B}_u^* \Phi_{\lambda_k^p} \right|^2 e^{-2\mathrm{Re}(\lambda_k^p)(T-\varepsilon)} \int_0^{\frac{\varepsilon}{2}} e^{2\mathrm{Re}(\lambda_k^p)(T-t)} dt \\
&\quad + C \sum_{|k| \geq k_0} \left| b_k k^{\frac{1}{2}} \mathcal{B}_u^* \Phi_{\lambda_k^h} \right|^2 \\
&\leq C \sum_{k \geq k_0} |a_k|^2 e^{2\mathrm{Re}(\lambda_k^p)(T-\varepsilon)} + C \sum_{|k| \geq k_0} |b_k|^2,
\end{aligned}$$

thanks to the observation estimate (4.73b). On the other hand, using the Riesz basis property of the eigenfunctions (see Corollary 4.3.1), we obtain

$$\|(\sigma_2(\varepsilon), v_2(\varepsilon))\|^2_{(\dot{H}^{\frac{1}{2}}_{\sharp})'\times L^2} \geq C \sum_{k\geq k_0} |a_k|^2 e^{2\mathrm{Re}(\lambda^p_k)(T-\varepsilon)} + C \sum_{|k|\geq k_0} |b_k|^2 .$$

Combining the above estimates, the weak admissibility inequality (4.101) follows. With this, we get from (4.100) that

$$\int_0^T |\mathcal{F}(t)|^2 \, dt \geq C \int_0^{\frac{\varepsilon}{2}} \left|\mathcal{B}^*_u(\sigma_2(t), v_2(t))\right|^2 dt. \tag{4.102}$$

We now introduce the finite dimensional space generated by the (generalized) eigenfunctions

$$\mathcal{X} := \mathrm{span}\left\{\Phi_{j_0}, \tilde{\Phi}_{j_0}\right\}$$

and define the norms on $\mathcal{X}$ as

$$\left\|(\hat{\sigma}_{T,1}, \hat{v}_{T,1})\right\|^2_1 : = \int_0^{\frac{\varepsilon}{2}} \left|e^{\lambda_{j_0}(T-t)} \left(c_{j_0}\mathcal{B}^*_u\Phi_{j_0} + (T-t)\tilde{c}_{j_0}\mathcal{B}^*_u\tilde{\Phi}_{j_0}\right)\right|^2 dt, \tag{4.103}$$

$$\left\|(\hat{\sigma}_{T,1}, \hat{v}_{T,1})\right\|_2 : = \|(\hat{\sigma}_1(0), \hat{v}_1(0))\|_{(\dot{H}^{\frac{1}{2}}_{\sharp})'\times L^2}, \tag{4.104}$$

where $(\hat{\sigma}_1(t), \hat{v}_1(t)) = e^{\lambda_{j_0}(T-t)}\left(c_{j_0}\Phi_{j_0} + \tilde{c}_{j_0}\tilde{\Phi}_{j_0}\right)$ for $t \in (0,T)$ is the solution of the adjoint system (4.15) with the terminal data $(\hat{\sigma}_{T,1}, \hat{v}_{T,1}) \in \mathcal{X}$ and $(f, g) = (0,0)$. In fact, the norms (4.103) and (4.104) are well-defined since we have $\mathcal{B}^*\Phi_{j_0}, \mathcal{B}^*\tilde{\Phi}_{j_0} \neq 0$ and $(\hat{\sigma}_1(0), \hat{v}_1(0)) = (0,0)$ implies $\Phi_{j_0} = \tilde{\Phi}_{j_0} = 0$. Moreover, as any two norms in a finite dimensional space are equivalent, we deduce that

$$\int_0^{\frac{\varepsilon}{2}} \left|e^{\lambda_{j_0}(T-t)} \left(c_{j_0}\mathcal{B}^*_u\Phi_{j_0} + (T-t)\tilde{c}_{j_0}\mathcal{B}^*_u\tilde{\Phi}_{j_0}\right)\right|^2 dt \geq C \|(\hat{\sigma}_1(0), \hat{v}_1(0))\|^2_{(\dot{H}^{\frac{1}{2}}_{\sharp})'\times L^2}.$$

As a consequence, we obtain (recall the function $\mathcal{F}$ defined by (4.97))

$$\|(\hat{\sigma}_1(0), \hat{v}_1(0))\|^2_{(\dot{H}^{\frac{1}{2}}_{\sharp})'\times L^2} \leq C \int_0^{\frac{\varepsilon}{2}} |\mathcal{F}(t)|^2 \, dt + C \int_0^{\frac{\varepsilon}{2}} \left|\mathcal{B}^*_u(\sigma_2(t), v_2(t))\right|^2 dt \leq C \int_0^T |\mathcal{F}(t)|^2 \, dt,$$

thanks to the lower bound (4.102). This inequality together with (4.99), we deduce that

$$\int_0^T |\mathcal{F}(t)|^2 \, dt \geq C \|(\sigma(0) + \hat{\sigma}_1(0), v(0) + \hat{v}_1(0))\|^2_{(\dot{H}^{\frac{1}{2}}_{\sharp})'\times L^2}. \tag{4.105}$$

In a similar way, we can add the remaining finitely many terms in the above inequality. As a result, we eventually get for $T > 1$,

$$\int_0^T |\mathcal{B}^*_u(\sigma(t), v(t))|^2 dt \geq C\|(\sigma(0), v(0))\|^2_{(\dot{H}^{\frac{1}{2}}_{\sharp})'\times L^2}, \tag{4.106}$$

for given data $(\sigma_T, v_T) \in D(A^*)$.

This is a necessary and sufficient for the null-controllability of system (4.4) with given initial data $(\rho_0, u_0) \in \dot{H}^{\frac{1}{2}}_{\sharp}(0,1) \times L^2(0,1)$, when $T > 1$, which proves the first part of Theorem 4.1.1.

Lack of null-controllability for less regular initial states. Consider $(\sigma_{T,k}, v_{T,k}) = \Phi_{\lambda^h_k}$ for $|k| \geq k_0$. Then, the solution to the adjoint system (4.15) reads as

$$(\sigma_k(t,x), v_k(t,x)) = e^{\lambda^h_k(T-t)}\Phi_{\lambda^h_k}(x), \quad \forall |k| \geq k_0, \quad (t,x) \in (0,T)\times(0,1).$$

Now, in one hand we have

$$\left\|\Phi_{\lambda^h_k}\right\|_{(\dot{H}^s_{\sharp})'\times L^2} \geq \frac{C}{|k|^s}, \quad \forall |k| \geq k_0,$$

by Lemma 4.3.2–eq. (4.32), and thus

$$\|(\sigma_k(0), v_k(0))\|^2_{(\dot{H}^s_\sharp)'\times L^2} \geq \frac{C}{|k|^{2s}}, \quad \forall |k| \geq k_0.$$

for all $k \in \mathbb{Z}^*$, since the real part of λ^h_k is bounded. On the other hand, we have the following upper bounds of the observation terms, namely

$$\int_0^T \left|\mathcal{B}^*_u(\sigma_k(t), v_k(t))\right|^2 dt \leq \frac{C}{|k|}, \quad \forall |k| \geq k_0,$$

in view of Lemma 4.4.3–eq. (4.73b). Thus, if the observability inequality (4.106) holds, we would have

$$\frac{C}{|k|^{2s}} \leq \frac{C}{|k|} \implies |k|^{1-2s} \leq C,$$

which is not possible since $0 \leq s < \frac{1}{2}$. Therefore, the system (4.4) is not null-controllable at any time T whenever $0 < s < \frac{1}{2}$.

This concludes the proof of Theorem 4.1.1. □

4.7 Null-controllability for the density case

This section is devoted to prove the null-controllability of the system (4.5), more precisely Theorem 4.1.2. The proof is made of two steps:

- First, we use the Ingham-type inequality (4.13) (introduced as before) to show the null-controllability of (4.5) in the space $\dot{L}^2(0,1) \times H^1_0(0,1)$.
- Secondly, by developing the moments method for parabolic-hyperbolic coupled system (due to [Han94]), we prove that the same system (4.5) is null-controllable in the space $\dot{H}^s_\sharp(0,1) \times L^2(0,1)$ for any $s > \frac{1}{2}$.

As a consequence, we conclude the null-controllability of our system (4.6) in the space $\dot{L}^2(0,1)\times L^2(0,1)$.

Before proceeding, we first write the following lemma, which gives an equivalent criterion for the null-controllability of system (4.5).

Lemma 4.7.1. *Let $s_1, s_2 \geq 0$ be given. The system (4.5) is null-controllable at time $T > 0$ in $\dot{H}^{s_1}_\sharp(0,1)\times H^{s_2}_0(0,1)$ if and only if there exists a control $p \in L^2(0,T)$ such that*

$$\left\langle \begin{pmatrix} \sigma(0) \\ v(0) \end{pmatrix}, \begin{pmatrix} \rho_0 \\ u_0 \end{pmatrix} \right\rangle_{(\dot{H}^{s_1}_\sharp)'\times H^{-s_2}, \dot{H}^{s_1}_\sharp \times H^{s_2}_0} = -\int_0^T \overline{\sigma(t,1)} p(t) dt, \tag{4.107}$$

where (σ, v) is the solution to the adjoint system (4.14) with $(f,g) = (0,0)$ and any given final data $(\sigma_T, v_T) \in D(A^*)$.

4.7.1 Null-controllability in $\dot{L}^2 \times H^1_0$: using Ingham-type inequality

We first write the following result, the proof of which is similar to the velocity case (Theorem 4.6.1) and so we omit the details here.

Theorem 4.7.1. *The system (4.5) is null-controllable at time $T > 0$ in the space $\dot{L}^2(0,1) \times H^1_0(0,1)$ if and only if the following observability inequality*

$$\int_0^T \left|\mathcal{B}^*_\rho(\sigma(t), v(t))\right|^2 dt \geq C \, \|(\sigma(0), v(0))\|^2_{\dot{L}^2 \times H^{-1}} \tag{4.108}$$

hold for every $(\sigma_T, v_T) \in D(A^*)$.

Let $(\sigma_T, v_T) \in \dot{L}^2(0,1) \times H^{-1}(0,1)$ be given. Since the set of (generalized) eigenfunctions

$$\left\{k\ \Phi_{\lambda_k^p},\ k \geq k_0\right\} \cup \left\{\Phi_{\lambda_k^h},\ |k| \geq k_0\right\} \cup \left\{\Phi_\lambda^i;\ \lambda \in \Lambda_0,\ i = 0, ..., m_\lambda - 1\right\}$$

forms a Riesz basis of $\dot{L}^2(0,1) \times H^{-1}(0,1)$, thanks to Corollary 4.3.1, we can write (σ_T, v_T) as

$$(\sigma_T, v_T) = \sum_{k \geq k_0} a_k k\ \Phi_{\lambda_k^p} + \sum_{|k| \geq k_0} b_k \Phi_{\lambda_k^h} + \sum_{\lambda \in \Lambda_0} \sum_{j=0}^{m_\lambda - 1} c_{\lambda,j} \Phi_\lambda^j.$$

Therefore, the solution to the adjoint system (4.14) with this terminal data (σ_T, v_T) and $(f, g) = (0,0)$, can be written as

$$(\sigma(t), v(t)) = \sum_{k \geq k_0} a_k k\ e^{\lambda_k^p (T-t)} \Phi_{\lambda_k^p} + \sum_{|k| \geq k_0} b_k e^{\lambda_k^h (T-t)} \Phi_{\lambda_k^h} + \sum_{\lambda \in \Lambda_0} \sum_{j=0}^{m_\lambda - 1} c_{\lambda,j} (T-t)^j e^{\lambda (T-t)} \Phi_\lambda^j,$$

for $t \in [0, T]$. Note that

$$\mathcal{B}_\rho^*(\sigma(t), v(t)) = \sum_{k \geq k_0} a_k k\ e^{\lambda_k^p (T-t)} \mathcal{B}_\rho^* \Phi_{\lambda_k^p} + \sum_{|k| \geq k_0} b_k e^{\lambda_k^h (T-t)} \mathcal{B}_\rho^* \Phi_{\lambda_k^h} + \sum_{\lambda \in \Lambda_0} \sum_{j=0}^{m_\lambda - 1} c_{\lambda,j} (T-t)^j e^{\lambda (T-t)} \mathcal{B}_\rho^* \Phi_\lambda^j,$$

for all $t \in (0, T)$. Since $T > 1$, we use the Ingham-type inequality (4.13) to obtain

$$\begin{aligned}
&\int_0^T \left| \sum_{k \geq k_0} a_k k\, e^{\lambda_k^p (T-t)} \mathcal{B}_\rho^* \Phi_{\lambda_k^p} + \sum_{|k| \geq k_0} b_k\, e^{\lambda_k^h (T-t)} \mathcal{B}_\rho^* \Phi_{\lambda_k^h} \right|^2 dt \\
&\geq C_1 \left(\sum_{k \geq k_0} \left| a_k k \mathcal{B}_\rho^* \Phi_{\lambda_k^p} \right|^2 e^{2\mathrm{Re}(\lambda_k^p) T} + \sum_{|k| \geq k_0} \left| b_k \mathcal{B}_\rho^* \Phi_{\lambda_k^h} \right|^2 \right) \\
&\geq C_1 \left(\sum_{k \geq k_0} |a_k|^2 e^{2\mathrm{Re}(\lambda_k^p) T} + \sum_{|k| \geq k_0} |b_k|^2 \right),
\end{aligned} \tag{4.109}$$

for some $C_1 > 0$, where we also have used the observation estimates from Lemma 4.4.2.

On the other hand, we have

$$\begin{aligned}
&\left\| \sum_{k \geq k_0} a_k k\ e^{\lambda_k^p T} \Phi_{\lambda_k^p} + \sum_{|k| \geq k_0} b_k e^{\lambda_k^h T} \Phi_{\lambda_k^h} \right\|_{\dot{L}^2 \times H^{-1}}^2 \\
&\leq C_2 \left(\sum_{k \geq k_0} |a_k|^2 e^{2\mathrm{Re}(\lambda_k^p) T} + \sum_{|k| \geq k_0} |b_k|^2 e^{2\mathrm{Re}(\lambda_k^h) T} \right),
\end{aligned}$$

for some $C_2 > 0$, thanks to the Riesz basis property (Corollary 4.3.1).

Thus we deduce that

$$\begin{aligned}
&\int_0^T \left| \sum_{k \geq k_0} a_k k\, e^{\lambda_k^p (T-t)} \mathcal{B}_\rho^* \Phi_{\lambda_k^p} + \sum_{|k| \geq k_0} b_k\, e^{\lambda_k^h (T-t)} \mathcal{B}_\rho^* \Phi_{\lambda_k^h} \right|^2 dt \\
&\geq C \left\| \sum_{k \geq k_0} a_k k\ e^{\lambda_k^p T} \Phi_{\lambda_k^p} + \sum_{|k| \geq k_0} b_k e^{\lambda_k^h T} \Phi_{\lambda_k^h} \right\|_{\dot{L}^2 \times H^{-1}}^2,
\end{aligned} \tag{4.110}$$

for some $C > 0$.

On the other hand, since $c^4 + 8c^2 + 5 < 4\pi^2$, all the observation terms $\mathcal{B}_\rho^* \Phi \neq 0$ for any (generalized) eigenfunction Φ of A^* and hence it is enough to consider only the large frequencies of eigenvalues. In fact, the lower frequencies can be added one by one by proceeding in a similar way as in the proof of Theorem 4.1.1 to deduce the required observability inequality

$$\int_0^T |\mathcal{B}_\rho^*(\sigma(t), v(t))|^2 dt \geq C \, \|(\sigma(0), v(0))\|_{\dot{L}^2 \times H^{-1}}^2, \tag{4.111}$$

for given data $(\sigma_T, v_T) \in D(A^*)$ provided $T > 1$.

This proves the null-controllability of the system (4.5) at time $T > 1$ for given initial data $(\rho_0, u_0) \in \dot{L}^2(0,1) \times H_0^1(0,1)$.

4.7.2 Null-controllability in $\dot{H}^s_\sharp \times L^2$, $s > \frac{1}{2}$ by moments method

To prove the null-controllability of system (4.5) at $T > 1$ in the space $\dot{H}^s_\sharp(0,1) \times L^2(0,1)$ for $s > \frac{1}{2}$, we shall formulate and solve a set of moments problem using the strategy developed in [Han94]. For the sake of completeness, we recall the main results from [Han94] and use these results with respect to our setting.

4.7.2.1 Parabolic-hyperbolic joint moments problem: results by S. W. Hansen

Let us first recall some important results by S. W. Hansen [Han94] which will be used to prove the required null-controllability result of the system (4.5) in the space $\dot{H}^s_\sharp(0,1) \times L^2(0,1)$ for $s > \frac{1}{2}$.

The author in [Han94] made the following assumptions in his work.

Hypothesis 4.7.2. *Let $\{\lambda_k\}_{k\in\mathbb{N}^*}$ and $\{\gamma_k\}_{k\in\mathbb{Z}}$ be two sequences in $\mathbb{C}$ with the following properties:*

(H1) for all $k, j \in \mathbb{Z}$, $\gamma_k \neq \gamma_j$ unless $j = k$,

(H2) $\gamma_k = \beta + bk\pi i + \nu_k$ for all $k \in \mathbb{Z}$,

where $\beta \in \mathbb{C}$, $b > 0$ and $\{\nu_k\}_{k\in\mathbb{Z}} \in \ell_2$.

Also, there exist positive constants $A_0, B_0, \delta, \epsilon$ and $0 \le \theta < \pi/2$ for which $\{\lambda_k\}_{k\in\mathbb{N}^*}$ satisfies

(P1) $|\arg(-\lambda_k)| \le \theta$ for all $k \in \mathbb{N}^*$,

(P2) $\left|\lambda_k - \lambda_j\right| \ge \delta \left|k^2 - j^2\right|$ for all $k \neq j$, $k, j \in \mathbb{N}^*$,

(P3) $\epsilon(A_0 + B_0 k^2) \le |\lambda_k| \le A_0 + B_0 k^2$ for all $k \in \mathbb{N}^*$.

We also assume that the families are disjoint, i.e.,

$$\{\gamma_k,\ k \in \mathbb{Z}\} \cap \{\lambda_k,\ k \in \mathbb{N}^*\} = \emptyset.$$

Then, he introduced the following spaces: for any $a < d$,

$$\begin{aligned} W_{[a,d]} &= \text{closed span } \{e^{\gamma_k t}\}_{k\in\mathbb{Z}} \text{ in } L^2(a,d), \\ E_{[a,d]} &= \text{closed span } \{e^{-\lambda_k t}\}_{k\in\mathbb{N}^*} \text{ in } L^2(a,d). \end{aligned}$$

With these, the author in [Han94] has proved the following results.

Theorem 4.7.3. *Assume that the Hypothesis 4.7.2 holds true. Then, for each $T > 2/b$, where b is defined as in Hypothesis 4.7.2, the spaces $W_{[0,T]}$ and $E_{[0,T]}$ are uniformly separated. This does not hold for $T \le 2/b$.*

The proof mainly relies upon the following lemma. Hereinafter, we denote $t_b = 2/b$.

Lemma 4.7.2. *For any $a \in \mathbb{R}$, $W_{[a,a+t_b]} = L^2(a, a + t_b)$. Furthermore, for $T \ge t_b$, $\{e^{\gamma_k t}\}_{k\in\mathbb{Z}}$ forms a Riesz basis for each of the spaces $W_{[a,a+T]}$.*

We refer to the work [Han94] for the proofs of Theorem 4.7.3 and Lemma 4.7.2.

Let us write the following set of moments problem,

$$p_k = \int_0^T e^{\lambda_k t} f(t)dt, \qquad k \in \mathbb{N}^*, \tag{4.112}$$

$$h_k = \int_0^T e^{\gamma_k t} f(t)dt, \qquad k \in \mathbb{Z}. \tag{4.113}$$

The space of all sequences $\{p_k\}_{k\in\mathbb{N}^*} \cup \{h_k\}_{k\in\mathbb{Z}}$ for which there exists a $f \in L^2(0,T)$ that solves the set of equations (4.112)–(4.113) is called the moment space.

Now, we recall the following results from the same paper which relate Theorem 4.7.3 to the moments problem (4.112)–(4.113).

Proposition 4.7.1. *Let $\{h_k\}_{k\in\mathbb{Z}} \in \ell^2$. Then, for any $T \geq t_b$, there exists $f \in W_{[0,T]}$, which solves the moment problem (4.113). Moreover, any $\widetilde{f} \in L^2(0,T)$ given by $\widetilde{f} = f + \widehat{f}$ with $\widehat{f} \in W_{[0,T]}^{\perp}$ also solves (4.113).*

The proof follows as a consequence of Lemma 4.7.2.

Proposition 4.7.2. *Assume that for any $r > 0$, the sequence $\{p_k\}_{k\in\mathbb{N}^*}$ satisfies*

$$|p_k|e^{rk} \to 0 \quad \text{as } k \to +\infty. \tag{4.114}$$

Then, for any given $\tau > 0$, there exists $g \in E_{[0,\tau]}$, which solves the moment problem (4.112). Moreover, any $\widetilde{g} \in L^2(0,\tau)$ given by $\widetilde{g} = g + \widehat{g}$ with $\widehat{g} \in E_{[0,\tau]}^{\perp}$ also solves (4.112).

The proof of the above proposition is standard. It relies on the existence of bi-orthogonal family in the space $E_{[0,\tau]}$ to the family of exponentials $\{e^{\lambda_k t}\}_{k\in\mathbb{N}^*}$; see [Han91] for a proof.

Let us now present the main theorem that tells the solvability of the mixed moment problems (4.112)–(4.113).

Theorem 4.7.4. *Let any $T > t_b$ be given. Then, under Hypothesis 4.7.2, given any sequence $\{p_k\}_{k\in\mathbb{N}^*}$ satisfying (4.114) and any $\{h_k\}_{k\in\mathbb{Z}} \in \ell^2$, there exists a function $f \in L^2(0,T)$ that simultaneously solves the set of moments problem (4.112)–(4.113). This does not hold for $T \leq t_b$.*

The proof of above theorem can be found in [Han94, Theorem 4.11]. For the sake of completeness, we give the proof below.

Proof. For $T \leq t_b$, the set of moments problem (4.112)–(4.113) does not necessarily have a solution. Thus, we start with $T > t_b$. By Theorem 4.7.3, the spaces $E := E_{[0,T]}$ and $W := W_{[0,T]}$ are uniformly separated. Thus the space $V := E + W$ is closed in $L^2(0,T)$ with its norm $\|\cdot\|_V := \|\cdot\|_{L^2(0,T)}$ and so $V := E \oplus W$. Moreover, the orthogonal complements $E^{\perp}$ and $W^{\perp}$ of E and W (resp.) in V are also uniformly separated using a result by T. Kato [Kat95, Chap. 4, §4] and therefore, $V = E^{\perp} \oplus W^{\perp}$. From this, one can show that the restrictions $P_E|_{W^{\perp}}$ and $P_W|_{E^{\perp}}$ are isomorphisms, where P_E and P_W are the orthogonal projections respectively onto E and W in V. By Propositions 4.7.2 and 4.7.1, there exist functions $f_1 \in E$ and $f_2 \in W$ which solve the equations (4.112) and (4.113) respectively. Set,

$$f = (P_E|_{W^{\perp}})^{-1} f_1 + (P_W|_{E^{\perp}})^{-1} f_2,$$

which simultaneously solves the equations (4.112)–(4.113) and moreover $f \in L^2(0,T)$. □

4.7.2.2 Formulation of the parabolic-hyperbolic moments problem

Let us recall that the set of eigenvalues $\sigma(A^*)$, given by (4.22).

The sequence $\{\lambda_k^h\}_{|k|\geq k_0}$ satisfies (H1) and (H2) of Hypothesis 4.7.2 with

$$\beta = -c^2, \ b = 2, \ \nu_k = O(|k|^{-1}).$$

Moreover, it is easy to observe that $\{\lambda_k^p\}_{k\geq k_0}$ satisfies the properties (P1), (P2), (P3) of Hypothesis 4.7.2.

Thus, the spectrum $\sigma(A^*)$ satisfies Hypothesis 4.7.2 except for the finite set $\{\lambda_0\} \cup \{\widehat{\lambda}_n\}_{n=1}^{n_0}$. But this will not lead any problem to construct and solve the associated moments equations. Let us go to the detail.

General setting We first recall Theorem 4.7.3 and Theorem 4.7.2. As per those results, our goal is to find uniformly separated spaces $\mathcal{W}_{[0,T]}$ and $\mathscr{E}_{[0,T]}$ in $L^2(0,T)$ for $T > t_b = 1$ (where $t_b = 2/b$ as introduced in Section 4.7.2.1 and in our case $b = 2$).

We start with $T > 1$. Then, we pick a subset of complex numbers $\{\widehat{\lambda}_{n_l}\}_{l=1}^{l_0}$ in such a way that

$$\mathcal{W}_{[a,a+1]} := \text{closed span } \left(\{e^{\lambda_k^h t}\}_{|k|\geq k_0} \cup \{e^{\widehat{\lambda}_{n_l} t}\}_{l=1}^{l_0}\right) \text{ in } L^2(a,a+1), \text{ for any } a \in \mathbb{R}, \tag{4.115}$$

equals the space $L^2(a,a+1)$; and moreover the above set forms a *Riesz basis* for the space $\mathcal{W}_{[a,a+T]}$ for each $T \geq 1$.

In particular,

$$\mathcal{W}_{[0,T]} = \text{closed span } \left(\{e^{\lambda_k^h t}\}_{|k|\geq k_0} \cup \{e^{\widehat{\lambda}_{n_l} t}\}_{l=1}^{l_0}\right) \text{ in } L^2(0,T). \tag{4.116}$$

Next, we consider the space

$$\mathscr{E}_{[0,T]} = \text{closed span } \left(\{e^{-\lambda_k^p t}\}_{k\geq k_0} \cup \{e^{-\lambda t}\}_{\lambda\in\Lambda_0} \cup \{1\}\right) \text{ in } L^2(0,T). \tag{4.117}$$

Then, we have the following result which follows from Theorem 4.7.3.

Lemma 4.7.3. *The spaces $\mathcal{W}_{[0,T]}$ and $\mathscr{E}_{[0,T]}$ defined by (4.116) and (4.117) respectively, are uniformly separated in $L^2(0,T)$ for $T > 1$. This does not hold for $T \leq 1$.*

The set of moments problem To begin with, let us recall that the eigenvalues for parabolic and hyperbolic parts, namely Λ_p and Λ_h given by (4.20) are simple. Also, recall that the set of eigenfunctions

$$\mathcal{E}(A^*) = \left\{\Phi_{\lambda_k^p},\, k \geq k_0\right\} \cup \left\{k^s \Phi_{\lambda_k^h},\, |k| \geq k_0\right\} \cup \left\{\Phi_\lambda^i;\, \lambda \in \Lambda_0,\, i = 0, ..., m_\lambda - 1\right\}$$

of A^* defines a Riesz basis in $(\dot{H}^s_\sharp(0,1))' \times L^2(0,1)$ for any $s > 0$, thanks to Corollary 4.3.1. Thus, it is enough to check the control problem (4.107) for the eigenfunctions of A^*. In what follows, the problem (4.5) is null-controllable at given time $T > 1$ if and only if there exists some $p \in L^2(0,T)$ such that we have the following:

$$\begin{cases} -\displaystyle\int_0^T \overline{e^{\lambda_k^p (T-t)}}\, p(t)\, dt & = m_{1,k}, \quad \forall k \geq k_0, \\ -\displaystyle\int_0^T (T-t)^j \overline{e^{\lambda(T-t)}}\, p(t)\, dt & = m_\lambda^j, \quad \forall \lambda \in \Lambda_0,\ j = 0, 1, \ldots, m_\lambda - 1, \end{cases} \tag{4.118}$$

and

$$-\int_0^T \overline{e^{\lambda_k^h (T-t)}}\, p(t)\, dt = m_{2,k}, \quad \forall |k| \geq k_0, \tag{4.119}$$

where

$$\begin{cases} m_{1,k} = \dfrac{\overline{e^{\lambda_k^p T}} \left\langle \begin{pmatrix} \xi_{\lambda_k^p} \\ \eta_{\lambda_k^p} \end{pmatrix}, \begin{pmatrix} \rho_0 \\ u_0 \end{pmatrix} \right\rangle_{(\dot{H}^s_\sharp)' \times L^2, \dot{H}^s_\sharp \times L^2}}{\overline{\xi_{\lambda_k^p}(1)}}, \quad \forall k \geq k_0, \\ m_\lambda^j = \dfrac{\overline{e^{\lambda T}} \left\langle \begin{pmatrix} \xi_\lambda^j \\ \eta_\lambda^j \end{pmatrix}, \begin{pmatrix} \rho_0 \\ u_0 \end{pmatrix} \right\rangle_{(\dot{H}^s_\sharp)' \times L^2, \dot{H}^s_\sharp \times L^2}}{\overline{\xi_\lambda^j(1)}}, \quad \forall \lambda \in \Lambda_0,\ j = 0, 1 \ldots, m_\lambda - 1, \end{cases} \tag{4.120}$$

and

$$m_{2,k} = \frac{\overline{e^{\lambda_k^h T}} \left\langle \begin{pmatrix} \xi_{\lambda_k^h} \\ \eta_{\lambda_k^h} \end{pmatrix}, \begin{pmatrix} \rho_0 \\ u_0 \end{pmatrix} \right\rangle_{(\dot{H}^s_\sharp)' \times L^2, \dot{H}^s_\sharp \times L^2}}{\overline{\xi_{\lambda_k^h}(1)}}, \quad \forall |k| \geq k_0. \tag{4.121}$$

$$\tag{4.122}$$

The above set of equations (4.118)–(4.119) are the so-called moments problem which are well-defined since $\mathcal{B}_\rho^* \Phi = \xi(1) \neq 0$ for any (generalized) eigenfunction $\Phi \in \mathcal{E}(A^*)$ as proved in Proposition 4.4.1–Part 1 under the assumption $c^4 + 8c^2 + 5 < 4\pi^2$. Let us now study the solvability of those equations.

Proof of the null-controllability result in $\dot{H}^s_\sharp \times L^2$, $s > \frac{1}{2}$ Let any parameter $s > 1/2$, initial data $(\rho_0, u_0) \in H^s_\sharp(0,1) \times L^2(0,1)$ and time $T > 1$ be given. We now consider the finitely many complex numbers $(\hat{\lambda}_{n_l})_{l=1}^{l_0}$ introduced earlier (see eq. (4.115)) in the above moments problem (hyperbolic part)

$$-\int_0^T \overline{e^{\hat{\lambda}_{n_l}(T-t)}} p(t) dt = m_{2,l}, \quad \forall\, l = 1, \ldots, l_0, \tag{4.123}$$

where $m_{2,l} \in \mathbb{C}$ for all $l = 1, \ldots, l_0$. Then, our goal is to apply the result of Theorem 4.7.4 to solve the set of moments problem (4.118)–(4.119)-(4.123). To do this, it suffices to show the following facts: for any $r > 0$

$$|m_{1,k}| e^{rk} \to 0 \quad \text{as} \quad k \to +\infty, \tag{4.124}$$

and

$$\sum_{|k| \geq k_0} |m_{2,k}|^2 < +\infty. \tag{4.125}$$

- Recall the expression of $m_{1,k}$ for $k \geq k_0$ from (4.120). We have

$$\begin{aligned} |m_{1,k}| &\leq C \|(\rho_0, u_0)\|_{\dot{H}^s_\sharp \times L^2} \, e^{\mathrm{Re}(\lambda_k^p) T} \frac{\|\xi_{\lambda_k^p}\|_{(\dot{H}^s_\sharp)'} + \|\eta_{\lambda_k^p}\|_{L^2}}{|\overline{\xi_{\lambda_k^p}(1)}|} \\ &\leq C \|(\rho_0, u_0)\|_{\dot{H}^s_\sharp \times L^2} \, e^{-k^2 \pi^2 T} k\pi \big(k^{-s-1} + 1\big), \end{aligned} \tag{4.126}$$

  thanks to the bounds of the eigenfunctions (4.31) and observation estimate (4.72a). Indeed, the bound (4.126) directly implies the Claim (4.124) due to the presence of $e^{-k^2 \pi^2 T}$ in the right hand side of (4.126).

  Thus, in view of Proposition 4.7.2, there exists a function $p_1 \in \mathscr{E} := \mathscr{E}_{[0,T]}$ that solves the set of equations (4.118) for the case of simple eigenvalues. To add the finitely many generalized eigenfunctions, one can adapt the strategy developed for instance in [FCGBdT10] or [BBGBO14], where the authors have proved the existence of bi-orthogonal family for a general sequence of type $\{t^j e^{\lambda_n t}\}_{j=0,\cdots,J; n \geq 1}$ for any $J \in \mathbb{N}^*$, where $\{\lambda_n\}_{n \geq 1}$ verifies the properties like (P1) and (P2) at least for large index $n \in \mathbb{N}^*$. As a consequence, we can find a $p_1 \in \mathscr{E}_{0,T}$ solving the parabolic moment problem (4.118).

- On the other hand, we show that $\{m_{2,k}\}_{|k| \geq k_0} \in \ell^2$. In this regard, we recall the bounds of the eigenfunctions given by (4.32) and the observation estimate (4.72b), which yields

$$\sum_{|k| \geq k_0} |m_{2,k}|^2 \leq C \|(\rho_0, u_0)\|^2_{\dot{H}^s_\sharp \times L^2} \sum_{|k| \geq k_0} \frac{\|\xi_{\lambda_k^h}\|^2_{(\dot{H}^s_\sharp)'} + \|\eta_{\lambda_k^h}\|^2_{L^2}}{|\overline{\xi_{\lambda_k^h}(1)}|^2}$$

$$\leq C\|(\rho_0,u_0)\|^2_{\dot{H}^s_\sharp\times L^2}\sum_{|k|\geq k_0}\left(|k|^{-2s}+|k|^{-2}\right)$$
$$\leq C\|(\rho_0,u_0)\|^2_{\dot{H}^s_\sharp\times L^2}.$$

The above series converges due to the sharp choice $s>1/2$ and indeed, it is clear that for $s\leq 1/2$, the series $\displaystyle\sum_{|k|\geq k_0}\frac{1}{|k|^{2s}}$ diverges.

Therefore, in view of Proposition 4.7.1, there exists a function $p_2\in\mathcal{W}:=\mathcal{W}_{[0,T]}$ that solves the set of equations (4.119)–(4.123).

Now, as consequence of Lemma 4.7.3, the space

$$\mathcal{V}:=\mathcal{E}+\mathcal{W} \tag{4.127}$$

is closed and thus a Hilbert space with $\|\cdot\|_{\mathcal{V}}:=\|\cdot\|_{L^2(0,T)}$, so $\mathcal{V}=\mathcal{E}\oplus\mathcal{W}$. Likewise, we have $\mathcal{V}:=\mathcal{E}^\perp\oplus\mathcal{W}^\perp$. Therefore, the restrictions $P_{\mathcal{E}}|_{\mathcal{W}^\perp}$ and $P_{\mathcal{W}}|_{\mathcal{E}^\perp}$ are isomorphisms, where $P_{\mathcal{E}}$ and $P_{\mathcal{W}}$ denote the orthogonal projections from $\mathcal{V}$ onto $\mathcal{E}$ and $\mathcal{W}$ respectively. Let us set

$$p:=(P_{\mathcal{E}}|_{\mathcal{W}^\perp})^{-1}p_1+(P_{\mathcal{W}}|_{\mathcal{E}^\perp})^{-1}p_2, \tag{4.128}$$

which certainly belongs to the space $L^2(0,T)$ and simultaneously solves the set of moments problem (4.118)–(4.119)–(4.123) for $T>1$ and any $\rho_0\in\dot{H}^s_\sharp(0,1)$ for $s>1/2$, $u_0\in L^2(0,1)$. This concludes the proof of the result of this section.

4.7.3 Null-controllability result with $\dot{L}^2\times L^2$ initial data

Proof of Theorem 4.1.2. We start with $c^4+8c^2+5<4\pi^2$ and pick any initial data $(\rho_0,u_0)\in\dot{L}^2(0,1)\times L^2(0,1)$ for the system (4.5). We express the initial data as

$$(\rho_0,u_0)=(\rho_0,0)+(0,u_0),$$

and consider the following two systems

$$\begin{cases}\rho_{1,t}+\rho_{1,x}+cu_{1,x}=0, & \text{in } (0,T)\times(0,1),\\ u_{1,t}-u_{1,xx}+u_{1,x}+c\rho_{1,x}=0, & \text{in } (0,T)\times(0,1),\\ \rho_1(t,0)=\rho_1(t,1)+p_1(t), & \text{for } t\in(0,T),\\ u_1(t,0)=0,\quad u_1(t,1)=0, & \text{for } t\in(0,T),\\ \rho_1(0,x)=\rho_0(x),\quad u_1(0,x)=0, & \text{in } (0,1),\end{cases} \tag{4.129}$$

and

$$\begin{cases}\rho_{2,t}+\rho_{2,x}+cu_{2,x}=0, & \text{in } (0,T)\times(0,1),\\ u_{2,t}-u_{2,xx}+u_{2,x}+c\rho_{2,x}=0, & \text{in } (0,T)\times(0,1),\\ \rho_2(t,0)=\rho_2(t,1)+p_2(t), & \text{for } t\in(0,T)\\ u_2(t,0)=0,\quad u_2(t,1)=0, & \text{for } t\in(0,T),\\ \rho_2(0,x)=0,\quad u_2(0,x)=u_0(x), & \text{in } (0,1).\end{cases} \tag{4.130}$$

Here $p_1,p_2\in L^2(0,T)$ are boundary controls which are to be determined.

Now, from the analysis pursued in Section 4.7.1, if we start with initial data $(\rho_0,0)$ with $\rho_0\in\dot{L}^2(0,1)$, then there exists a control $p_1\in L^2(0,T)$ such that the solution (ρ_1,u_1) to the system (4.129) verifies

$$(\rho_1(T,\cdot),u_1(T,\cdot))=(0,0),\quad \text{in } (0,1).$$

On the other hand, it is also known from Section 4.7.2.2 that, starting with initial data $(0,u_0)$ with $u_0\in L^2(0,1)$, we can find a control $p_2\in L^2(0,T)$ such that the solution (ρ_2,u_2) to the system (4.130) satisfies

$$(\rho_2(T,\cdot),u_2(T,\cdot))=(0,0),\quad \text{in } (0,1).$$

Let us define $p(t) = p_1(t) + p_2(t)$ for $t \in (0,T)$. Then $p \in L^2(0,T)$, and the solution (ρ, u) to the main system (4.5), with this control p and the prescribed initial state $(\rho_0, u_0) \in \dot{L}^2(0,1) \times L^2(0,1)$, satisfies

$$(\rho(T), u(T)) = (0,0) \quad \text{in } (0,1).$$

□

Proof of Theorem 4.1.3. Let $T > 1$. We have already shown the existence of a null control $p \in L^2(0,T)$ for the system (4.5). Now, to prove the existence of a null control $h \in L^2(0,T)$ for the control problem (4.6), all we need to show that $\rho(\cdot,1) \in L^2(0,T)$, where ρ is the solution component of the system (4.5) associated with the control function $p \in L^2(0,T)$. But the proof for $\rho(\cdot,1) \in L^2(0,T)$ follows from a hidden regularity result given in Appendix A.1 (Lemma A.1.1). Hence, we define $h(t) = \rho(t,1) + p(t)$ for all $t \in (0,T)$, which plays the role of a Dirichlet (null) control function for the main system (4.6).

On the other hand, when $0 < T < 1$, the system (4.6) cannot be null controllable at time T in $L^2(0,1) \times L^2(0,1)$. If so, then we can find a null control $h \in L^2(0,T)$ for the system (4.6). By defining $p(t) := \rho(t,1) + [h(t) - \rho(t,1)]$ for $t \in (0,T)$, we see that $p \in L^2(0,T)$ and is a null control for the system (4.5), which is a contradiction to Proposition 4.1.1 (see below for the proof of Proposition 4.1.1).

The proof is complete. □

4.7.4 Lack of null-controllability at small time

This section is devoted to prove the lack of null-controllability result of the system (4.5) for $0 < T < 1$, that is precisely Proposition 4.1.1. In this regard, we mention the work [BKLB20] where the authors proved the lack of null-controllability for a transport-parabolic system with localized interior control. Similar result has been treated in [CDM23] in the context of boundary controllability for a transport-elliptic system (the so-called creeping flow model).

Proof of Proposition 4.1.1. Let $0 < T < 1$. Consider the transport equation

$$\begin{cases} \tilde{\sigma}_t(t,x) + \tilde{\sigma}_x(t,x) - c^2\tilde{\sigma}(t,x) = 0, & (t,x) \in (0,T) \times (0,1), \\ \tilde{\sigma}(t,0) = \tilde{\sigma}(t,1), & t \in (0,T), \\ \tilde{\sigma}(T,x) = \tilde{\sigma}_T(x), & x \in (0,1), \end{cases} \tag{4.131}$$

where $\tilde{\sigma}_T \in L^2(0,1)$. Since $T < 1$, there exists a nontrivial function $\tilde{\sigma}_T \in C^\infty(0,1)$ with $\operatorname{supp}(\tilde{\sigma}_T) \subset (T,1)$ such that the associated solution $\tilde{\sigma}$ of (4.131) satisfies $\tilde{\sigma}(t,0) = \tilde{\sigma}(t,1) = 0$ for all $t \in (0,T)$ and $\tilde{\sigma} \neq 0$ in $(0,T) \times (0,1)$. Let $N > 0$ be a fixed integer. We define the polynomial

$$P^N(x) := \prod_{l=-N}^{N} (x - l), \quad x \in (0,1)$$

and the function

$$\tilde{\sigma}_T^N := P^N\left(-i\frac{d}{dx}\right)\tilde{\sigma}_T.$$

We now write the terminal state $\tilde{\sigma}_T \in L^2(0,1)$ as

$$\tilde{\sigma}_T(x) := \sum_{n\in\mathbb{Z}} a_n e^{2in\pi x}, \quad x \in (0,1).$$

Then, the above function $\tilde{\sigma}_T^N$ becomes

$$\tilde{\sigma}_T^N(x) = \sum_{n\in\mathbb{Z}} a_n \prod_{l=-N}^{N} \left(-i\frac{d}{dx} - l\right) e^{2in\pi x} = \sum_{n\in\mathbb{Z}} a_n \prod_{l=-N}^{N} (n-l)\, e^{2in\pi x} = \sum_{n\in\mathbb{Z}} a_n P^N(n) e^{2in\pi x},$$

for $x \in (0,1)$. Note that $P^N(n) = 0$ for all $|n| \le N$ and therefore

$$\tilde{\sigma}_T^N(x) = \sum_{|n|\ge N+1} a_n P^N(n) e^{2in\pi x}.$$

With this $\tilde{\sigma}_T^N$, let us now consider the following system

$$\begin{cases} \tilde{\sigma}_t(t,x) + \tilde{\sigma}_x(t,x) - c^2\tilde{\sigma}(t,x) = 0, & (t,x) \in (0,T) \times (0,1), \\ \tilde{\sigma}(t,0) = \tilde{\sigma}(t,1), & t \in (0,T), \\ \tilde{\sigma}(T,x) = \tilde{\sigma}_T^N(x), & x \in (0,1). \end{cases} \tag{4.132}$$

Since $\mathrm{supp}(\tilde{\sigma}_T^N) \subset \mathrm{supp}(\tilde{\sigma}_T) \subset (T,1)$, the solution $\tilde{\sigma}$ to (4.132) satisfies $\tilde{\sigma}^N(t,0) = \tilde{\sigma}^N(t,1) = 0$ for all $t \in (0,T)$. We now consider the following adjoint system

$$\begin{cases} \sigma_t(t,x) + \sigma_x(t,x) + cv_x(t,x) = 0, & (t,x) \in (0,T) \times (0,1), \\ v_t(t,x) - v_{xx}(t,x) + v_x(t,x) + c\sigma_x(t,x) = 0, & (t,x) \in (0,T) \times (0,1), \\ \sigma(t,0) = \sigma(t,1), & t \in (0,T), \\ v(t,0) = 0, \quad v(t,1) = 0, & t \in (0,T), \\ \sigma(T,x) = \tilde{\sigma}_T^N(x), \quad v(T,x) = v_T^N(x), & x \in (0,1), \end{cases} \tag{4.133}$$

where we choose v_T^N such that

$$(\tilde{\sigma}_T^N, v_T^N)^\dagger = \sum_{|n| \geq N+1} \tilde{a}_n^h \Phi_{\lambda_n^h}$$

with $\tilde{a}_n^h := \frac{a_n P^N(n)}{\xi_{\lambda_n^h}(1)}$ for all $|n| \geq N+1$ (note that $\xi_{\lambda_n^h}(1) \neq 0$, thanks to the eigen equation). We write the solutions to the systems (4.132) and (4.133) respectively as

$$\tilde{\sigma}^N(t,x) = \sum_{|n| \geq N+1} a_n P^N(n) e^{(-2in\pi - c^2)(T-t)} e^{2in\pi x}, \tag{4.134}$$

$$\sigma^N(t,x) = \sum_{|n| \geq N+1} \frac{a_n P^N(n)}{\xi_{\lambda_n^h}(1)} e^{\lambda_n^h (T-t)} \xi_{\lambda_n^h}, \tag{4.135}$$

$$v^N(t,x) = \sum_{|n| \geq N+1} \frac{a_n P^N(n)}{\xi_{\lambda_n^h}(1)} e^{\lambda_n^h (T-t)} \eta_{\lambda_n^h}, \tag{4.136}$$

for $(t,x) \in [0,T] \times [0,2\pi]$. We prove that the solution component σ^N of (4.133) approximates the solution $\tilde{\sigma}^N$ of (4.132) at the point $x = 1$. Indeed,

$$\begin{aligned} &\left\|\sigma^N(\cdot,1) - \tilde{\sigma}^N(\cdot,1)\right\|_{L^2(0,T)}^2 \\ &\leq \sum_{|n| \geq N+1} |a_n|^2 \left|P^N(n)\right|^2 \left\|e^{\lambda_n^h(T-t)} - e^{(-2in\pi - c^2)(T-t)} e^{2in\pi}\right\|_{L^2(0,T)}^2 \\ &\leq \sum_{|n| \geq N+1} |a_n|^2 \left|P^N(n)\right|^2 \left\|e^{O(\frac{1}{n})(T-t)} - 1\right\|_{L^2(0,T)}^2 \\ &\leq \sum_{|n| \geq N+1} \frac{1}{|n|^2} |a_n|^2 \left|P^N(n)\right|^2, \end{aligned}$$

and therefore

$$\left\|\sigma^N(\cdot,1) - \tilde{\sigma}^N(\cdot,1)\right\|_{L^2(0,T)}^2 \leq \frac{C}{|N|^2} \sum_{|n| \geq N+1} |a_n|^2 \left|P^N(n)\right|^2.$$

Let us now suppose that the following observability inequality holds

$$\int_0^T \left|\sigma^N(t,1)\right|^2 dt \geq C \left\|(\sigma^N(0), v^N(0))\right\|_{(L^2(0,1))^2}^2. \tag{4.137}$$

Then, we have

$$\left\|(\sigma^N(0), v^N(0))\right\|_{(L^2(0,1))^2}^2 \leq C \int_0^T \left|\sigma^N(t,1)\right|^2 dt$$

$$
\begin{aligned}
&\leq C \int_0^T \left(\left|\sigma^N(t,1) - \tilde{\sigma}^N(t,1)\right|^2 + \left|\tilde{\sigma}^N(t,1)\right|^2 \right) dt \\
&\leq \frac{C}{N^2} \sum_{|n| \geq N+1} |a_n|^2 \left|P^N(n)\right|^2,
\end{aligned}
$$

as we have $\tilde{\sigma}^N(t,0) = 0 = \tilde{\sigma}^N(t,1)$ for all $t \in (0,T)$. Thus we get

$$
\left\|\sigma^N(0)\right\|^2_{L^2(0,1)} \leq \left\|(\sigma^N(0), v^N(0))\right\|^2_{(L^2(0,1))^2} \leq \frac{C}{N^2} \sum_{|n| \geq N+1} |a_n|^2 \left|P^N(n)\right|^2 \leq \frac{C}{N^2} \left\|\sigma^N(0)\right\|^2_{L^2(0,1)},
$$

since $\mathrm{Re}(\nu_n^h)$ is bounded and $\left\|\xi_{\lambda_n^h}\right\|_{L^2(0,1)} \geq C$, $\left|\xi_{\lambda_n^h}(0)\right| \geq C$. Therefore, $1 \leq \frac{C}{N^2}$ for all N and hence the above inequality is not true. This shows that the observability inequality (4.137) cannot hold; as a consequence, the system (4.5) is not null controllable at time T. This completes the proof. □

4.8 Detailed spectral analysis of the adjoint operator

In this section, we study the detailed spectral analysis of the adjoint operator A^*. We hereby recall the eigenvalue problem (4.18) from Section 4.3 which has been rewritten below,

$$
\begin{aligned}
\xi'(x) + c\eta'(x) &= \lambda\xi(x), \quad x \in (0,1), \\
\eta''(x) + \eta'(x) + c\xi'(x) &= \lambda\eta(x), \quad x \in (0,1), \\
\xi(0) &= \xi(1), \\
\eta(0) = 0, \quad \eta(1) &= 0.
\end{aligned}
\tag{4.138}
$$

We divide the analysis into several steps. Let us begin by the following results.

Proof of point (ii)-Proposition 4.3.1: all non-trivial eigenvalues have negative real parts Multiplying the first equation of (4.138) by $\overline{\xi}$, the second one by $\overline{\eta}$ and then integrating, we obtain

$$
\begin{aligned}
\int_0^1 \overline{\xi(x)}\xi'(x)dx + c\int_0^1 \overline{\xi(x)}\eta'(x)dx &= \lambda\int_0^1 |\xi(x)|^2 dx \\
\int_0^1 \overline{\eta(x)}\eta''(x)dx + \int_0^1 \overline{\eta(x)}\eta'(x)dx + c\int_0^1 \overline{\eta(x)}\xi'(x)dx &= \lambda\int_0^1 |\eta(x)|^2 dx.
\end{aligned}
$$

Adding these two equations, we get

$$
\begin{aligned}
\int_0^1 \overline{\xi(x)}\xi'(x)dx + \int_0^1 \overline{\eta(x)}\eta'(x)dx + c\int_0^1 \overline{\xi(x)}\eta'(x)dx + c\int_0^1 \overline{\eta(x)}\xi'(x)dx \\
+ \int_0^1 \overline{\eta(x)}\eta''(x)dx = \lambda\int_0^1 |\xi(x)|^2 dx + \lambda\int_0^1 |\eta(x)|^2 dx,
\end{aligned}
\tag{4.139}
$$

where we have used the following fact,

$$
\int_0^1 \overline{\xi(x)}\xi'(x)dx = \frac{1}{2}\int_0^1 \frac{d}{dx}|\xi(x)|^2 dx + i\int_0^1 \mathrm{Im}(\overline{\xi(x)}\xi'(x))dx = i\int_0^1 \mathrm{Im}(\overline{\xi(x)}\xi'(x))dx,
\tag{4.140}
$$

thanks to the boundary condition $\xi(0) = \xi(1)$.

Similarly, we can obtain

$$
\int_0^1 \overline{\eta(x)}\eta'(x)dx = i\int_0^1 \mathrm{Im}(\overline{\eta(x)}\eta'(x))dx.
\tag{4.141}
$$

Using the relations (4.140), (4.141) in (4.139) and performing an integration by parts, we deduce that

$$
i\int_0^1 \left(\mathrm{Im}(\overline{\xi(x)}\xi'(x)) + \mathrm{Im}(\overline{\eta(x)}\eta'(x))\right) dx + c\int_0^1 \xi'(x)\overline{\eta(x)}dx - c\int_0^1 \overline{\xi'(x)}\eta(x)dx - \int_0^1 |\eta'(x)|^2 dx
$$

$$= \lambda \int_0^1 |\xi(x)|^2 dx + \lambda \int_0^1 |\eta(x)|^2 dx,$$

from which it is clear that

$$\mathrm{Re}(\lambda) = -\frac{\|\eta'\|_{L^2}^2}{\|\xi\|_{L^2}^2 + \|\eta\|_{L^2}^2} < 0, \tag{4.142}$$

since $\eta' = 0$ is not possible. If yes, then from the boundary condition $\eta(0) = \eta(1) = 0$, we have $\eta = 0$ and this yields that ξ =constant, which is possible if and only if $\lambda = 0$. Therefore, when $\lambda \neq 0$, then one has the condition (4.142).

Remark 4.8.1. *It can be easily seen that the first component ξ satisfies $\int_0^1 \xi = 0$ provided $\lambda \neq 0$.*

Proof of point (iii)- Proposition 4.3.1: compactness of the resolvent to the adjoint operator In this section, we are going to prove the part (iii) of Proposition 4.3.1.

For any $\lambda \notin \sigma(A^*)$, denote the resolvent operator associated to A^* by $R(\lambda, A^*) := (\lambda I - A^*)^{-1}$ (where $\sigma(A^*)$ is the spectrum of A^* defined by (4.22)).

Let $\{Y_n\}_n = \{(f_n, g_n)\}_n$ be a bounded sequence in $\mathrm{Z} := L^2(0,1) \times L^2(0,1)$. Our claim is to prove that for any $\lambda > 0$ the sequence $\left\{R(\lambda; A^*)Y_n\right\}_n$ contains a convergent subsequence. Let $X_n = (\sigma_n, v_n) = R(\lambda; A^*)Y_n \in D(A^*)$, that is

$$(\lambda I - A^*)X_n = Y_n. \tag{4.143}$$

More explicitly,

$$\begin{cases} \lambda\sigma_n - (\sigma_n)_x - c(v_n)_x = f_n & \text{in } (0,1), \\ \lambda v_n - c(\sigma_n)_x - (v_n)_x - (v_n)_{xx} = g_n & \text{in } (0,1), \\ \sigma_n(0) = \sigma_n(1), \quad v_n(0) = v_n(1) = 0. \end{cases} \tag{4.144}$$

Taking inner product with X_n in the equation (4.143), we get

$$\lambda \left\langle X_n, X_n \right\rangle_{\mathrm{Z}} - \left\langle A^* X_n, X_n \right\rangle_{\mathrm{Z}} = \left\langle X_n, Y_n \right\rangle_{\mathrm{Z}}.$$

Considering only the real parts, we see

$$\lambda \|X_n\|_{\mathrm{Z}}^2 - \mathrm{Re}(\langle A^* X_n, X_n\rangle_{\mathrm{Z}}) = \mathrm{Re}(\langle X_n, Y_n\rangle_{\mathrm{Z}}).$$

Now, recall that the operator A^* is dissipative, i.e., $\mathrm{Re}(\langle A^* X_n, X_n\rangle_{\mathrm{Z}}) \leq 0$; in what follows, we have

$$\lambda \|X_n\|_{\mathrm{Z}}^2 \leq \mathrm{Re}(\langle X_n, Y_n\rangle_{\mathrm{Z}}) \leq |\langle X_n, Y_n\rangle_{\mathrm{Z}}| \leq \frac{\lambda}{2} \|X_n\|_{\mathrm{Z}}^2 + \frac{1}{2\lambda} \|Y_n\|_{\mathrm{Z}}^2.$$

In other words,

$$\|X_n\|_{\mathrm{Z}}^2 \leq \frac{1}{\lambda^2} \|Y_n\|_{\mathrm{Z}}^2.$$

Thus, the sequence $\{X_n\}_n$ is bounded in Z. We now prove that $\{X_n\}_n$ is in fact bounded in $H^1_\sharp(0,1) \times H^1_0(0,1)$. Multiplying the second equation of (4.144) by v_n, we get

$$\lambda \int_0^1 |v_n|^2 \, dx - c \int_0^1 (\sigma_n)_x \bar{v}_n dx - \int_0^1 (v_n)_{xx} \bar{v}_n dx = \int_0^1 g_n \bar{v}_n dx.$$

Performing an integration by parts, we obtain

$$\lambda \int_0^1 |v_n|^2 \, dx + c \int_0^1 \sigma_n (\bar{v}_n)_x dx + \int_0^1 |(v_n)_x|^2 \, dx = \int_0^1 g_n \bar{v}_n dx,$$

from which, it follows that

$$\lambda \int_0^1 |v_n|^2 \, dx + \int_0^1 |(v_n)_x|^2 \, dx = \mathrm{Re}\left(\int_0^1 g_n \bar{v}_n dx\right) - c\mathrm{Re}\left(\int_0^1 \sigma_n (\bar{v}_n)_x dx\right)$$

$$\leq \left|\int_0^1 g_n \bar{v}_n dx\right| + c\left|\int_0^1 \sigma_n (\bar{v}_n)_x dx\right|$$
$$\leq \frac{1}{2\lambda}\int_0^1 |g_n|^2\,dx + \frac{\lambda}{2}\int_0^1 |v_n|^2\,dx + \frac{c^2}{2}\int_0^1 |\sigma_n|^2\,dx + \frac{1}{2}\int_0^1 |(v_n)_x|^2\,dx.$$

After simplification, we have

$$\frac{\lambda}{2}\int_0^1 |v_n|^2\,dx + \frac{1}{2}\int_0^1 |(v_n)_x|^2\,dx \leq \frac{1}{2\lambda}\int_0^1 |g_n|^2\,dx + \frac{c^2}{2}\int_0^1 |\sigma_n|^2\,dx,$$

that is, the sequence $\{v_n\}_n$ is bounded in $H_0^1(0,1)$. Then, the first equation of (4.144) gives

$$(\sigma_n)_x = \lambda\sigma_n - c(v_n)_x - f_n,$$

which shows that the sequence $\{(\sigma_n)_x\}_n$ is bounded in $L^2(0,1)$.

So, we have proved that $\{X_n\}_n$ is a bounded sequence in $H_\sharp^1(0,1)\times H_0^1(0,1)$ (which is compactly embedded in Z) and therefore, $\{X_n\}_n$ is relatively compact in Z.

This completes the proof.

Proof of point (iv)-Proposition 4.3.1: all eigenvalues are geometrically simple. Let $c > 0$ be such that $c^4 + 8c^2 + 5 < 4\pi^2$. On contrary, let us assume that for any eigenvalue λ, there are two distinct eigenfunctions $\Phi_1 := (\xi_1, \eta_1)$ and $\Phi_2 := (\xi_2, \eta_2)$ of A^*. We prove that Φ_1 and Φ_2 are linearly dependent.

Let be $\alpha_1, \alpha_2 \in \mathbb{C}\setminus\{0\}$ and consider the linear combination $\Phi := \alpha_1\Phi_1 + \alpha_2\Phi_2$. Then $\Phi := (\xi, \eta)$ also satisfies the eigenvalue problem (4.138). We now choose α_1, α_2 in such a way that $\xi(0) = 0$ (a particular choice is $\alpha_1 = -\frac{\alpha_2\xi_2(0)}{\xi_1(0)}$). Then, in the same spirit of Proposition 4.4.1–Part 1, we can conclude that $\Phi = 0$.

This ensures the assumption that each eigenvalue of A^* has geometric multiplicity 1.

4.8.1 Determining the eigenvalues for large modulus

We write the set of equations (4.138) satisfied by ξ and η into a single equation of η as obtained in (4.40), given by

$$\eta'''(x) - (\lambda + c^2 - 1)\eta''(x) - 2\lambda\eta'(x) + \lambda^2\eta(x) = 0, \quad \forall x \in (0,1), \tag{4.145a}$$
$$\eta(0) = \eta(1) = 0, \quad \eta''(0) - (c^2-1)\eta'(0) = \eta''(1) - (c^2-1)\eta'(1). \tag{4.145b}$$

Then, the auxiliary equation associated to (4.145a) is

$$m^3 - (\lambda + c^2 - 1)m^2 - 2\lambda m + \lambda^2 = 0. \tag{4.146}$$

Introduce $\mu = -\lambda \in \mathbb{C}$ and $a_1 = \mu - c^2 + 1$, $a_2 = 2\mu$, $a_3 = \mu^2$, so that the roots of cubic polynomial (4.146) are given by

$$\begin{cases} m_1 &= -\frac{1}{3}\left(a_1 + C + \frac{D_0}{C}\right), \\ m_2 &= -\frac{1}{3}\left(a_1 + \frac{(-1+i\sqrt{3})}{2}C + \frac{(-1-i\sqrt{3})}{2}\frac{D_0}{C}\right), \\ m_3 &= -\frac{1}{3}\left(a_1 + \frac{(-1-i\sqrt{3})}{2}C + \frac{(-1+i\sqrt{3})}{2}\frac{D_0}{C}\right), \end{cases} \tag{4.147}$$

with

$$D_0 = a_1^2 - 3a_2, \quad D_1 = 2a_1^3 - 9a_1a_2 + 27a_3, \quad C = \left(\frac{D_1 + \sqrt{D_1^2 - 4D_0^3}}{2}\right)^{1/3}.$$

Exerting the values of a_1, a_2, a_3, we can find

$$\begin{aligned}
D_0 &= \mu^2 + (c^2-1)^2 - 2(2+c^2)\mu,\\
D_1 &= 2(\mu - c^2 + 1)^3 - 9(\mu - c^2 + 1)2\mu + 27\mu^2\\
&= 2(\mu^3 - c^6 + 1 - 3c^2\mu^2 + 3\mu^2 + 3c^4\mu + 3\mu + 3c^4 - 3c^2 - 6c^2\mu)\\
&\quad - 18\mu^2 + 18c^2\mu - 18\mu + 27\mu^2\\
&= 2\mu^3 + (15-6c^2)\mu^2 + (6c^4 + 6c^2 - 12)\mu - 2c^6 + 6c^4 - 6c^2 + 2.
\end{aligned}$$

From the above expressions, we calculate

$$\begin{aligned}
D_1^2 - 4D_0^3 &= [2\mu^3 + (15-6c^2)\mu^2 + (6c^4+6c^2-12)\mu - 2c^6 + 6c^4 - 6c^2 + 2]^2\\
&\quad - 4[\mu^2 - 2(c^2+2)\mu + (c^2-1)^2]^3\\
&= 4\mu^6 + 4(15-6c^2)\mu^5 + (60c^4 - 156c^2 + 177)\mu^4 + O(\mu^3)\\
&\quad - 4[\mu^6 - 6(c^2+2)\mu^5 + (15c^4 + 42c^2 + 51)\mu^4 + O(\mu^3)]\\
&= 108\mu^5 - (324c^2 + 27)\mu^4 + O(\mu^3).
\end{aligned}$$

Using the binomial expansion and approximating for large $|\mu|$, we obtain

$$\begin{aligned}
\sqrt{D_1^2 - 4D_0^3} &= \left[108\mu^5 - (324c^2+27)\mu^4 + O(\mu^3)\right]^{1/2}\\
&= 6\sqrt{3}\mu^{5/2}\left[1 - \left(\frac{12c^2+1}{4\mu} + O(\mu^{-2})\right)\right]^{1/2}\\
&= 6\sqrt{3}\mu^{5/2}\left[1 - \frac{1}{2}\left(\frac{12c^2+1}{4\mu} + O(\mu^{-2})\right) + O(\mu^{-2})\right]\\
&= 6\sqrt{3}\mu^{5/2}\left[1 - \frac{12c^2+1}{8\mu} + O(\mu^{-2})\right]\\
&= 6\sqrt{3}\mu^{5/2} - \frac{6\sqrt{3}}{8}(12c^2+1)\mu^{3/2} + O(\mu^{1/2}).
\end{aligned}$$

In terms of the above quantities, we have

$$C = \left[\mu^3 + 3\sqrt{3}\mu^{5/2} + \frac{(15-6c^2)}{2}\mu^2 - \frac{3\sqrt{3}}{8}(12c^2+1)\mu^{3/2} + O(\mu)\right]^{1/3}$$

Now, using binomial expansion and simplifying, one can obtain for large modulus of μ, that

$$\begin{aligned}
C &= \mu\left[1 + 3\sqrt{3}\mu^{-1/2} + \frac{(15-6c^2)}{2}\mu^{-1} - \frac{3\sqrt{3}}{8}(12c^2+1)\mu^{-3/2} + O(\mu^{-2})\right]^{1/3}\\
&= \mu\Bigg[1 + \frac{1}{3}\left(3\sqrt{3}\mu^{-1/2} + \frac{(15-6c^2)}{2}\mu^{-1} - \frac{3\sqrt{3}}{8}(12c^2+1)\mu^{-3/2} + O(\mu^{-2})\right)\\
&\quad - \frac{1}{9}\left(3\sqrt{3}\mu^{-1/2} + \frac{(15-6c^2)}{2}\mu^{-1} - \frac{3\sqrt{3}}{8}(12c^2+1)\mu^{-3/2} + O(\mu^{-2})\right)^2\\
&\quad + \frac{5}{81}\left(3\sqrt{3}\mu^{-1/2} + \frac{(15-6c^2)}{2}\mu^{-1} - \frac{3\sqrt{3}}{8}(12c^2+1)\mu^{-3/2} + O(\mu^{-2})\right)^3 + O(\mu^{-2})\Bigg]\\
&= \mu\Bigg[1 + \left(\sqrt{3}\mu^{-1/2} + \frac{(15-6c^2)}{6}\mu^{-1} - \frac{\sqrt{3}}{8}(12c^2+1)\mu^{-3/2} + O(\mu^{-2})\right)\\
&\quad - \frac{1}{9}\left(27\mu^{-1} + 3\sqrt{3}(15-6c^2)\mu^{-3/2} + O(\mu^{-2})\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{5}{81}\left(81\sqrt{3}\mu^{-3/2}+O(\mu^{-2})\right)+O(\mu^{-2})\Bigg]\\
&=\mu\left[1+\sqrt{3}\mu^{-1/2}-\frac{2c^2+1}{2}\mu^{-1}+\frac{\sqrt{3}}{8}(4c^2-1)\mu^{-3/2}+O(\mu^{-2})\right]\\
&=\mu+\sqrt{3}\mu^{1/2}-\frac{2c^2+1}{2}+\frac{\sqrt{3}}{8}(4c^2-1)\mu^{-1/2}+O(\mu^{-1}).
\end{aligned}
$$

Similarly we have,

$$
\begin{aligned}
\frac{D_0}{C}&=\left(\frac{D_1-\sqrt{D_1^2-4D_0^3}}{2}\right)^{\frac{1}{3}}\\
&=\left[\mu^3-3\sqrt{3}\mu^{5/2}+\frac{(15-6c^2)}{2}\mu^2+\frac{3\sqrt{3}}{8}(12c^2+1)\mu^{3/2}+O(\mu)\right]^{1/3}\\
&=\mu\left[1-3\sqrt{3}\mu^{-1/2}+\frac{(15-6c^2)}{2}\mu^{-1}+\frac{3\sqrt{3}}{8}(12c^2+1)\mu^{-3/2}+O(\mu^{-2})\right]^{1/3}\\
&=\mu\Bigg[1+\frac{1}{3}\left(-3\sqrt{3}\mu^{-1/2}+\frac{(15-6c^2)}{2}\mu^{-1}+\frac{3\sqrt{3}}{8}(12c^2+1)\mu^{-3/2}+O(\mu^{-2})\right)\\
&\qquad-\frac{1}{9}\left(-3\sqrt{3}\mu^{-1/2}+\frac{(15-6c^2)}{2}\mu^{-1}+\frac{3\sqrt{3}}{8}(12c^2+1)\mu^{-3/2}+O(\mu^{-2})\right)^2\\
&\qquad+\frac{5}{81}\left(-3\sqrt{3}\mu^{-1/2}+\frac{(15-6c^2)}{2}\mu^{-1}+\frac{3\sqrt{3}}{8}(12c^2+1)\mu^{-3/2}+O(\mu^{-2})\right)^3+O(\mu^{-2})\Bigg]\\
&=\mu\Bigg[1+\left(-\sqrt{3}\mu^{-1/2}+\frac{(15-6c^2)}{6}\mu^{-1}+\frac{\sqrt{3}}{8}(12c^2+1)\mu^{-3/2}+O(\mu^{-2})\right)\\
&\qquad-\frac{1}{9}\left(27\mu^{-1}-3\sqrt{3}(15-6c^2)\mu^{-3/2}+O(\mu^{-2})\right)\\
&\qquad+\frac{5}{81}\left(-81\sqrt{3}\mu^{-3/2}+O(\mu^{-2})\right)+O(\mu^{-2})\Bigg]\\
&=\mu\left[1-\sqrt{3}\mu^{-1/2}-\frac{2c^2+1}{2}\mu^{-1}-\frac{\sqrt{3}}{8}(4c^2-1)\mu^{-3/2}+O(\mu^{-2})\right]\\
&=\mu-\sqrt{3}\mu^{1/2}-\frac{2c^2+1}{2}-\frac{\sqrt{3}}{8}(4c^2-1)\mu^{-1/2}+O(\mu^{-1}).
\end{aligned}
$$

So, the characteristic roots are (recall (4.147))

$$
\begin{aligned}
m_1&=-\frac{1}{3}\Bigg[\mu-c^2+1+\left(\mu+\sqrt{3}\mu^{1/2}-\frac{2c^2+1}{2}+\frac{\sqrt{3}}{8}(4c^2-1)\mu^{-1/2}+O(\mu^{-1})\right)\\
&\qquad+\left(\mu-\sqrt{3}\mu^{1/2}-\frac{2c^2+1}{2}-\frac{\sqrt{3}}{8}(4c^2-1)\mu^{-1/2}+O(\mu^{-1})\right)\Bigg]\\
&=-\frac{1}{3}\left(3\mu-3c^2+O(\mu^{-1})\right)\\
&=-\mu+c^2+O(\mu^{-1}),
\end{aligned}
$$

$$
m_2=-\frac{1}{3}\Bigg[\mu-c^2+1+\frac{-1+i\sqrt{3}}{2}\left(\mu+\sqrt{3}\mu^{1/2}-\frac{2c^2+1}{2}+\frac{\sqrt{3}}{8}(4c^2-1)\mu^{-1/2}+O(\mu^{-1})\right)
$$

$$+\frac{-1-i\sqrt{3}}{2}\left(\mu-\sqrt{3}\mu^{1/2}-\frac{2c^2+1}{2}-\frac{\sqrt{3}}{8}(4c^2-1)\mu^{-1/2}+O(\mu^{-1})\right)\Bigg]$$
$$=-\frac{1}{3}\left[\frac{3}{2}+3i\mu^{1/2}+O(\mu^{-1/2})\right]$$
$$=-\frac{1}{2}-i\mu^{1/2}+O(\mu^{-1/2}),$$

$$m_3=-\frac{1}{3}\left(a+\frac{(-1-i\sqrt{3})}{2}C+\frac{(-1+i\sqrt{3})}{2}\frac{D_0}{C}\right)$$
$$=-\frac{1}{3}\Bigg[\mu-c^2+1+\frac{-1-i\sqrt{3}}{2}\left(\mu+\sqrt{3}\mu^{1/2}-\frac{2c^2+1}{2}+\frac{\sqrt{3}}{8}(4c^2-1)\mu^{-1/2}+O(\mu^{-1})\right)$$
$$+\frac{-1+i\sqrt{3}}{2}\left(\mu-\sqrt{3}\mu^{1/2}-\frac{2c^2+1}{2}-\frac{\sqrt{3}}{8}(4c^2-1)\mu^{-1/2}+O(\mu^{-1})\right)\Bigg]$$
$$=-\frac{1}{3}\left[\frac{3}{2}-3i\mu^{1/2}+O(\mu^{-1/2})\right]$$
$$=-\frac{1}{2}+i\mu^{1/2}+O(\mu^{-1/2}).$$

Together, we write

$$\begin{cases} m_1=-\mu+c^2+O(\mu^{-1}),\\ m_2=-\dfrac{1}{2}-i\mu^{1/2}+O(\mu^{-1/2}),\\ m_3=-\dfrac{1}{2}+i\mu^{1/2}+O(\mu^{-1/2}),\end{cases}\tag{4.148}$$

with $\mu=-\lambda$ as mentioned earlier. Since, for large modulus of μ, the roots m_1, m_2 and m_3 are distinct, we can write the general solution to the equation (4.145a) as

$$\eta(x)=C_1e^{m_1x}+C_2e^{m_2x}+C_3e^{m_3x},\quad x\in(0,1),\tag{4.149}$$

for some constants $C_1, C_2, C_3\in\mathbb{C}$.

Using the boundary conditions (4.145b), we get a system of linear equations in C_1, C_2 and C_3, given by

$$\begin{cases} C_1+C_2+C_3=0,\\ C_1e^{m_1}+C_2e^{m_2}+C_3e^{m_3}=0,\\ C_1m_1^2\left(1-e^{m_1}\right)+C_2m_2^2\left(1-e^{m_2}\right)+C_3m_3^2\left(1-e^{m_3}\right)=0.\end{cases}\tag{4.150}$$

These system of equations (4.150) has a nontrivial solution if and only if

$$\det\begin{pmatrix}1&1&1\\ e^{m_1}&e^{m_2}&e^{m_3}\\ m_1^2\left(1-e^{m_1}\right)&m_2^2\left(1-e^{m_2}\right)&m_3^2\left(1-e^{m_3}\right)\end{pmatrix}=0.$$

Expanding the determinant, we obtain

$$m_1^2\left(1-e^{m_1}\right)\left(e^{m_3}-e^{m_2}\right)+m_2^2\left(1-e^{m_2}\right)\left(e^{m_1}-e^{m_3}\right)+m_3^2\left(1-e^{m_3}\right)\left(e^{m_2}-e^{m_1}\right)=0.\tag{4.151}$$

We shall now compute the determinant term by term for large $|\mu|$.

- Plugging the values of m_1, m_2 and m_3 as given in (4.148), we obtain

$$m_1^2\left(1-e^{m_1}\right)\left(e^{m_3}-e^{m_2}\right)\tag{4.152}$$

$$
\begin{aligned}
&= \left(-\mu + c^2 + O(\mu^{-1/2})\right)^2 \left(1 - e^{-\mu+c^2+O(\mu^{-1/2})}\right)\left(e^{-1/2+i\mu^{1/2}+O(\mu^{-1/2})} - e^{-1/2-i\mu^{1/2}+O(\mu^{-1/2})}\right)\\
&= \left(\mu^2 - 2c^2\mu + O(\mu^{1/2})\right)\left(1 - e^{-\mu+c^2+O(\mu^{-1})}\right)\left(e^{-1/2+O(\mu^{-1/2})}\left(\cos(\mu^{1/2}) + i\sin(\mu^{1/2})\right)\right.\\
&\qquad \left. -e^{-1/2+O(\mu^{-1/2})}\left(\cos(\mu^{1/2}) - i\sin(\mu^{1/2})\right)\right)\\
&= \left(\mu^2 - 2c^2\mu + O(\mu^{1/2})\right)\left(1 - e^{-\mu+c^2+O(\mu^{-1})}\right)\Big[O(\mu^{-1/2})e^{-1/2+O(\mu^{-1/2})}\cos(\mu^{1/2})\\
&\qquad +i(2+O(\mu^{-\frac{1}{2}}))e^{-1/2+O(\mu^{-1/2})}\sin(\mu^{1/2})\Big],
\end{aligned}
$$

where we have used the facts that

$$
e^{-1/2+O(\mu^{-1/2})} - e^{-1/2+O(\mu^{-1/2})} = e^{-1/2+O(\mu^{-1/2})}\left(1 - e^{O(\mu^{-\frac{1}{2}})}\right) = e^{-1/2+O(\mu^{-1/2})} \times O(\mu^{-\frac{1}{2}}),
$$

and

$$
e^{-1/2+O(\mu^{-1/2})} + e^{-1/2+O(\mu^{-1/2})} = e^{-1/2+O(\mu^{-1/2})}\left(1 + e^{O(\mu^{-\frac{1}{2}})}\right) = e^{-1/2+O(\mu^{-1/2})} \times (2 + O(\mu^{-\frac{1}{2}})).
$$

- We also compute

$$
\begin{aligned}
&m_2^2\,(1 - e^{m_2})\,(e^{m_1} - e^{m_3})\\
&= \left(-\frac{1}{2} - i\mu^{1/2} + O(\mu^{-\frac{1}{2}})\right)^2\left(1 - e^{-1/2-i\mu^{1/2}+O(\mu^{-1/2})}\right)\left(e^{-\mu+c^2+O(\mu^{-1})} - e^{-\frac{1}{2}+i\mu^{\frac{1}{2}}+O(\mu^{-\frac{1}{2}})}\right)\\
&= \left(-\mu + i\mu^{\frac{1}{2}} + O(1)\right)\left(e^{-\mu+c^2+O(\mu^{-1})} + e^{-1+O(\mu^{-\frac{1}{2}})} - e^{-\mu+c^2-\frac{1}{2}-i\mu^{\frac{1}{2}}+O(\mu^{-\frac{1}{2}})} - e^{-\frac{1}{2}+i\mu^{\frac{1}{2}}+O(\mu^{-\frac{1}{2}})}\right)\\
&= \left(-\mu + i\mu^{\frac{1}{2}} + O(1)\right)\Big[e^{-\mu+c^2+O(\mu^{-1})} + e^{-1+O(\mu^{-\frac{1}{2}})} - e^{-\mu+c^2-\frac{1}{2}+O(\mu^{-\frac{1}{2}})}\left(\cos(\mu^{\frac{1}{2}}) - i\sin(\mu^{\frac{1}{2}})\right)\\
&\qquad\qquad -e^{-\frac{1}{2}+O(\mu^{-\frac{1}{2}})}\left(\cos(\mu^{\frac{1}{2}}) + i\sin(\mu^{\frac{1}{2}})\right)\Big].
\end{aligned}
$$

- Finally, we have

$$
\begin{aligned}
&m_3^2\,(1 - e^{m_3})\,(e^{m_2} - e^{m_1})\\
&= \left(-\frac{1}{2} + i\mu^{1/2} + O(\mu^{-\frac{1}{2}})\right)^2\left(1 - e^{-\frac{1}{2}+i\mu^{\frac{1}{2}}+O(\mu^{-\frac{1}{2}})}\right)\times\left(e^{-\frac{1}{2}-i\mu^{\frac{1}{2}}+O(\mu^{-\frac{1}{2}})} - e^{-\mu+c^2+O(\mu^{-1})}\right)\\
&= \left(-\mu - i\mu^{\frac{1}{2}} + O(1)\right)\Big[-e^{-\mu+c^2+O(\mu^{-1})} - e^{-1+O(\mu^{-\frac{1}{2}})} + e^{-\mu+c^2-\frac{1}{2}+O(\mu^{-\frac{1}{2}})}\left(\cos(\mu^{\frac{1}{2}}) + i\sin(\mu^{\frac{1}{2}})\right)\\
&\qquad\qquad +e^{-\frac{1}{2}+O(\mu^{-\frac{1}{2}})}\left(\cos(\mu^{\frac{1}{2}}) - i\sin(\mu^{\frac{1}{2}})\right)\Big].
\end{aligned}
$$

- We add now the last two terms, in what follows

$$
\begin{aligned}
&m_2^2\,(1 - e^{m_2})\,(e^{m_1} - e^{m_3}) + m_3^2\,(1 - e^{m_3})\,(e^{m_2} - e^{m_1}) \qquad (4.153)\\
&= -\mu\left(e^{-\mu+c^2+O(\mu^{-1})} + e^{-1+O(\mu^{-\frac{1}{2}})} - e^{-\mu+c^2+O(\mu^{-\frac{1}{2}})} - e^{-1+O(\mu^{-\frac{1}{2}})}\right) + i\mu^{\frac{1}{2}}\left(2 + O(\mu^{-\frac{1}{2}})\right)e^{-\mu+c^2+O(\mu^{-1})}\\
&+ i\mu^{\frac{1}{2}}(2 + O(\mu^{-\frac{1}{2}}))e^{-1+O(\mu^{-\frac{1}{2}})} + \cos\mu^{\frac{1}{2}}\Big[\big(-\mu - i\mu^{\frac{1}{2}} + O(1)\big)\Big(e^{-\mu+c^2-\frac{1}{2}+O(\mu^{-\frac{1}{2}})} + e^{-\frac{1}{2}+O(\mu^{-\frac{1}{2}})}\Big)\\
&\qquad - \big(-\mu + i\mu^{\frac{1}{2}} + O(1)\big)\Big(e^{-\mu+c^2-\frac{1}{2}+O(\mu^{-\frac{1}{2}})} + e^{-\frac{1}{2}+O(\mu^{-\frac{1}{2}})}\Big)\Big]\\
&\quad + i\sin\mu^{\frac{1}{2}}\Big[\big(-\mu - i\mu^{\frac{1}{2}} + O(1)\big)\Big(e^{-\mu+c^2-\frac{1}{2}+O(\mu^{-\frac{1}{2}})} - e^{-\frac{1}{2}+O(\mu^{-\frac{1}{2}})}\Big)\\
&\qquad + \big(-\mu + i\mu^{\frac{1}{2}} + O(1)\big)\Big(e^{-\mu+c^2-\frac{1}{2}+O(\mu^{-\frac{1}{2}})} - e^{-\frac{1}{2}+O(\mu^{-\frac{1}{2}})}\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
=& -\mu O(\mu^{-\frac{1}{2}})e^{-\mu+c^2+O(\mu^{-1})} - \mu O(\mu^{-\frac{1}{2}})e^{-1+O(\mu^{-\frac{1}{2}})} + \big(2i\mu^{\frac{1}{2}} + O(1)\big)e^{-\mu+c^2+O(\mu^{-1})} \\
&+ \big(2i\mu^{\frac{1}{2}} + O(1)\big)e^{-1+O(\mu^{-\frac{1}{2}})} + i\sin\mu^{\frac{1}{2}}\Big[\big(-2\mu + O(\mu^{\frac{1}{2}})\big)e^{-\mu+c^2-\frac{1}{2}+O(\mu^{-\frac{1}{2}})} \\
&+ i\mu^{\frac{1}{2}}O(\mu^{-\frac{1}{2}})e^{-\mu+c^2-\frac{1}{2}+O(\mu^{-\frac{1}{2}})} + \big(2\mu + O(\mu^{\frac{1}{2}})\big)e^{-\frac{1}{2}+O(\mu^{-\frac{1}{2}})} + i\mu^{\frac{1}{2}}O(\mu^{-\frac{1}{2}})e^{-\frac{1}{2}+O(\mu^{-\frac{1}{2}})}\Big] \\
&+ \cos\mu^{\frac{1}{2}}\Big[\big(\mu + O(1)\big)O(\mu^{-\frac{1}{2}})e^{-\mu+c^2-\frac{1}{2}+O(\mu^{-\frac{1}{2}})} + \big(\mu + O(1)\big)O(\mu^{-\frac{1}{2}})e^{-\frac{1}{2}+O(\mu^{-\frac{1}{2}})} \\
&\quad - i\mu^{\frac{1}{2}}(2 + O(\mu^{-\frac{1}{2}}))e^{-\mu+c^2-\frac{1}{2}+O(\mu^{-\frac{1}{2}})} - i\mu^{\frac{1}{2}}(2 + O(\mu^{-\frac{1}{2}}))e^{-\frac{1}{2}+O(\mu^{-\frac{1}{2}})}\Big].
\end{aligned}
$$

We get after adding (4.152) and (4.153),

$$
\begin{aligned}
& m_1^2\,(1-e^{m_1})\,(e^{m_3}-e^{m_2}) + m_2^2\,(1-e^{m_2})\,(e^{m_1}-e^{m_3}) + m_3^2\,(1-e^{m_3})\,(e^{m_2}-e^{m_1}) \\
=& \cos\mu^{\frac{1}{2}}(\mu^2 - 2c^2\mu + O(\mu^{\frac{1}{2}}))O(\mu^{-\frac{1}{2}})\left(e^{-\frac{1}{2}+O(\mu^{-\frac{1}{2}})} - e^{-\mu+c^2-\frac{1}{2}+O(\mu^{-\frac{1}{2}})}\right) \\
&+ i\sin\mu^{\frac{1}{2}}(2 + O(\mu^{-\frac{1}{2}}))(\mu^2 - 2c^2\mu + O(\mu^{-\frac{1}{2}}))\left(e^{-\frac{1}{2}+O(\mu^{-\frac{1}{2}})} - e^{-\mu+c^2-\frac{1}{2}+O(\mu^{-\frac{1}{2}})}\right) \\
&- \mu O(\mu^{-\frac{1}{2}})e^{-\mu+c^2+O(\mu^{-1})} - \mu O(\mu^{-\frac{1}{2}})e^{-1+O(\mu^{-\frac{1}{2}})} \\
&+ \big(2i\mu^{\frac{1}{2}} + O(1)\big)e^{-\mu+c^2+O(\mu^{-1})} + \big(2i\mu^{\frac{1}{2}} + O(1)\big)e^{-1+O(\mu^{-\frac{1}{2}})} \\
&+ \cos\mu^{\frac{1}{2}}\Big[\big(\mu + O(1)\big)O(\mu^{-\frac{1}{2}})e^{-\mu+c^2-\frac{1}{2}+O(\mu^{-\frac{1}{2}})} + \big(\mu + O(1)\big)O(\mu^{-\frac{1}{2}})e^{-\frac{1}{2}+O(\mu^{-\frac{1}{2}})} \\
&- i\mu^{\frac{1}{2}}(2 + O(\mu^{-\frac{1}{2}}))e^{-\mu+c^2-\frac{1}{2}+O(\mu^{-\frac{1}{2}})} - i\mu^{\frac{1}{2}}(2 + O(\mu^{-\frac{1}{2}}))e^{-\frac{1}{2}+O(\mu^{-\frac{1}{2}})}\Big] \\
&\quad + i\sin\mu^{\frac{1}{2}}\Big[\big(-2\mu + O(\mu^{\frac{1}{2}})\big)e^{-\mu+c^2-\frac{1}{2}+O(\mu^{-\frac{1}{2}})} + i\mu^{\frac{1}{2}}O(\mu^{-\frac{1}{2}})e^{-\mu+c^2-\frac{1}{2}+O(\mu^{-\frac{1}{2}})} \\
&\quad + \big(2\mu + O(\mu^{\frac{1}{2}})\big)e^{-\frac{1}{2}+O(\mu^{-\frac{1}{2}})} + i\mu^{\frac{1}{2}}O(\mu^{-\frac{1}{2}})e^{-\frac{1}{2}+O(\mu^{-\frac{1}{2}})}\Big] \\
=& -\mu O(\mu^{-\frac{1}{2}})e^{-\mu+c^2+O(\mu^{-1})} - \mu O(\mu^{-\frac{1}{2}})e^{-1+O(\mu^{-\frac{1}{2}})} \\
&\quad + \big(2i\mu^{\frac{1}{2}} + O(1)\big)e^{-\mu+c^2+O(\mu^{-1})} + \big(2i\mu^{\frac{1}{2}} + O(1)\big)e^{-1+O(\mu^{-\frac{1}{2}})} \\
&\quad + i\sin\mu^{\frac{1}{2}}\left[\left(-2\mu^2 + O(\mu^{\frac{3}{2}})\right)e^{-\mu+c^2-\frac{1}{2}+O(\mu^{-\frac{1}{2}})} + \left(2\mu^2 + O(\mu^{\frac{3}{2}})\right)e^{-\frac{1}{2}+O(\mu^{-\frac{1}{2}})}\right] \\
&\quad + \cos\mu^{\frac{1}{2}}\left[\left(-\mu^2O(\mu^{-\frac{1}{2}}) + (2c^2+1)\mu O(\mu^{-\frac{1}{2}}) - 2i\mu^{\frac{1}{2}} + O(1)\right)e^{-\mu+c^2-\frac{1}{2}+O(\mu^{-\frac{1}{2}})}\right. \\
&\qquad\qquad \left.+ \left(\mu^2O(\mu^{-\frac{1}{2}}) - \mu O(\mu^{-\frac{1}{2}}) - 2i\mu^{\frac{1}{2}} + O(1)\right)e^{-\frac{1}{2}+O(\mu^{-\frac{1}{2}})}\right].
\end{aligned}
$$

Now, replacing the above quantity in the equation (4.151), and then dividing it by μ^2 (since $\mu \neq 0$), we obtain the equation

$$
F(\mu) = 0, \tag{4.154}
$$

where

$$
\begin{aligned}
F(\mu) =& -2\sin\mu^{\frac{1}{2}}\left(e^{-\mu+c^2} - 1\right) + O(\mu^{-\frac{1}{2}})\sin\mu^{\frac{1}{2}}\,e^{-\mu+c^2+O(\mu^{-\frac{1}{2}})} + O(\mu^{-\frac{1}{2}})\sin\mu^{\frac{1}{2}}\,e^{O(\mu^{-\frac{1}{2}})} \\
&+ \cos\mu^{\frac{1}{2}}\left[O(\mu^{-\frac{1}{2}})e^{-\mu+c^2+O(\mu^{-\frac{1}{2}})} + O(\mu^{-\frac{1}{2}})e^{O(\mu^{-\frac{1}{2}})}\right] \\
&+ O(\mu^{-\frac{3}{2}})e^{-\mu+c^2+\frac{1}{2}+O(\mu^{-\frac{1}{2}})} + O(\mu^{-\frac{3}{2}})e^{-\frac{1}{2}+O(\mu^{-\frac{1}{2}})}.
\end{aligned}
$$

Application of Rouche's theorem Let G be a function of μ, defined as

$$G(\mu) = -2\sin(\mu^{\frac{1}{2}})\left(e^{-\mu+c^2} - 1\right).$$

Then

$$\begin{aligned} F(\mu) - G(\mu) =& \underbrace{\sin(\mu^{\frac{1}{2}})\left(O(\mu^{-\frac{1}{2}})e^{-\mu+c^2+O(\mu^{-\frac{1}{2}})} + O(\mu^{-\frac{1}{2}})e^{O(\mu^{-\frac{1}{2}})}\right)}_{I_1} \\ & \underbrace{+\cos(\mu^{\frac{1}{2}})\left(O(\mu^{-\frac{1}{2}})e^{-\mu+c^2+O(\mu^{-\frac{1}{2}})} + O(\mu^{-\frac{1}{2}})e^{O(\mu^{-\frac{1}{2}})}\right)}_{I_2} \\ & + \underbrace{O(\mu^{-\frac{3}{2}})e^{-\mu+c^2+\frac{1}{2}+O(\mu^{-\frac{1}{2}})} + O(\mu^{-\frac{3}{2}})e^{-\frac{1}{2}+O(\mu^{-\frac{1}{2}})}}_{I_3} \\ :=& I_1 + I_2 + I_3. \end{aligned}$$

Since the function G has two branches of zeros, we will calculate them separately and in each case, we use the Rouche's theorem to talk about the zeros of the function F.

Case 1. We first observe that $\mu = k^2\pi^2$ is a zero of G for each $k \in \mathbb{N}^*$ and consider the following region in the complex plane

$$\mathcal{R}_k = \left\{z = x + iy \in \mathbb{C} \;:\; k\pi - \frac{\pi}{2} \le x \le k\pi + \frac{\pi}{2},\;\; -\frac{\pi}{2} \le y \le \frac{\pi}{2}\right\}, \quad \text{for } k \in \mathbb{N}^*. \tag{4.155}$$

Our goal is to prove that $|F(\mu) - G(\mu)| < |G(\mu)|$ on $\partial\mathcal{R}_k$. It is sufficient to prove that

$$\left|\frac{F(\mu) - G(\mu)}{G(\mu)}\right| \to 0 \quad \text{for } \mu \in \partial\mathcal{R}_k \text{ such that } \mathrm{Re}(\mu) \to +\infty. \tag{4.156}$$

To avoid difficulties in notations, we denote $w = \mu^{\frac{1}{2}}$ and without loss of generality, we simply write I_1, I_2 and I_3 as the functions w. Note that

$$\left|\frac{I_1(w)}{G(w)}\right| = \left|\frac{O(w^{-1})e^{-w^2+c^2+O(w^{-1})} + O(w^{-1})e^{O(w^{-1})}}{e^{-w^2+c^2} - 1}\right| \le \frac{C}{|w|}\frac{\left|e^{-w^2+c^2}\right| + 1}{\left|e^{-w^2+c^2} - 1\right|},$$

and since $\frac{\left|e^{-w^2+c^2}\right|+1}{\left|e^{-w^2+c^2}-1\right|}$ is bounded when $\mathrm{Re}(w) \to +\infty$, therefore

$$\left|\frac{I_1(w)}{G(w)}\right| \to 0, \quad \text{as } \mathrm{Re}(w) \to +\infty.$$

We now compute

$$\left|\frac{I_2(w)}{G(w)}\right| = \left|\frac{\cos(w)}{\sin(w)}\right| \frac{\left|O(w^{-1})e^{-w^2+c^2+O(w^{-1})} + O(w^{-1})e^{O(w^{-1})}\right|}{\left|e^{-w^2+c^2} - 1\right|} \le \frac{C}{|w|}\left|\frac{\cos(w)}{\sin(w)}\right| \frac{\left|e^{-w^2+c^2}\right| + 1}{\left|e^{-w^2+c^2} - 1\right|},$$

which yields

$$\left|\frac{I_2(w)}{G(w)}\right| \to 0, \quad \text{for } w \in \partial\mathcal{R}_k \text{ such that } \mathrm{Re}(w) \to +\infty,$$

because of the fact that $\left|\frac{\cos(w)}{\sin(w)}\right|$ is bounded on $\partial\mathcal{R}_k$. We can say similarly for the third term that

$$\left|\frac{I_3(w)}{G(w)}\right| \to 0, \quad \text{for } w \in \partial\mathcal{R}_k \text{ such that } \mathrm{Re}(w) \to +\infty,$$

as we have

$$\left|\frac{I_3(w)}{G(w)}\right| \leq \frac{C}{|w|^3}\left|\frac{1}{\sin(w)}\right| \frac{\left|e^{-w^2+c^2+\frac{1}{2}}\right|+1}{\left|e^{-w^2+c^2}-1\right|}.$$

Case 2. When $\sin(\mu^{\frac{1}{2}}) \neq 0$, $G(\mu)=0$ gives $e^{-\mu+c^2}-1=0$, that is $\mu = c^2+2ik\pi$ for $k\in\mathbb{Z}$. In this case, we consider the following region in the complex plane

$$\mathcal{S}_k = \left\{z = x+iy \in \mathbb{C} \ : \ c^2-\frac{\pi}{2} \leq x \leq c^2+\frac{\pi}{2}, \quad 2k\pi - \frac{\pi}{2} \leq y \leq 2k\pi+\frac{\pi}{2}\right\}. \tag{4.157}$$

We need to show that $|F(\mu)-G(\mu)| < |G(\mu)|$ on $\partial\mathcal{S}_k$. In particular, we prove that

$$\left|\frac{F(\mu)-G(\mu)}{G(\mu)}\right| \to 0 \ \text{ for } \mu\in\partial\mathcal{S}_k \text{ such that } \mathrm{Im}(\mu)\to+\infty.$$

We compute

$$\left|\frac{I_1(\mu)}{G(\mu)}\right| = \frac{1}{\left|\sin(\mu^{\frac{1}{2}})\right|}\left|\frac{O(\mu^{-\frac{1}{2}})e^{-\mu+c^2+O(\mu^{-\frac{1}{2}})}+O(\mu^{-1})e^{O(\mu^{-\frac{1}{2}})}}{e^{-\mu+c^2}-1}\right| \leq \frac{C}{|\mu|^{\frac{1}{2}}}\frac{1}{\left|\sin(\mu^{\frac{1}{2}})\right|}\frac{\left|e^{-\mu+c^2}\right|+1}{\left|e^{-\mu+c^2}-1\right|},$$

$$\left|\frac{I_2(\mu)}{G(\mu)}\right| = \left|\frac{\cos(\mu^{\frac{1}{2}})}{\sin(\mu^{\frac{1}{2}})}\right| \frac{\left|O(\mu^{-\frac{1}{2}})e^{-\mu+c^2+O(\mu^{-\frac{1}{2}})}+O(\mu^{-\frac{1}{2}})e^{O(\mu^{-\frac{1}{2}})}\right|}{\left|e^{-\mu+c^2}-1\right|} \leq \frac{C}{|\mu|^{\frac{1}{2}}}\left|\frac{\cos(\mu^{\frac{1}{2}})}{\sin(\mu^{\frac{1}{2}})}\right|\frac{\left|e^{-\mu+c^2}\right|+1}{\left|e^{-\mu+c^2}-1\right|},$$

and

$$\left|\frac{I_3(\mu)}{G(\mu)}\right| \leq \frac{C}{|\mu|^{\frac{3}{2}}}\frac{1}{\left|\sin(\mu^{\frac{1}{2}})\right|}\frac{\left|e^{-\mu+c^2+\frac{1}{2}}\right|+1}{\left|e^{-\mu+c^2}-1\right|}.$$

On $\partial\mathcal{S}_k$, $\left|\cos(\mu^{\frac{1}{2}})\right|$ and $\left|\sin(\mu^{\frac{1}{2}})\right|$ has both lower and upper bounds and $\frac{\left|e^{-\mu+c^2}\right|+1}{\left|e^{-\mu+c^2}-1\right|}$, $\frac{\left|e^{-\mu+c^2+\frac{1}{2}}\right|+1}{\left|e^{-\mu+c^2}-1\right|}$ are bounded. Therefore, for each $j=1,2,3$, we have

$$\left|\frac{I_j(\mu)}{G(\mu)}\right| \to 0, \quad \text{for } \mu\in\partial\mathcal{S}_k \text{ such that } \mathrm{Im}(\mu)\to+\infty.$$

Thus, combining the above two cases, we conclude that there exists some $k_0\in\mathbb{N}^*$ sufficiently large, such that

$$|F(\mu)-G(\mu)| < |G(\mu)|, \quad \forall\mu\in\partial\mathcal{R}_k\cup\partial\mathcal{S}_k \text{ and for large } k. \tag{4.158}$$

Since any two regions $\mathcal{R}_k$ and $\mathcal{R}_l$ are disjoint for $k\neq l$ and in each region $\mathcal{R}_k$, there is exactly one root of G (more precisely, the square-root of μ), the same is true for the function F, thanks to the Rouche's theorem. Similar phenomenon holds true in the region $\mathcal{S}_k$. To be more precise, we have the following.

On the region $\mathcal{R}_k$: parabolic part. For $k\geq k_0$, the function F has a unique root in $\mathcal{R}_k$ of the form

$$\mu_k^{\frac{1}{2}} = (k\pi+c_k)+id_k,$$

with $|c_k|$, $|d_k| \leq \frac{\pi}{2}$. Therefore, the first set of eigenvalues are given by

$$\lambda_k^p := -\mu_k := -k^2\pi^2 - 2c_k k\pi - 2id_k k\pi - (c_k^2-d_k^2) - 2ic_k d_k, \quad \forall k\geq k_0. \tag{4.159}$$

On the region $\mathcal{S}_k$: hyperbolic part. On the other hand, for $|k|\geq k_0$, the function F has a unique root in $\mathcal{S}_k$ of the form

$$\tilde{\mu}_k = c^2+\alpha_{1,k}+i(2k\pi+\alpha_{2,k}),$$

with $\left|\alpha_{1,k}\right|$, $\left|\alpha_{2,k}\right| \leq \frac{\pi}{2}$.

Therefore, the second set of eigenvalues are given by

$$\lambda_k^h := -\tilde{\mu}_k := -c^2-\alpha_{1,k}-i(2k\pi+\alpha_{2,k}), \quad \forall|k|\geq k_0. \tag{4.160}$$

This indeed proves the results (4.19a) and (4.19b) of our Lemma 4.3.1.

4.8.2 Computing the eigenfunctions for large frequencies

From the set of equations (4.150), one can obtain the following values of C_1, C_2, C_3

$$\begin{cases} C_1 = e^{m_2} - e^{m_3}, \\ C_2 = e^{m_3} - e^{m_1}, \\ C_3 = e^{m_1} - e^{m_2}. \end{cases} \tag{4.161}$$

Note that C_1, C_2 and C_3 cannot be simultaneously zero for large $|\mu|$. Once we have that, one can easily obtain the function $\eta(x)$, defined by (4.149),

$$\eta(x) = (e^{m_2} - e^{m_3})e^{m_1 x} + (e^{m_3} - e^{m_1})e^{m_2 x} + (e^{m_1} - e^{m_2})e^{m_3 x}, \quad \forall x \in (0,1). \tag{4.162}$$

We now compute the first and second derivatives of η which will let us obtain the other component ξ of the set of equations (4.138). We see

$$\eta'(x) = m_1(e^{m_2} - e^{m_3})e^{m_1 x} + m_2(e^{m_3} - e^{m_1})e^{m_2 x} + m_3(e^{m_1} - e^{m_2})e^{m_3 x},$$

$$\eta''(x) = m_1^2(e^{m_2} - e^{m_3})e^{m_1 x} + m_2^2(e^{m_3} - e^{m_1})e^{m_2 x} + m_3^2(e^{m_1} - e^{m_2})e^{m_3 x}.$$

Now, from equation (4.138), one can obtain

$$\eta''(x) + (1 - c^2)\eta'(x) + c\lambda\xi(x) = \lambda\eta(x),$$

and therefore, (writing $\mu = -\lambda$)

$$\begin{aligned} \xi(x) =& \frac{\eta''(x) + (1 - c^2)\eta'(x) + \mu\eta(x)}{c\mu} \\ =& \Big(\frac{m_1^2 + (1 - c^2)m_1 + \mu}{c\mu}\Big)(e^{m_2} - e^{m_3})e^{m_1 x} + \Big(\frac{m_2^2 + (1 - c^2)m_2 + \mu}{c\mu}\Big)(e^{m_3} - e^{m_1})e^{m_2 x} \\ &+ \Big(\frac{m_3^2 + (1 - c^2)m_3 + \mu}{c\mu}\Big)(e^{m_1} - e^{m_2})e^{m_3 x}. \end{aligned} \tag{4.163}$$

Set of eigenfunctions associated with λ_k^p For the set of eigenvalues $\{\lambda_k^p\}_{k \geq k_0}$ given by (4.159), we denote the eigenfunctions by $\Phi_{\lambda_k^p}$, $\forall k \geq k_0$, where we shall use the notation

$$\Phi_{\lambda_k^p}(x) = \begin{pmatrix} \xi_{\lambda_k^p}(x) \\ \eta_{\lambda_k^p}(x) \end{pmatrix}, \quad \forall k \geq k_0. \tag{4.164}$$

Computing $\eta_{\lambda_k^p}$. Let us recall the values of m_1, m_2 and m_3 from (4.148) and observe that $O(\mu_k^{-1/2}) = O(k^{-1})$. In what follows, we have their explicit expressions for all $k \geq k_0$ large enough, given by

$$\begin{cases} m_1 = -k^2\pi^2 - 2c_k k\pi - 2id_k k\pi + O(1), \\ m_2 = -\dfrac{1}{2} + d_k - i(k\pi + c_k) + O(k^{-1}), \\ m_3 = -\dfrac{1}{2} - d_k + i(k\pi + c_k) + O(k^{-1}). \end{cases} \tag{4.165}$$

where we have used the expression of $\mu = \mu_k$ from (4.159).

Recall the values of m_1, m_2, m_3, given by (4.165) and from the expression (4.162), we get that

$$\begin{aligned} \eta_{\lambda_k^p}(x) =& \left(e^{-\frac{1}{2}+d_k-i(k\pi+c_k)+O(k^{-1})} - e^{-\frac{1}{2}-d_k+i(k\pi+c_k)+O(k^{-1})}\right) e^{x\left(-k^2\pi^2-2c_k k\pi-2id_k k\pi+O(1)\right)} \\ &+ \left(e^{-\frac{1}{2}-d_k+i(k\pi+c_k)+O(k^{-1})} - e^{-k^2\pi^2-2c_k k\pi-2id_k k\pi+O(1)}\right) e^{x\left(-i(k\pi+c_k)-\frac{1}{2}+d_k+O(k^{-1})\right)} \end{aligned} \tag{4.166}$$

$$+\left(e^{-k^2\pi^2-2c_kk\pi-2id_kk\pi+O(1)}-e^{-\frac{1}{2}+d_k-i(k\pi+c_k)+O(k^{-1})}\right)e^{x\left(i(k\pi+c_k)-\frac{1}{2}-d_k+O(k^{-1})\right)},$$

for all $x \in (0,1)$ and for all $k \geq k_0$ large enough.

Computing $\xi_{\lambda_k^p}$. By using the values of m_1, m_2, m_3 from (4.165), we have

$$\begin{cases} m_1^2 = \left(-k^2\pi^2 - 2c_kk\pi - 2id_kk\pi + O(1)\right)^2 \\ = k^4\pi^4 + 4c_kk^3\pi^3 + 4id_kk^3\pi^3 + O(k^2), \\ m_2^2 = \left(-\frac{1}{2} + d_k - i(k\pi + c_k) + O(k^{-1})\right)^2 \\ = -k^2\pi^2 - 2c_kk\pi + ik\pi - 2id_kk\pi + O(1), \\ m_3^2 = \left(-\frac{1}{2} - d_k + i(k\pi + c_k) + O(k^{-1})\right)^2 \\ = -k^2\pi^2 - 2c_kk\pi - ik\pi - 2id_kk\pi + O(1), \end{cases}$$

for all $k \geq k_0$ large enough.

Also recall that, $\mu_k = -\lambda_k^p = k^2\pi^2 + 2c_kk\pi + 2id_kk\pi + O(1)$, using which we find

$$\frac{m_1^2 + (1-c^2)m_1 + \mu_k}{c\mu_k} = \frac{k^4\pi^4 + 4c_kk^3\pi^3 + 4id_kk^3\pi^3 + O(k^2)}{c(k^2\pi^2 + 2c_kk\pi + 2id_kk\pi + O(1))} = \frac{1}{c}k^2\pi^2 + O(k), \tag{4.167}$$

$$\frac{m_2^2 + (1-c^2)m_2 + \mu_k}{c\mu_k} = \frac{ic^2k\pi + O(1)}{c(k^2\pi^2 + 2c_kk\pi + 2id_kk\pi + O(1))} = \frac{ic}{k\pi} + O(k^{-2}), \tag{4.168}$$

$$\frac{m_3^2 + (1-c^2)m_3 + \mu_k}{c\mu_k} = -\frac{ic}{k\pi} + O(k^{-2}), \tag{4.169}$$

for all $k \geq k_0$ large enough.

Now, by using the quantities (4.167), (4.168) and (4.169) in the expression (4.163), we obtain

$$\begin{aligned} \xi_{\lambda_k^p}(x) = &\left(\frac{k^2\pi^2}{c} + O(k)\right)\left(e^{-i(k\pi+c_k)-\frac{1}{2}+d_k+O(k^{-1})} - e^{i(k\pi+c_k)-\frac{1}{2}-d_k+O(k^{-1})}\right) \times e^{x\left(-k^2\pi^2-2c_kk\pi-2id_kk\pi+O(1)\right)} \\ &+ \left(\frac{ic}{k\pi} + O(\frac{1}{k^2})\right)\left(e^{i(k\pi+c_k)+O(k^{-1})-\frac{1}{2}-d_k} - e^{-k^2\pi^2-2c_kk\pi-2id_kk\pi+O(1)}\right)e^{x\left(-i(k\pi+c_k)-\frac{1}{2}+d_k+O(k^{-1})\right)} \\ &- \left(\frac{ic}{k\pi} + O(\frac{1}{k^2})\right)\left(e^{-k^2\pi^2-2c_kk\pi-2id_kk\pi+O(1)} - e^{-i(k\pi+c_k)-\frac{1}{2}+d_k+O(k^{-1})}\right)e^{x\left(i(k\pi+c_k)-\frac{1}{2}-d_k+O(k^{-1})\right)}. \end{aligned} \tag{4.170}$$

Set of eigenfunctions associated with λ_k^h For the set of eigenvalues $\{\lambda_k^h\}_{|k|\geq k_0}$ given by (4.160), we denote the eigenfunctions by $\Phi_{\lambda_k^h}$, where we shall use the notation

$$\Phi_{\lambda_k^h}(x) = \begin{pmatrix} \xi_{\lambda_k^h}(x) \\ \eta_{\lambda_k^h}(x) \end{pmatrix}, \quad \forall |k| \geq k_0. \tag{4.171}$$

Computing $\eta_{\lambda_k^h}$. Recall that $\tilde{\mu}_k = -\lambda_k^h = c^2 + \alpha_{1,k} + i(2k\pi + \alpha_{2,k})$, for all $|k| \geq k_0$, so that Let us compute $\tilde{\mu}_k^{1/2}$. Assume $\tilde{\mu}_k^{1/2} = p_k + iq_k$, $p_k, q_k \in \mathbb{R}$ and $\tilde{\mu}_k = a_k + ib_k$, $a_k, b_k \in \mathbb{R}$, so that

$$(p_k + iq_k)^2 = (p_k^2 - q_k^2) + i2p_kq_k = a_k + ib_k,$$

From the fact that $q_k = \frac{b_k}{2p_k}$, we have

$$p_k^2 - q_k^2 = a_k \Longrightarrow 4p_k^4 - 4a_k p_k^2 - b_k^2 = 0,$$

and that yields

$$p_k = \left(\frac{\sqrt{a_k^2 + b_k^2} + a_k}{2} \right)^{\frac{1}{2}}, \quad q_k = \left(\frac{\sqrt{a_k^2 + b_k^2} - a_k}{2} \right)^{\frac{1}{2}}.$$

Now,

$$\sqrt{a_k^2 + b_k^2} = \left[(c^2 + \alpha_{1,k})^2 + (2k\pi + \alpha_{2,k})^2 \right]^{\frac{1}{2}} = \left[4k^2\pi^2 + O(k) \right]^{\frac{1}{2}} = 2|k\pi| + O(1), \quad \forall |k| \geq k_0.$$

Thus, it follows that

$$p_k = \sqrt{|k\pi|} + O(|k|^{-\frac{1}{2}}), \quad q_k = \pm\sqrt{|k\pi|} + O(|k|^{-\frac{1}{2}}), \quad \forall |k| \geq k_0. \tag{4.172}$$

we get

$$\tilde{\mu}_k^{1/2} = \sqrt{|k\pi|} + i\,\mathrm{sgn}(k)\sqrt{|k\pi|} + O(|k|^{-\frac{1}{2}}), \quad \forall |k| \geq k_0, \tag{4.173}$$

(the sign function sgn has been defined by (4.29)).

Then, using the characteristic roots m_1, m_2, m_3, given by (4.148), we get that

$$\begin{cases} m_1 = -\alpha_{1,k} - i(2k\pi + \alpha_{2,k}) + O(|k|^{-1}), \\ m_2 = -\dfrac{1}{2} + \mathrm{sgn}(k)\sqrt{|k\pi|} - i\sqrt{|k\pi|} + O(|k|^{-\frac{1}{2}}), \\ m_3 = -\dfrac{1}{2} - \mathrm{sgn}(k)\sqrt{|k\pi|} + i\sqrt{|k\pi|} + O(|k|^{-\frac{1}{2}}), \end{cases} \tag{4.174}$$

for all $|k| \geq k_0$ large enough.

Using the above information, we now write the expression of $\eta_{\lambda_k^h}(x)$ (we take the formulation after dividing by $k\pi e^{\sqrt{|k\pi|} + \frac{1}{\sqrt{|k|}}}$), given by

$$\begin{aligned} \eta_{\lambda_k^h}(x) = {} & \frac{1}{k\pi e^{\sqrt{|k\pi|} + \frac{1}{\sqrt{|k|}}}} \left(e^{\mathrm{sgn}(k)\sqrt{|k\pi|} - \frac{1}{2} - i\sqrt{|k\pi|} + O(|k|^{-\frac{1}{2}})} - e^{-\mathrm{sgn}(k)\sqrt{|k\pi|} - \frac{1}{2} + i\sqrt{|k\pi|} + O(|k|^{-\frac{1}{2}})} \right) \\ & \times e^{-x\left(\alpha_{1,k} + i(2k\pi + \alpha_{2,k}) + O(|k|^{-1})\right)} \\ & + \frac{1}{k\pi e^{\sqrt{|k\pi|} + \frac{1}{\sqrt{|k|}}}} \left(e^{-\mathrm{sgn}(k)\sqrt{|k\pi|} - \frac{1}{2} + i\sqrt{|k\pi|} + O(|k|^{-\frac{1}{2}})} - e^{-\alpha_{1,k} - i(2k\pi + \alpha_{2,k}) + O(|k|^{-1})} \right) \\ & \times e^{x\left(\mathrm{sgn}(k)\sqrt{|k\pi|} - \frac{1}{2} - i\sqrt{|k\pi|} + O(|k|^{-\frac{1}{2}})\right)} \\ & + \frac{1}{k\pi e^{\sqrt{|k\pi|} + \frac{1}{\sqrt{|k|}}}} \left(e^{-\alpha_{1,k} - i(2k\pi + \alpha_{2,k}) + O(|k|^{-1})} - e^{\mathrm{sgn}(k)\sqrt{|k\pi|} - \frac{1}{2} - i\sqrt{|k\pi|} + O(|k|^{-\frac{1}{2}})} \right) \\ & \times e^{x\left(-\mathrm{sgn}(k)\sqrt{|k\pi|} - \frac{1}{2} + i\sqrt{|k\pi|} + O(|k|^{-\frac{1}{2}})\right)}, \end{aligned} \tag{4.175}$$

for all $x \in (0, 1)$ and for all $|k| \geq k_0$.

Computing $\xi_{\lambda_k^h}$. By using the values of m_1, m_2, m_3 from (4.174), we calculate the following quantities for all $|k| \geq k_0$ large enough, namely

$$\begin{cases} m_1^2 = (-\alpha_{1,k} - i(2k\pi + \alpha_{2,k}) + O(|k|^{-1}))^2 \\ = -4k^2\pi^2 + 4ik\pi\alpha_{1,k} + O(k), \\ m_2^2 = (-\frac{1}{2} + \operatorname{sgn}(k)\sqrt{|k\pi|} - i\sqrt{|k\pi|} + O(|k|^{-\frac{1}{2}}))^2 \\ = -\operatorname{sgn}(k)\sqrt{|k\pi|} - 2i\operatorname{sgn}(k)|k\pi| + i\sqrt{|k\pi|} + O(1), \\ m_3^2 = (-\frac{1}{2} - \operatorname{sgn}(k)\sqrt{|k\pi|} + i\sqrt{|k\pi|} + O(|k|^{-\frac{1}{2}}))^2 \\ = \operatorname{sgn}(k)\sqrt{|k\pi|} - 2i\operatorname{sgn}(k)|k\pi| - i\sqrt{|k\pi|} + O(1). \end{cases}$$

Next, we compute the following: for all $|k| \geq k_0$ large enough,

$$\begin{aligned} \frac{m_1^2 + (1-c^2)m_1 + \tilde{\mu}_k}{c\tilde{\mu}_k} &= \frac{1}{c}\frac{(-4k^2\pi^2 + 4ik\pi\alpha_{1,k} + O(k))(c^2 + \alpha_{1,k} - i(2k\pi + \alpha_{2,k}))}{(c^2+\alpha_{1,k})^2 + (2k\pi + \alpha_{2,k})^2} \\ &= \frac{1}{c}\frac{-(c^2+\alpha_{1,k})4k^2\pi^2 + 8ik^3\pi^3 + O(k^2)}{4k^2\pi^2 + O(k)} \\ &= -\frac{\alpha_{1,k}}{c} + \frac{2ik\pi}{c} + O(1), \end{aligned} \tag{4.176}$$

$$\begin{aligned} \frac{m_2^2 + (1-c^2)m_2 + \tilde{\mu}_k}{c\tilde{\mu}_k} &= \frac{-c^2\operatorname{sgn}(k)\sqrt{|k\pi|} - 2i\operatorname{sgn}(k)|k\pi| + ic^2\sqrt{|k\pi|} + 2ik\pi + O(1)}{c(c^2 + \alpha_{1,k} + i(2k\pi + \alpha_{2,k}))} \\ = \frac{1}{c}&\frac{\Big(-c^2\operatorname{sgn}(k)\sqrt{|k\pi|} - 2i\operatorname{sgn}(k)|k\pi| + ic^2\sqrt{|k\pi|} + 2ik\pi + O(1)\Big)\Big(c^2 + \alpha_{1,k} - i(2k\pi + \alpha_{2,k})\Big)}{(c^2+\alpha_{1,k})^2 + (2k\pi + \alpha_{2,k})^2} \\ = \frac{1}{c}&\frac{2c^2(k\pi)^{3/2} + 2ic^2(k\pi)^{3/2} + O(k)}{4k^2\pi^2 + O(k)} \\ = \operatorname{sgn}&(k)\frac{c}{2\sqrt{|k\pi|}} + \frac{ic}{2\sqrt{|k\pi|}} + O\Big(\frac{1}{|k|}\Big), \end{aligned} \tag{4.177}$$

$$\frac{m_3^2 + (1-c^2)m_3 + \tilde{\mu}_k}{c\tilde{\mu}_k} \tag{4.178}$$

$$\begin{aligned} &= \frac{1}{c}\frac{c^2\Big(\operatorname{sgn}(k)\sqrt{|k\pi|} - 2i\operatorname{sgn}(k)|k\pi| - ic^2\sqrt{|k\pi|} + 2ik\pi + O(1)\Big)\Big(c^2 + \alpha_{1,k} - i(2k\pi + \alpha_{2,k})\Big)}{(c^2+\alpha_{1,k})^2 + (2k\pi + \alpha_{2,k})^2} \\ &= \frac{1}{c}\frac{-2c^2(k\pi)^{3/2} - 2ic^2(k\pi)^{3/2} + O(k)}{4k^2\pi^2 + O(k)} \\ &= -\operatorname{sgn}(k)\frac{c}{2\sqrt{|k\pi|}} - \frac{ic}{2\sqrt{|k\pi|}} + O\Big(\frac{1}{|k|}\Big). \end{aligned} \tag{4.179}$$

Using the quantities (4.176), (4.177) and (4.178) in the expression (4.163), we obtain the component $\xi_{\lambda_k^h}(x)$, for all $|k| \geq k_0$ (upon a division by $k\pi e^{\sqrt{|k\pi|} + \frac{1}{\sqrt{|k|}}}$),

$$\begin{aligned} \xi_{\lambda_k^h}(x) = &\Big(e^{\operatorname{sgn}(k)\sqrt{|k\pi|} - \frac{1}{2} - i\sqrt{|k\pi|} + O(|k|^{-\frac{1}{2}})} - e^{-\operatorname{sgn}(k)\sqrt{|k\pi|} - \frac{1}{2} + i\sqrt{|k\pi|} + O(|k|^{-\frac{1}{2}})}\Big) \\ &\times \frac{(-\alpha_{1,k} + 2ik\pi + O(1))}{ck\pi e^{\sqrt{|k\pi|} + \frac{1}{\sqrt{|k|}}}} \times e^{-x\big(\alpha_{1,k} + i(2k\pi + \alpha_{2,k}) + O(|k|^{-1})\big)} \\ &+ \Big(e^{-\operatorname{sgn}(k)\sqrt{|k\pi|} - \frac{1}{2} + i\sqrt{|k\pi|} + O(|k|^{-\frac{1}{2}})} - e^{-\alpha_{1,k} - i(2k\pi + \alpha_{2,k}) + O(|k|^{-1})}\Big) \\ &\times \frac{1}{k\pi e^{\sqrt{|k\pi|} + \frac{1}{\sqrt{|k|}}}}\Big(\operatorname{sgn}(k)\frac{c}{2\sqrt{|k\pi|}} + \frac{ic}{2\sqrt{|k\pi|}} + O\Big(\frac{1}{|k|}\Big)\Big) \times e^{x\big(\operatorname{sgn}(k)\sqrt{|k\pi|} - \frac{1}{2} - i\sqrt{|k\pi|} + O(|k|^{-\frac{1}{2}})\big)} \end{aligned}$$

$$+\left(e^{-\alpha_{1,k}-i(2k\pi+\alpha_{2,k})+O(|k|^{-1})}-e^{\mathrm{sgn}(k)\sqrt{|k\pi|}-\frac{1}{2}-i\sqrt{|k\pi|}+O(|k|^{-\frac{1}{2}})}\right)$$
$$\times\frac{1}{k\pi e^{\sqrt{|k\pi|}+\frac{1}{\sqrt{|k|}}}}\left(-\mathrm{sgn}(k)\frac{c}{2\sqrt{|k\pi|}}-\frac{ic}{2\sqrt{|k\pi|}}+O\Big(\frac{1}{|k|}\Big)\right)\times e^{x\left(-\mathrm{sgn}(k)\sqrt{|k\pi|}-\frac{1}{2}+i\sqrt{|k\pi|}+O(|k|^{-\frac{1}{2}})\right)},\tag{4.180}$$

We can now prove the last part of Lemma 4.3.1.

4.8.3 Proof of Lemma 4.3.1

We have already proved the existence of eigenvalues $\{\lambda_k^p\}_{k\geq k_0}$ (parabolic part) and $\{\lambda_k^h\}_{|k|\geq k_0}$ (hyperbolic part) by (4.159) and (4.160) respectively, which is the first part of Lemma 4.3.1.

It lefts to show the asymptotic properties of the sequences $\{c_k\}_{k\geq k_0}$, $\{d_k\}_{k\geq k_0}$ and $\{\alpha_{1,k}\}_{|k|\geq k_0}$, $\{\alpha_{1,k}\}_{|k|\geq k_0}$.

- Let us use the form of μ_k (i.e., of $-\lambda_k^p$) in the eigenvalue equation (4.154). Then, for large k, it is easy to observe that

$$\begin{aligned}F(\mu_k)&=2\sin(k\pi+c_k+id_k)+O(k^{-1})\\&=2(-1)^k\sin(c_k+id_k)+O(k^{-1}).\end{aligned}$$

  But μ_k is a root of F and thus

$$\sin(c_k+id_k)=O(k^{-1}),\quad\text{for large } k\geq k_0.\tag{4.181}$$

  Now, since $|\sin(c_k+id_k)|^2=\sin^2(c_k)+\sinh^2(d_k)$, we can write

$$\sin^2(c_k),\ \sinh^2(d_k)\leq\frac{C}{k^2},\quad\forall k\geq k_0\text{ large.}$$

  Therefore, $|c_k|^2,\ |d_k|^2\leq\frac{C}{k^2},\quad\forall k\geq k_0$, that is to say,

$$c_k,\ d_k=O(k^{-1}),\quad\text{for large } k\geq k_0,$$

  which gives the asymptotic formulation (4.19a) of λ_k^p given in Lemma 4.3.1.

- For the hyperbolic part $\{\lambda_k^h\}_{|k|\geq k_0}$, using the property $\xi_{\lambda_k^h}(0)=\xi_{\lambda_k^h}(1)$ ($\xi_{\lambda_k^h}$ is defined by (4.180)), we obtain that

$$\left(1-e^{-\alpha_{1,k}-i2k\pi-i\alpha_{2,k}+O(|k|^{-1})}\right)+O(|k|^{-1})=0,$$

  that is,

$$e^{-\alpha_{1,k}-i\alpha_{2,k}}=1+O(|k|^{-1}),\quad\text{for large } |k|\geq k_0.\tag{4.182}$$

  that is, there exists a $C>0$ such that

$$\left|e^{-\alpha_{1,k}-i\alpha_{2,k}}\right|\leq\left(1+\frac{C}{|k|}\right),\quad\forall\,|k|\geq k_0\text{ large.}$$

  As a consequence,

$$e^{-\alpha_{1,k}-i\alpha_{2,k}}\longrightarrow1,\quad\text{as }|k|\longrightarrow+\infty.$$

  But both $\alpha_{1,k}$ and $\{\alpha_{2,k}\}$ is bounded and therefore

$$\alpha_{1,k},\ \alpha_{2,k}\longrightarrow0,\quad\text{as }|k|\longrightarrow\infty.\tag{4.183}$$

Since $\left|e^{-\alpha_{1,k}-i\alpha_{2,k}}\right| = e^{-\alpha_{1,k}}$, we have $\left|\alpha_{1,k}\right| \leq \dfrac{C}{|k|}$, $\forall\,|k| \geq k_0$ large and that is

$$\alpha_{1,k} = O(k^{-1}), \quad \text{for large } |k| \geq k_0.$$

Using the above result, we get

$$e^{-i\alpha_{2,k}} = 1 + O(k^{-1}), \quad \text{for large } |k| \geq k_0.$$

But, one has $\left|e^{-i\alpha_{2,k}} - 1\right| = 2|\sin(\alpha_{2,k}/2)|$ and therefore,

$$\left|\alpha_{2,k}\right| \leq \frac{C}{|k|}, \quad \text{for large } |k| \geq k_0.$$

that is, $\alpha_{2,k} = O(|k|^{-1})$. This yields the asymptotic formulation (4.19b) of λ_k^h given in Lemma 4.3.1.

Finally, we recall that the existence of lower frequencies of eigenvalues are already given in Section 4.3.3.

Thus, the proof of Lemma 4.3.1 is complete.

4.8.4 Proof of Proposition 4.3.2–Part 1

In this portion, we shall simplify the expressions of the eigenfunctions (for large frequencies) using the properties of c_k, d_k, $\alpha_{1,k}, \alpha_{2,k}$ obtained in Section 4.8.3.

– *The parabolic part.* Recall the component $\xi_{\lambda_k^p}$ given by (4.170). By using the condition $\xi_{\lambda_k^p}(0) = \xi_{\lambda_k^p}(1)$, one can deduce that

$$\left(e^{-i(k\pi+c_k)-\frac{1}{2}+d_k+O(k^{-1})} - e^{i(k\pi+c_k)-\frac{1}{2}-d_k+O(k^{-1})}\right) = O\Big(\frac{1}{k^3}\Big), \quad \text{for large } k \geq k_0.$$

We further observe that (since c_k and d_k are or order $O(1/k)$)

$$\begin{aligned}
&e^{i(1-x)(k\pi+c_k+id_k)+O(k^{-1})} - e^{-i(1-x)(k\pi+c_k+id_k)+O(k^{-1})}\\
&= 2i\sin((1-x)(k\pi+c_k+id_k)) + O(k^{-1})\\
&\sim_{+\infty} 2i\sin(k\pi(1-x)) + O(k^{-1}).
\end{aligned}$$

Using the above ingredients in the expressions of $\eta_{\lambda_k^p}$ and $\xi_{\lambda_k^p}$ given by (4.166) and (4.170), we conclude that

$$\begin{aligned}
\eta_{\lambda_k^p}(x) &= e^{-\frac{1}{2}(1+x)}\sin(k\pi(1-x)) + O\left(\frac{1}{k}\right),\\
\xi_{\lambda_k^p}(x) &= \frac{ic}{k\pi}e^{-\frac{1}{2}(1+x)}\cos(k\pi(1-x)) + e^{x\left(-k^2\pi^2+O(1)\right)} \times O\left(\frac{1}{k}\right)\\
&\quad + O\left(\frac{1}{k^2}\right)e^{x\left(-k^2\pi^2-2c_kk\pi-2id_kk\pi+O(1)\right)} \times O\big(\frac{1}{k}\big)\\
&\quad + \left(\frac{ic}{k\pi} + O\big(\frac{1}{k^2}\big)\right)\left(e^{i(k\pi+c_k)+O(k^{-1})-\frac{1}{2}-d_k} - e^{-k^2\pi^2-2c_kk\pi-2id_kk\pi+O(1)}\right)e^{x\left(-i(k\pi+c_k)-\frac{1}{2}+d_k+O(k^{-1})\right)}\\
&\quad + \left(-\frac{ic}{k\pi} + O\big(\frac{1}{k^2}\big)\right)\left(e^{-k^2\pi^2-2c_kk\pi-2id_kk\pi+O(1)} - e^{-i(k\pi+c_k)-\frac{1}{2}+d_k+O(k^{-1})}\right)e^{x\left(i(k\pi+c_k)-\frac{1}{2}-d_k+O(k^{-1})\right)},
\end{aligned}$$

for all $x \in (0,1)$.

– *The hyperbolic part.* For the hyperbolic part, we simply use the fact: $\alpha_{1,k} = O(|k|^{-1})$, $\alpha_{2,k} = O(|k|^{-1})$ in the expression of the eigenfunctions (4.180) and (4.175), to obtain the required formulations (4.27) and (4.28).

4.8.5 Proof of Lemma 4.3.2: bounds of the eigenfunctions

In this section, we shall give the sketch of the estimates for $\xi_{\lambda_k^p}$, $\eta_{\lambda_k^p}$ for $k \geq k_0$ and $\xi_{\lambda_k^h}$, $\eta_{\lambda_k^h}$ for $|k| \geq k_0$. We use the interpolation results of Sobolev spaces to find the $(H^s_\sharp)'$ and H^{-s}-norms of the eigen-components.

We present the proof for $0 < s < 1$. In a similar way, one can prove the estimates for $s \geq 1$.

– *The parabolic part.* Recall the expressions of $\xi_{\lambda_k^p}$ and $\eta_{\lambda_k^p}$ from (4.24) and (4.25) respectively. Note that

$$\left\|\xi_{\lambda_k^p}\right\|_{L^2(0,1)} \leq \frac{C}{k} \quad \text{and} \quad \left\|\xi_{\lambda_k^p}\right\|_{(H^1_\sharp(0,1))'} \leq \frac{C}{k^2}, \quad \text{for } k \geq k_0 \text{ large.}$$

Therefore, using the interpolation between $(H^1_\sharp(0,1))'$ and L^2 spaces, we get for any $0 < s < 1$ (since $-s = s \times (-1) + (1-s) \times 0$),

$$\left\|\xi_{\lambda_k^p}\right\|_{(H^s_\sharp(0,1))'} \leq \left\|\xi_{\lambda_k^p}\right\|^{1-s}_{L^2(0,1)} \left\|\xi_{\lambda_k^p}\right\|^{s}_{(H^1_\sharp(0,1))'} \leq \frac{C}{|k|^{1+s}}, \quad \text{for } k \geq k_0 \text{ large.}$$

We also have

$$\left\|\eta_{\lambda_k^p}\right\|_{L^2(0,1)} \leq C \quad \text{and} \quad \left\|\eta_{\lambda_k^p}\right\|_{H^{-1}(0,1)} \leq \frac{C}{k}, \quad \text{for } k \geq k_0 \text{ large.}$$

Thus, for any $0 < s < 1$, we deduce that

$$\left\|\eta_{\lambda_k^p}\right\|_{H^{-s}(0,1)} \leq \left\|\eta_{\lambda_k^p}\right\|^{1-s}_{L^2(0,1)} \left\|\eta_{\lambda_k^p}\right\|^{s}_{H^{-1}(0,1)} \leq \frac{C}{|k|^{s}}, \quad \text{for } k \geq k_0 \text{ large.}$$

On the other hand, to find the lower bounds, first we observe that

$$\left\|\xi_{\lambda_k^p}\right\|_{L^2(0,1)} \geq \frac{C}{k} \quad \text{and} \quad \left\|\xi_{\lambda_k^p}\right\|_{H^1_\sharp(0,1)} \geq C, \quad \text{for } k \geq k_0 \text{ large.}$$

Now, using the interpolation between $(H^s_\sharp(0,1))'$ for $0 < s < 1$ and $H^1_\sharp(0,1)$, we obtain that (as $0 = \frac{1}{1+s} \times (-s) + \frac{s}{1+s} \times 1$)

$$\left\|\xi_{\lambda_k^p}\right\|_{L^2(0,1)} \leq \left\|\xi_{\lambda_k^p}\right\|^{\frac{1}{1+s}}_{(H^s_\sharp(0,1))'} \left\|\xi_{\lambda_k^p}\right\|^{\frac{s}{1+s}}_{H^1_\sharp(0,1)},$$

and therefore

$$\left\|\xi_{\lambda_k^p}\right\|_{(H^s_\sharp(0,1))'} \geq \left\|\xi_{\lambda_k^p}\right\|^{1+s}_{L^2(0,1)} \left\|\xi_{\lambda_k^p}\right\|^{-s}_{H^1_\sharp(0,1)} \geq \frac{C}{k^{1+s}},$$

for $k \geq k_0$ large enough.

Next, we have

$$\left\|\eta_{\lambda_k^p}\right\|_{L^2(0,1)} \geq C \quad \text{and} \quad \left\|\eta_{\lambda_k^p}\right\|_{H^1_0(0,1)} \geq Ck, \quad \text{for } k \geq k_0 \text{ large,}$$

and thus, by following the similar strategy as previous, we deduce that

$$\left\|\eta_{\lambda_k^p}\right\|_{H^{-s}(0,1)} \geq \left\|\eta_{\lambda_k^p}\right\|^{1+s}_{L^2(0,1)} \left\|\eta_{\lambda_k^p}\right\|^{-s}_{H^1_0(0,1)} \geq \frac{C}{k^{s}},$$

for $k \geq k_0$ large enough.

– *The hyperbolic part.* The steps will be exactly same as we analysed for the parabolic part. In this case, we have the following estimates:

$$C_1 \leq \left\|\xi_{\lambda_k^h}\right\|_{L^2(0,1)} \leq C_2, \quad \left\|\xi_{\lambda_k^h}\right\|_{(H^1_\sharp(0,1))'} \leq \frac{C}{|k|}, \quad \left\|\xi_{\lambda_k^h}\right\|_{H^1_\sharp(0,1)} \geq C\,|k|,$$
$$\frac{C_1}{|k|} \leq \left\|\eta_{\lambda_k^h}\right\|_{L^2(0,1)} \leq \frac{C_2}{|k|}, \quad \left\|\eta_{\lambda_k^h}\right\|_{H^{-1}(0,1)} \leq \frac{C}{|k|^2}, \quad \left\|\eta_{\lambda_k^h}\right\|_{H^1_0(0,1)} \geq C,$$

for large enough $k \geq k_0$.

Then, by following the interpolation arguments as previous, we can determine the required norm-estimates of $\xi_{\lambda_k^h}$ and $\eta_{\lambda_k^h}$, that is (4.32).

This completes the proof of Lemma 4.3.2.

4.9 Further remarks and conclusion

In the present chapter, we have proved the boundary null-controllability of our linearized 1D compressible Navier-Stokes system when a control acting either on the velocity or density part. For the velocity case, we have shown that when the initial states are chosen from the space $\dot{H}^{\frac{1}{2}}_\sharp(0,1)\times L^2(0,1)$, the system (4.4) is null-controllable at time $T > 1$. Moreover, for $0 \leq s < \frac{1}{2}$, the system fails to verify the null-controllability at any $T > 0$ in the space $\dot{H}^s_\sharp(0,1)\times L^2(0,1)$. Thus, the space is $\dot{H}^{\frac{1}{2}}_\sharp(0,1)\times L^2(0,1)$ is optimal w.r.t. the null-controllability of the system (4.4).

For the density case, we can even allow the $\dot{L}^2(0,1) \times L^2(0,1)$ initial states for the systems (4.5) and (4.6) to be null-controllable at time $T > 1$. We further proved that for small time, that is when $0 < T < 1$, the system (4.5) is no more null-controllable in the space $L^2(0,1) \times L^2(0,1)$.

In view of the above discussion, one immediate open question is the (non) null-controllability of the velocity case (the system (4.4)) or the full Dirichlet density case (system (4.6)) in small time $0 < T < 1$. We also cannot conclude the (non) null-controllability of the systems (4.4), (4.5) or (4.6) at the optimal time $T = 1$.

Let us make some final remarks related to our work.

- **Backward uniqueness and approximate controllability.** The backward uniqueness property tells that when the solution of a system (without any control) vanishes at some time $T > 0$, then it is identically zero at all time. This property plays an important role in the context of unique continuation and controllability.

  In this regard, we mention that the backward uniqueness is well-known for the cases when the associated operator forms a C^0-group (hyperbolic case), for instance the system

$$\begin{cases} \rho_t + \rho_x = 0, & \text{in } (0,T)\times(0,1), \\ \rho(t,0) = \rho(t,1), & t \in (0,T), \\ \rho(0,x) = \rho_0(x), & x \in (0,1), \end{cases}$$

  or an analytic semigroup (parabolic case), for instance the system

$$\begin{cases} u_t - u_{xx} = 0, & \text{in } (0,T)\times(0,1), \\ u(t,0) = u(t,1) = 0, & t \in (0,T), \\ u(0,x) = u_0(x), & x \in (0,1). \end{cases}$$

Let us come to our problem. Consider the following system without any control input,

$$\begin{cases} \rho_t + \rho_x + c u_x = 0 & \text{in } (0,T)\times(0,1), \\ u_t - u_{xx} + u_x + c\rho_x = 0 & \text{in } (0,T)\times(0,1), \\ \rho(t,0) = \rho(t,1) & \text{for } t\in(0,T), \\ u(t,0)=0, \quad u(t,1)=0 & \text{for } t\in(0,T), \\ \rho(0,x)=\rho_0(x), \quad u(0,x)=u_0(x) & \text{for } x\in(0,1). \end{cases} \tag{4.184}$$

Since the system (4.184) is of mixed nature (coupling between parabolic and hyperbolic components), the backward uniqueness question is interesting from the mathematical point of view. In fact, it has been indicated in [LRT01, AT08, AT10], that the backward uniqueness property is a delicate issue for the coupled parabolic-hyperbolic systems.

But in our case, the advantage is that the (generalized) eigenfunctions of the operator A forms a Riesz basis in $L^2(0,1)\times L^2(0,1)$ (see Remark 4.3.1). Also, we have that $(A, D(A))$ defines a strongly continuous semigroup in $L^2(0,1)\times L^2(0,1)$. As a result, we have the following: if the solution (ρ,u) to the system (4.184) satisfies

$$\rho(T,\cdot) = u(T,\cdot) = 0 \quad \text{in} \quad (0,1),$$

then we necessarily have

$$\rho_0 = u_0 = 0, \quad \text{in } (0,1), \quad \text{i.e.,} \quad \rho(t,x) = u(t,x) = 0 \quad \text{in} \quad (0,T)\times(0,1).$$

The above *backward uniqueness property* of (4.184), that is the free system of (4.4) (resp. (4.5)), together with the null-controllability of (4.4) (resp. (4.5)), we deduce the approximate controllability of the system (4.4) (resp. (4.5)) at time $T>1$ in the space $\dot{H}^{\frac{1}{2}}_{\sharp}(0,1)\times L^2(0,1)$ (resp. $\dot{L}^2(0,1)\times L^2(0,1)$).

Finally, the approximate controllability of the system (4.6) at time $T>1$ in the space $\dot{L}^2(0,1)\times L^2(0,1)$ follows from the null-controllability result Theorem 4.1.3 and the backward uniqueness of the free system associated to (4.6) (as proved in [Ren15]).

- **Growth bound of the semigroup and a stability result when** $(\rho_0,u_0)\in \dot{L}^2(0,1)\times L^2(0,1)$**.** Recall the space

$$\dot{L}^2(0,1) := \left\{\phi\in L^2(0,1) : \int_0^1 \phi = 0\right\}.$$

We shall point out some stability result associated with the system (4.184) (that is, without any control) when the initial data $(\rho_0,u_0)\in \dot{L}^2(0,1)\times L^2(0,1)$.

In this case, the operator A with its formal expression (4.7) has the domain

$$\mathcal{D}(A) = \left\{\Phi = (\xi,\eta)\in \dot{H}^1(0,1)\times H^2(0,1) : \xi(0)=\xi(1), \ \eta(0)=\eta(1)=0\right\}, \tag{4.185}$$

where $\dot{H}^1(0,1)$ contains all the functions in $H^1(0,1)$ with mean zero. Similarly, A^* has its formal expression as (4.9) with the same domain $\mathcal{D}(A^*)=\mathcal{D}(A)$ as of (4.185).

It is enough to obtain the growth bound of the semigroup $\{S^*(t)\}_{t\geq 0}$ generated by $(A^*,\mathcal{D}(A^*))$ in $L^2(0,1)\times L^2(0,1)$. Then, using the fact $\|S(t)\| = \|S^*(t)\|$ we can deduce the growth of the semigroup $\{S(t)\}_{t\geq 0}$ generated by $(A,\mathcal{D}(A))$ (in $L^2(0,1)\times L^2(0,1)$).

We first ensure that $\lambda=0$ cannot be an eigenvalue of A^* (or A) with the domain (4.185). If yes, then the associated eigenfunction will be $(1,0)$, but this is not possible since $(1,0)\notin\mathcal{D}(A^*)$. Also, observe that the first component of the eigenfunction of A^* (or A) corresponding to any eigenvalue has mean zero (in the light of Remark 4.8.1). As a consequence, in this case we can prove that the set of eigenfunctions of A^* (or A) with the domain given by (4.185) forms a Riesz basis for $\dot{L}^2(0,1) \times L^2(0,1)$ (using Theorem 4.3.1). So, $(A^*, \mathcal{D}(A^*))$ (or $(A, \mathcal{D}(A))$) is indeed a Riesz-spectral operator since there is no accumulation point of the set of eigenvalues of A^* (or A), see [CZ20, Chapter 3].

Now in one hand, since $\lambda \neq 0$, all the eigenvalues of A^* with domain (4.185) have negative real parts (see (4.142)), i.e.,

$$\mathrm{Re}(\lambda) < 0, \quad \forall \lambda \in \sigma(A^*).$$

On the other hand, thanks to Lemma 4.3.1, the set of parabolic and hyperbolic branches of the eigenvalues of A^* with domain (4.185) have the following asymptotics properties:

$$\begin{aligned} \lambda_k^p &= -k^2\pi^2 + O(1), \qquad \text{for large } k \geq k_0, \\ \lambda_k^h &= -c^2 - 2ik\pi + O(|k|^{-1}), \quad \text{for large } |k| \geq k_0. \end{aligned}$$

Thus, there exists some $\omega_0 \in [-c^2, 0)$ such that

$$\omega_0 = \sup \left\{ \mathrm{Re}(\lambda) : \lambda \in \sigma(A) \right\} < 0.$$

Now recall that $(A^*, \mathcal{D}(A^*))$ is a Riesz-spectral operator and so the semigroup $\{S^*(t)\}_{t\geq 0}$ generated by $(A^*, \mathcal{D}(A^*))$ has the following growth

$$\|S^*(t)\| \leq Ce^{\omega_0 t}, \quad \forall t \geq 0.$$

But, $\|S(t)\| = \|S^*(t)\|$ and therefore

$$\|S(t)\| \leq Ce^{\omega_0 t}, \quad \forall t \geq 0.$$

with $-c^2 \leq \omega_0 < 0$, which gives the exponential stability of the system (4.184) with initial data $(\rho_0, u_0) \in \dot{L}^2(0,1) \times L^2(0,1)$.

- **Characterization of the coefficient c.** We have proved the null-controllability of linearized compressible Navier-Stokes systems (4.4), (4.5) and (4.6) at a large time provided the coefficient b is small, in particular $b^4 + 8b^2 + 5 < 4\pi^2$. This condition ensures that all the eigenvalues of A^* has geometric multiplicity 1, thanks to Proposition 4.3.1-Part (iv). However, this is not a necessary condition for achieving null-controllability of the systems (4.4), (4.5) and (4.6). To be more precise, characterization of all $b > 0$ such that the systems (4.4), (4.5) and (4.6) are null-controllable at a large time is not obtained and it is a very difficult problem due to the complicated cubic polynomial (4.41). Equivalently, one can say that finding all $b > 0$ such that all the eigenvalues of A^* are geometrically simple is unknown.

- **A Dirichlet-Dirichlet system with control on velocity.** Recall that, when we considered a Dirichlet boundary control on velocity, then we have the assumption $\rho(t,0) = \rho(t,1)$ for the density part. It would be really interesting to deal with the full Dirichlet case when a control q acts on the velocity, that is the following system

$$\begin{cases} \rho_t + \rho_x + cu_x = 0 & \text{in } (0,T) \times (0,1), \\ u_t - u_{xx} + u_x + c\rho_x = 0 & \text{in } (0,T) \times (0,1), \\ \rho(t,0) = 0 & \text{for } t \in (0,T), \\ u(t,0) = 0, \ u(t,1) = q(t) & \text{for } t \in (0,T), \\ \rho(0,x) = \rho_0(x), \ u(0,x) = u_0(x) & \text{for } x \in (0,1). \end{cases} \tag{4.186}$$

This is really a challenging open problem to handle because of the difficulty in analyzing the spectral properties of the associated adjoint operator. This can be considered as a future work.

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Chapter **5**

Nonlinear Two-Parabolic System

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Abstract

This article is concerned with the local boundary null-controllability of a 1-D system of two-parabolic nonlinear equations (often referred as reaction-diffusion system) with coupled boundary conditions by means of a scalar control. The control force is exerted on one of the two state components through a Neumann condition at the left end of the boundary while the other component simply satisfies the homogeneous Neumann condition at that point. On the other hand, at the right end of the boundary, the states are coupled through the so-called δ'-type condition. Upon linearization around the stationary point $(0,0)$, we apply the well-known *moments method* to prove the global null-controllability of the associated linearized system with explicit control cost $Me^{M/T}$ as $T \to 0^+$. Then, we show the local null-controllability of the main system by employing the source term method developed in [LTT13] followed by the Banach fixed point theorem.

5.1 Introduction and main results

5.1.1 The system under consideration

In this paper, we address the boundary null-controllability result of a 2×2 nonlinear parabolic system with coupled boundary conditions by means of one Neumann boundary control. More precisely, for given finite time $T > 0$, we consider the following system

$$\begin{cases} y_t - y_{xx} = f\big(y, z, \int_0^1 y, \int_0^1 z\big), & \text{in } (0,T)\times(0,1), \\ z_t - z_{xx} = g\big(y, z, \int_0^1 y, \int_0^1 z\big), & \text{in } (0,T)\times(0,1), \\ y_x(t,0) = q(t), \quad z_x(t,0) = 0, & \text{for } t \in (0,T), \\ y_x(t,1) = z_x(t,1), & \text{for } t \in (0,T), \\ y(t,1) + z(t,1) + \alpha y_x(t,1) = 0, & \text{for } t \in (0,T), \\ y(0,x) = y_0(x), \quad z(0,x) = z_0(x), & \text{in } (0,1), \end{cases} \tag{5.1}$$

where $\alpha \geq 0$ is some real parameter and (y_0, z_0) is the given initial data which we choose from the space $[L^2(0,1)]^2$.

In the above system, a control function $q \in L^2(0,T)$ (to be determined) is applied through the Neumann condition of only one state (namely y) while the other state z simply satisfies the homogeneous Neumann boundary condition at the point $x = 0$. On the other hand, the states are coupled at the boundary point $x = 1$ in terms of the "equality condition of their normal derivatives" and a "combined Robin-type condition". In the literature, this kind of combined conditions (appearing at the point $x = 1$) is typically called the δ'-type condition, see for instance [BK13, p. 26, Chapter 1.4.4] or [Exn96]. In fact, it has been addressed in [Exn96] that the wavefunction of a quantum mechanical particle living on a graph often satisfies the δ'-type boundary conditions at the junction points.

The nonlinear functions f and g in (5.1) are given by

$$\begin{cases} f\big(y, z, \int_0^1 y, \int_0^1 z\big) & = -yz + ay^2 + bz^2 + r_1(t)y, \\ g\big(y, z, \int_0^1 y, \int_0^1 z\big) & = yz + cy^2 + dz^2 + r_2(t)z, \end{cases} \tag{5.2}$$

where a, b, c, d are $L^\infty((0,T)\times(0,1))$ functions and

$$\begin{cases} r_1(t) = \alpha_1 \displaystyle\int_0^1 \Big(\psi_{1,1}(x)y(t,x) + \psi_{2,1}(x)z(t,x)\Big)\, dx, \\ r_2(t) = \alpha_2 \displaystyle\int_0^1 \Big(\psi_{1,2}(x)y(t,x) + \psi_{2,2}(x)z(t,x)\Big)\, dx, \end{cases} \tag{5.3}$$

with α_1, α_2 are real constants and $\psi_{1,j}, \psi_{2,j} \in L^\infty(0,1)$ for $j = 1, 2$.

Observe that the nonlinear model (5.1)–(5.2) is actually a reaction-diffusion system which often describes several biological phenomenon or chemical reactions. In the literature, such system is commonly known as "Lotka-Volterra" model with diffusion (without any boundary conditions and control for the moment, let say), that sometimes characterize the dynamics of a biological system where two species: *prey* and *predator* interact between each other; see for instance [Per15, Jos14, Mur02]. In our model, we consider that the two species are interacting in the reference domain (through the nonlinear functions f, g) as well as at one boundary end (through the coupled conditions at $x = 1$). Then, our goal is to put an external control force only on one species from the other boundary end to locally control both the species at a given time T. In this regard, we refer the very detailed work [RBZ22], where several results concerning the controllability of reaction-diffusion systems in biology and social sciences have been addressed.

5.1.2 Bibliographic comments

The parabolic boundary control systems with less number of control(s) than equations can be a delicate issue in various situations and that there is lack of enough mathematical tools to tackle with these systems. In fact, unlike the scalar problems the boundary controllability for such systems is no longer equivalent with the distributed one, as it has been proven for instance in [FCGBdT10]. Moreover, the very powerful *Carleman technique* is often inefficient in that context. Among some fascinating works on coupled control systems, we point out [FCGBdT10] where the authors have proved a necessary and sufficient condition for boundary null-controllability of some 2×2 coupled parabolic system with single Dirichlet control. A more general result regarding the controllability to the trajectories of an $n\times n$ parabolic system with $m(< n)$ Dirichlet controls (applied on a part of a boundary) is available in [AKBGBdT11a]. In those works, the authors actually proved a general Kalman condition which is necessary and sufficient for their controllability results.

To the best of our knowledge, most of the boundary controllability results for a system with less controls than the equations are in 1-D and the reason behind is that the spectral analysis of the associated adjoint elliptic operator helps to deal with the so-called "moments technique" (initially developed by Fattorini and Russell [FR71, FR75]) to construct a control. In this regard, we mention that some multi-D (in cylindrical geometry) results have been developed in [BBGBO14, AB20], which need a sharp estimate of the control cost for the associated 1-D problem and a Lebeau-Robbiano spectral inequality for higher dimensions. We further refer to [AKBGBdT11b] where the authors made a survey of several recent results concerning the controllability of coupled parabolic systems.

The above references mainly address the parabolic systems with internal couplings. Let us mention that several systems with boundary couplings use to appear when one considers the system of pdes on metric graphs, e.g., [Lum80, KPS08, BK13]. Concerning the controllability issues for such systems, we first address [DZ06, Chapters 6, 8] where the authors have discussed some controllability results of wave, heat and Schrödinger systems in the network when some control(s) is (are) exerted on some of the vertices; see also the survey paper [Avd08]. We also refer the works [CIP18, CCV20, CCM20, ABP23] where several controllability results have been achieved in the setting of metric graph and certainly, in those works, the couplings are arisen in the junction points of the graph. Very recently, the boundary null-controllability of some interior-boundary coupled linear parabolic systems has been addressed in [BBHS21] where the boundary coupling is chosen by means of a Kirchhoff-type condition.

In the context of controllability of nonlinear systems, let us first mention [FI96, Sec. 4, Chap. I] by Fursikov and Imanuvilov where a small-time local null-controllability of semilinear heat equations has been proved using a perturbation argument. In 2000, Barbu [Bar00], independently Fernández-Cara and Zuazua [FCZ00] proved the small-time global null-controllability of semilinear heat equations where the nonlinear functions satisfy the growth condition $|s|\ln^{3/2}(1+|s|)$. More recently, the large-time global null-controllability has been established in [LB20a] for the nonlinearities F growing slower than $|s|\ln^2(1+|s|)$ verifying $sF(s)\geq 0$ and $\frac{1}{F}\in L^1([0,+\infty))$. Last but not the least, we mention [HSLB21] where the local null-controllability of a nonlocal semilinear heat equation has been intensively investigated along with numerical illustrations.

In the present work, we shall deal with the local null-controllability of the parabolic system (5.1) and, as far as we know, the δ'-type condition has not been treated in the literature from the control theoretic perspective. Moreover, we consider the nonlocal nonlinearities in this work.

5.1.3 Linearized system and functional setting

For any given boundary parameter $\alpha\geq 0$, the linearized system around the equilibrium point $(0,0)$ is given by

$$\begin{cases} y_t - y_{xx} = 0, & \text{in } (0,T)\times(0,1),\\ z_t - z_{xx} = 0, & \text{in } (0,T)\times(0,1),\\ y_x(t,0) = q(t), \quad z_x(t,0) = 0, & \text{for } t\in(0,T),\\ y_x(t,1) = z_x(t,1), & \text{for } t\in(0,T),\\ y(t,1)+z(t,1)+\alpha y_x(t,1) = 0, & \text{for } t\in(0,T),\\ y(0,x) = y_0(x), \quad z(0,x) = z_0(x), & \text{in } (0,1). \end{cases} \tag{5.4}$$

The free system, that is the set of equations (5.4) without any control input, can be written in the form of an infinite dimensional system of ordinary differential equations as follows

$$\begin{cases} Y'(t) + AY(t) = 0, \\ Y(0) = Y_0, \end{cases} \tag{5.5}$$

where $Y := (y, z)$, $Y_0 := (y_0, z_0)$ and the operator

$$A = \begin{pmatrix} -\partial_{xx} & 0 \\ 0 & -\partial_{xx} \end{pmatrix}, \tag{5.6}$$

with its domain

$$\mathcal{D}(A) = \Big\{(u, v) \in [H^2(0,1)]^2 \mid u'(0) = 0,\ v'(0) = 0,\ u'(1) = v'(1),$$
$$u(1) + v(1) + \alpha u'(1) = 0\Big\}.$$

Observe that the operator $(A, \mathcal{D}(A))$ is self-adjoint in nature but still we denote the adjoint of A by A^* for more clear presentation.

5.1.4 Notations

Throughout the paper, C denotes a generic positive constant that may change line to line but does not depend on the time T or on the initial data (y_0, z_0). We also denote the following Lebesgue spaces:

(i) $Z := [L^2(0,1)]^2$,

(ii) $\mathcal{H} := [H^1(0,1)]^2$,

(iii) $\mathcal{H}^*$ = dual of the space $\mathcal{H}$ with respect to the pivot space Z,

(iv) $H^1_{\{a\}}(0,1) = \{u \in H^1(0,1) : u(a) = 0\}$, for $a \in \{0,1\}$,

which shall be intensively used in the present work. The inner product in the space Z is simply denoted by $(\cdot,\cdot)_Z$ while we denote the dual product by $\langle \cdot,\cdot \rangle_{X^*,X}$ between the space X and its dual X^*. Sometimes, we write $\langle \cdot,\cdot \rangle_{\mathbb{R}^d}$ to denote the usual inner product in the space $\mathbb{R}^d$, $d \geq 1$. The characteristic function will be denoted by $\chi_{[a,b]}$ in the real interval $[a, b]$ with $a < b$.

5.1.5 Main results

We now write the main results of our present work.

5.1.5.1 Local null-controllability of the nonlinear system

We have the following controllability result for the system (5.1).

Theorem 5.1.1. *Let f and g be given by (5.2) and $\alpha \geq 0$. Then, the nonlinear system (5.1) is small-time locally null-controllable around the equilibrium $(0,0)$, that is to say, for any given time $T > 0$, there is a $\delta > 0$ such that for chosen initial state $(y_0, z_0) \in Z$ verifying $\|(y_0, z_0)\|_Z \leq \delta$, there exists a solution-control pair $((y,z), q)$ with $(y,z) \in C^0([0,T]; Z) \cap L^2(0,T;\mathcal{H})$ and $q \in L^2(0,T)$ to the system (5.1) satisfying*

$$(y(T,x), z(T,x)) = (0,0), \quad \forall x \in (0,1). \tag{5.7}$$

The strategy to prove Theorem 5.1.1 is the following:

- First, we prove the global boundary null-controllability result of the associated linear model (5.4) by using the method of moments ([FR75, FR71]) with a proper estimation of the control cost, precisely $Me^{M/T}\|(y_0, z_0)\|_Z$, where M is independent in T and (y_0, z_0).
- Next, by applying the source term method introduced in [LTT13], we prove a null-controllability result of the linearized model with additional source terms in $L^2(0, T; Z)$ which are exponentially decreasing as $t \to T^-$, and in this step, we notably use the precise control cost as prescribed earlier.
- Finally, we use the Banach fixed-point theorem to obtain the local (boundary) null-controllability for our nonlinear system (5.1).

5.1.5.2 Null-controllability of the linear system

Let us now state the global null-controllability result for the linearized system (5.4).

Theorem 5.1.2. *Let any $T > 0$, initial data $(y_0, z_0) \in Z$ and parameter $\alpha \geq 0$ be given. Then, there exists a control $q \in L^2(0, T)$ such that the solution (y, z) to the system (5.4) satisfies $(y(T, \cdot), z(T, \cdot)) = (0, 0)$ in $(0, 1)$. In addition, q satisfies the following estimate*

$$\|q\|_{L^2(0,T)} \leq Me^{M/T} \|(y_0, z_0)\|_Z, \tag{5.8}$$

where the constant $M > 0$ neither depends on T nor on (y_0, z_0).

5.1.6 Organization of the paper

- In Section 5.2, we discuss the required well-posedness results for the linear control problem (5.4) and its associated adjoint system (without any control input).
- Section 5.3 is devoted to prove the null-controllability of the linearized system (5.4). We study the spectral analysis for the associated adjoint operator in subsection 5.3.1, which is crucial to apply the method of moments to construct a null-control $q \in L^2(0, T)$ for the system (5.4) with a precise control cost as introduced earlier (see subsection 5.3.5).
- In Section 5.4, we prove the main result of our work, that is, Theorem 5.1.1.
- Finally, we conclude our paper by mentioning possible extension of this work to a more general internal-boundary coupled parabolic system related to the present paper, see Section 5.5.

5.2 Well-posedness of the linearized system

This section is devoted to prove the existence and uniqueness of solution to the linear control system (5.4).

5.2.1 Existence of analytic semigroup

Let us first prove the well-posedness of the following homogeneous system

$$\begin{cases} y_t - y_{xx} = g_1, & \text{in } (0, T) \times (0, 1), \\ z_t - z_{xx} = g_2, & \text{in } (0, T) \times (0, 1), \\ y_x(t, 0) = 0, \quad z_x(t, 0) = 0, & \text{for } t \in (0, T), \\ y_x(t, 1) = z_x(t, 1), & \text{for } t \in (0, T), \\ y(t, 1) + z(t, 1) + \alpha y_x(t, 1) = 0, & \text{for } t \in (0, T), \\ y(0, x) = y_0(x), \quad z(0, x) = z_0(x), & \text{in } (0, 1). \end{cases} \tag{5.9}$$

with given initial data $(y_0, z_0) \in Z$ and source term $(g_1, g_2) \in L^2(0,T;Z)$. We start by proving the existence of semigroup defined by $(-A, \mathcal{D}(A))$.

Proposition 5.2.1. *The operator $(-A, \mathcal{D}(A))$ defined in (5.6) forms an analytic semigroup in the space Z.*

Proof. We shall present the proof for the boundary parameter $\alpha > 0$. The case $\alpha = 0$ is simpler. We prove this result into two steps.

Step 1. Let us define the usual norm on $\mathcal{H}$, given by

$$\|(u,v)\|_{\mathcal{H}} = \left(\int_0^1 (|u(x)|^2 + |u'(x)|^2)dx + \int_0^1 (|v(x)|^2 + |v'(x)|^2)dx\right)^{\frac{1}{2}},$$

and the sesquilinear map $h : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ such that for any (u,v), $(\varphi,\psi) \in \mathcal{H}$

$$\begin{aligned} h((u,v),(\varphi,\psi)) = &\int_0^1 u'(x)\varphi'(x)dx + \int_0^1 v'(x)\psi'(x)dx \\ &+ \frac{1}{\alpha}[u(1)+v(1)][\varphi(1)+\psi(1)]. \end{aligned}$$

It follows that h is continuous on $\mathcal{H} \times \mathcal{H}$ with

$$|h((u,v),(\varphi,\psi))| \le c\,\|(u,v)\|_{\mathcal{H}}\,\|(\varphi,\psi)\|_{\mathcal{H}}, \quad \text{for all } (u,v),(\varphi,\psi) \in \mathcal{H},$$

where c is a positive constant depending on α. We also have

$$|h((u,v),(u,v))| \ge \|(u,v)\|^2_{\mathcal{H}} - \|(u,v)\|^2_{Z}, \quad \text{for all } (u,v) \in \mathcal{H}.$$

Therefore, by [Ouh05, Proposition 1.51 & Theorem 1.52], the *negative operator* associated with h generates an analytic semigroup in Z of angle $(\pi/2 - \arctan(c))$.

It remains to prove that the operator associated to h is indeed A with the domain $\mathcal{D}(A)$.

Step 2. Let us define the operator $(\tilde{A}, \mathcal{D}(\tilde{A}))$ associated with the map h as follows.

$$\begin{cases} \mathcal{D}(\tilde{A}) = \Big\{(\tilde{u},\tilde{v}) \in \mathcal{H} \mid \quad \exists\,(f_1,f_2) \in Z \ \text{ such that} \\ \qquad\qquad h((\tilde{u},\tilde{v}),(\varphi,\psi)) = ((f_1,f_2),(\varphi,\psi))_Z, \quad \forall(\varphi,\psi) \in \mathcal{H}\Big\}, \\ \tilde{A}(\tilde{u},\tilde{v}) := (f_1,f_2). \end{cases}$$

Part (i). Here we prove $\mathcal{D}(A) \subset \mathcal{D}(\tilde{A})$. Let $(u,v) \in \mathcal{D}(A)$. Then, for all $(\varphi,\psi) \in \mathcal{H}$, we have

$$h((u,v),(\varphi,\psi)) = \int_0^1 u'(x)\varphi'(x)dx + \int_0^1 v'(x)\psi'(x)dx + \frac{1}{\alpha}[u(1)+v(1)][\varphi(1)+\psi(1)].$$

Integrating by parts, we obtain

$$\begin{aligned} h((u,v),(\varphi,\psi)) = &-\int_0^1 u''(x)\varphi(x)dx - \int_0^1 v''(x)\psi(x)dx + u'(1)\varphi(1) + v'(1)\psi(1) \\ &+ \frac{1}{\alpha}[u(1)+v(1)][\varphi(1)+\psi(1)]. \end{aligned} \tag{5.10}$$

We also have that $u'(1) = v'(1)$ and $u(1) + v(1) = -\alpha u'(1)$. Therefore, we get from (5.10)

$$\begin{aligned} h((u,v),(\varphi,\psi)) &= -\int_0^1 u''(x)\varphi(x)dx - \int_0^1 v''(x)\psi(x)dx \\ &= (A(u,v),(\varphi,\psi))_Z. \end{aligned}$$

Thus, for given $(u,v) \in \mathcal{D}(A)$ we found a pair $(f_1,f_2) = A(u,v) \in Z$ such that $h((u,v),(\varphi,\psi)) = ((f_1,f_2),(\varphi,\psi))_Z$ for all $(\varphi,\psi) \in \mathcal{H}$. This implies $(u,v) \in \mathcal{D}(\tilde{A})$ and consequently, $\mathcal{D}(A) \subset \mathcal{D}(\tilde{A})$.

Part (ii). We now show that $\mathcal{D}(\tilde{A}) \subset \mathcal{D}(A)$. Let $(\tilde{u}, \tilde{v}) \in \mathcal{D}(\tilde{A})$. Then, there exists $(f_1, f_2) \in Z$ such that $h((\tilde{u}, \tilde{v}), (\varphi, \psi)) = ((f_1, f_2), (\varphi, \psi))_Z$, for all $(\varphi, \psi) \in \mathcal{H}$ with $\tilde{A}(\tilde{u}, \tilde{v}) = (f_1, f_2)$, and accordingly,

$$\int_0^1 \tilde{u}'(x)\varphi'(x)dx + \int_0^1 \tilde{v}'(x)\psi'(x)dx + \frac{1}{\alpha}[\tilde{u}(1) + \tilde{v}(1)][\varphi(1) + \psi(1)]$$
$$= \int_0^1 f_1(x)\varphi(x)dx + \int_0^1 f_2(x)\psi(x)dx,$$

for all $(\varphi, \psi) \in \mathcal{H}$. Since $f_1, f_2 \in L^2(0,1)$, by elliptic regularity theory, we have $u, v \in H^2(0,1)$. Thus, an integration by parts yields

$$\begin{aligned} &- \int_0^1 \tilde{u}''(x)\varphi(x)dx - \int_0^1 \tilde{v}''(x)\psi(x)dx + \tilde{u}'(1)\varphi(1) - \tilde{u}'(0)\varphi(0) + \tilde{v}'(1)\psi(1) \\ &- \tilde{v}'(0)\psi(0) + \frac{1}{\alpha}[\tilde{u}(1) + \tilde{v}(1)][\varphi(1) + \psi(1)] = \int_0^1 f_1(x)\varphi(x)dx + \int_0^1 f_2(x)\psi(x)dx, \end{aligned} \tag{5.11}$$

for all $(\varphi, \psi) \in \mathcal{H}$.

Let us first choose any $(\varphi, \psi) \in [H_0^1(0,1)]^2 \subset \mathcal{H}$ in (5.11) and as a result we deduce

$$f_1(x) = -\tilde{u}''(x), \quad f_2(x) = -\tilde{v}''(x), \quad \text{for a.a.} \quad x \in (0,1).$$

Once we have this, going back to (5.11), one has

$$\tilde{u}'(1)\varphi(1) - \tilde{u}'(0)\varphi(0) + \tilde{v}'(1)\psi(1) - \tilde{v}'(0)\psi(0) + \frac{1}{\alpha}[\tilde{u}(1) + \tilde{v}(1)][\varphi(1) + \psi(1)] = 0, \tag{5.12}$$

for all $(\varphi, \psi) \in \mathcal{H}$. Now consider any $(\varphi, \psi) \in H^1_{\{0\}}(0,1) \times H^1_0(0,1) \subset \mathcal{H}$, so that we have

$$\left(\tilde{u}'(1) + \frac{1}{\alpha}[\tilde{u}(1) + \tilde{v}(1)]\right)\varphi(1) = 0,$$

that is,

$$\tilde{u}(1) + \tilde{v}(1) + \alpha\tilde{u}'(1) = 0. \tag{5.13}$$

Next, by choosing any $(\varphi, \psi) \in H^1_{\{1\}}(0,1) \times H^1_0(0,1) \subset \mathcal{H}$ in (5.12) we obtain the condition

$$\tilde{u}'(0) = 0, \tag{5.14}$$

and similarly, the choice of any $(\varphi, \psi) \in H^1_0(0,1) \times H^1_{\{1\}}(0,1) \subset \mathcal{H}$ leads to the condition

$$\tilde{v}'(0) = 0. \tag{5.15}$$

Finally, by considering any $(\varphi, \psi) \in \mathcal{H}$ and utilizing the previous boundary conditions (5.13), (5.14) and (5.15), the equality (5.12) reduces to

$$(\tilde{v}'(1) - \tilde{u}'(1))\,\psi(1) = 0,$$

for all $\psi \in H^1(0,1)$ and this yields

$$\tilde{u}'(1) = \tilde{v}'(1). \tag{5.16}$$

Therefore $(\tilde{u}, \tilde{v}) \in \mathcal{D}(A)$, which proves $\mathcal{D}(\tilde{A}) \subset \mathcal{D}(A)$.

Hence, the operator associated with the sesquilinear form h is indeed $(A, \mathcal{D}(A))$. This completes the proof. □

We hereby denote the associated semigroup by $(e^{-tA})_{t\geq 0}$ and the following results hold.

Proposition 5.2.2. *Let any parameter $\alpha \geq 0$ be given. Then, for any $Y_0 := (y_0, z_0) \in \mathcal{D}(A)$ and $G := (g_1, g_2) \in C^1([0,T]; Z)$, there exists unique strong solution $Y := (y, z) \in C^0([0,T]; \mathcal{D}(A)) \cap C^1([0,T]; Z)$ to the system (5.9), given by*

$$Y(t) = e^{-tA} Y_0 + \int_0^t e^{-(t-s)A} G(s)\, ds. \tag{5.17}$$

Proposition 5.2.3. *Let any parameter $\alpha \geq 0$ be given. Then, for any $(y_0, z_0) \in Z$ and $(g_1, g_2) \in L^2(0,T;Z)$, there exists a unique weak solution*

$$(y, z) \in C^0([0,T]; Z) \cap L^2(0,T; \mathcal{H}) \cap H^1(0,T; \mathcal{H}^*)$$

to the system (5.9) which satisfies the following energy estimate

$$\begin{aligned} &\|(y,z)\|_{C^0([0,T];Z)} + \|(y,z)\|_{L^2(0,T;\mathcal{H})} + \|(y_t, z_t)\|_{L^2(0,T;\mathcal{H}^*)} \\ &\leq C e^{CT} \left(\|(y_0, z_0)\|_Z + \|(g_1, g_2)\|_{L^2(0,T;Z)} \right), \end{aligned} \tag{5.18}$$

where $C > 0$ is a constant that does not depend in $T > 0$.

Proof. For given initial state $(y_0, z_0) \in Z$ and source term $(g_1, g_2) \in L^2(0,T;Z)$, the existence of a unique weak solution $(y, z) \in C^0([0,T]; Z)$ can be ensured by applying Proposition 5.2.1. We just need to prove the energy estimate (5.18).

- We start with $(y_0, z_0) \in \mathcal{D}(A)$ and $(g_1, g_2) \in C^1([0,T]; Z)$. Then, the system (5.9) has a unique strong solution (y, z) in the space $C^0([0,T]; \mathcal{D}(A)) \cap C^1([0,T]; Z)$ as per Proposition 5.2.2. Taking the inner product in Z of (5.9) with (y, z), we get

  $$\frac{1}{2}\frac{d}{dt} \|(y(t), z(t))\|_Z^2 + (A(y(t), z(t)), (y(t), z(t)))_Z = ((g_1(t), g_2(t)), (y(t), z(t)))_Z, \quad \forall t \in [0,T].$$

  Integrating by parts w.r.t. space and by applying the Cauchy-Schwarz and Young's inequalities, we have

  $$\begin{aligned} &\frac{1}{2}\frac{d}{dt} \|(y(t), z(t))\|_Z^2 + \|(y(t), z(t))\|_{\mathcal{H}}^2 + \alpha |y'(t,1)|^2 \\ &\qquad \leq C \left(\|(g_1(t), g_2(t))\|_Z^2 + \|(y(t), z(t))\|_Z^2 \right), \quad \forall t \in [0,T]. \end{aligned} \tag{5.19}$$

  Here we recall that $\alpha \geq 0$, and then using Gronwall's lemma (see [Eva10, Appendix B.2]) one can obtain the required estimate (5.18) for the quantity $\|(y,z)\|_{C^0([0,T];Z)}$. Then, by integrating (5.19) over $[0,T]$ and using the previous estimate, we get the required bound for $\|(y,z)\|_{L^2(0,T;\mathcal{H})}$.

- To obtain the estimate for (y_t, z_t) in $L^2(0,T;\mathcal{H}^*)$, we consider any $(\varphi, \psi) \in \mathcal{H}$ and from (5.9) we have

  $$\big\langle (y_t(t), z_t(t)), (\varphi, \psi) \big\rangle_{\mathcal{H}^*, \mathcal{H}} + \big(A(y(t), z(t)), (\varphi, \psi)\big)_Z = \big((g_1(t), g_2(t)), (\varphi, \psi)\big)_Z, \quad \forall t \in [0,T],$$

  which implies

  $$\left|\big\langle (y_t(t), z_t(t)), (\varphi, \psi) \big\rangle_{\mathcal{H}^*, \mathcal{H}}\right| \leq C\Big(\|(y(t), z(t))\|_{\mathcal{H}} + \|(g_1(t), g_2(t))\|_Z \Big) \|(\varphi, \psi)\|_{\mathcal{H}}, \quad \forall t \in [0,T],$$

  and this gives the estimation of $\|(y_t, z_t)\|_{L^2(0,T;\mathcal{H}^*)}$ as stated in (5.18).

Finally, by applying the usual density argument, we shall obtain the same estimate (5.18) for given data $(y_0, z_0) \in Z$ and $(g_1, g_2) \in L^2(0,T;Z)$. The proof is finished. □

5.2.2 The homogeneous adjoint system: Backward in time

The adjoint system to the linearized model (5.9) is given by

$$\begin{cases} -\zeta_t - \zeta_{xx} = 0, & \text{in } (0,T)\times(0,1), \\ -\theta_t - \theta_{xx} = 0, & \text{in } (0,T)\times(0,1), \\ \zeta_x(t,0) = 0, \quad \theta_x(t,0) = 0, & \text{for } t \in (0,T), \\ \zeta_x(t,1) = \theta_x(t,1), & \text{for } t \in (0,T), \\ \zeta(t,1) + \theta(t,1) + \alpha\zeta_x(t,1) = 0, & \text{for } t \in (0,T), \\ \zeta(T,x) = \zeta_T(x), \quad \theta(T,x) = \theta_T(x), & \text{in } (0,1), \end{cases} \tag{5.20}$$

with given final data $(\zeta_T, \theta_T) \in Z$. In fact, we have the following result.

Proposition 5.2.4. *Let any parameter $\alpha \geq 0$ and final data $(\zeta_T, \theta_T) \in Z$ be given. Then, the system (5.20) possesses a unique weak solution*

$$(\zeta, \theta) \in C^0([0,T]; Z) \cap L^2(0,T;\mathcal{H}) \cap H^1(0,T;\mathcal{H}^*)$$

with the following energy estimate:

$$\|(\zeta,\theta)\|_{C^0([0,T];Z)} + \|(\zeta,\theta)\|_{L^2(0,T;\mathcal{H})} + \|(\zeta_t,\theta_t)\|_{L^2(0,T;\mathcal{H}^*)} \leq Ce^{CT}\|(\zeta_T,\theta_T)\|_Z, \tag{5.21}$$

where $C > 0$ is a constant independent in $T > 0$.

Thanks to Proposition 5.2.1, the adjoint operator $(-A^*, \mathcal{D}(A^*))$ (which is the same as $(-A, \mathcal{D}(A))$ but we use a different notation for better understanding) defines a strongly continuous semigroup in Z, which ensures the existence and uniqueness of solution $(\zeta,\theta) \in C^0([0,T]; Z)$ to (5.20) and moreover it can be expressed as

$$(\zeta,\theta)(t,x) = e^{-(T-t)A^*}(\zeta_T,\theta_T)(x), \quad \forall (t,x) \in (0,T)\times(0,1),$$

where $\left(e^{-tA^*}\right)_{t\geq 0}$ denotes the semigroup defined by $(-A^*, \mathcal{D}(A^*))$.

Then the energy estimate (5.21) can be obtained by applying similar technique as described in the proof of Proposition 5.2.3.

5.2.3 The nonhomogeneous linearized system

We now address the notion of solution to the following nonhomogeneous system (which is forward in time) in the sense of transposition as introduced in [Cor07, TW09]. Consider the system

$$\begin{cases} y_t - y_{xx} = g_1, & \text{in } (0,T)\times(0,1), \\ z_t - z_{xx} = g_2, & \text{in } (0,T)\times(0,1), \\ y_x(t,0) = q_1(t), \quad z_x(t,0) = q_2(t), & \text{for } t \in (0,T), \\ y_x(t,1) = z_x(t,1), & \text{for } t \in (0,T), \\ y(t,1) + z(t,1) + \alpha y_x(t,1) = 0, & \text{for } t \in (0,T), \\ y(0,x) = y_0(x), \quad z(0,x) = z_0(x), & \text{in } (0,1), \end{cases} \tag{5.22}$$

and we write the following definition.

Definition 5.2.1 (Solution by transposition)**.** *Let $\alpha \geq 0$ be a given parameter. Then, for given initial state $(y_0, z_0) \in Z$, boundary data $(q_1, q_2) \in L^2(0,T;\mathbb{R}^2)$ and source term $(g_1, g_2) \in L^2(0,T;Z)$, a function $(y,z) \in C^0([0,T];Z)$ is said to be a solution to the system (5.22), if for any $t \in [0,T]$ and $(\zeta_T,\theta_T) \in Z$, the following relation holds:*

$$\begin{aligned} \big((y(t),z(t)),(\zeta_T,\theta_T)\big)_Z = &\big((y_0,z_0), e^{-tA^*}(\zeta_T,\theta_T)\big)_Z + \int_0^t \big((g_1(s),g_2(s)), e^{-(t-s)A^*}(\zeta_T,\theta_T)\big)_Z \\ &- \int_0^t \big\langle (q_1(s),q_2(s)), \big(e^{-(t-s)A^*}(\zeta_T,\theta_T)\big)(0)\big\rangle_{\mathbb{R}^2}. \end{aligned} \tag{5.23}$$

Let us now write the following result.

Theorem 5.2.1. *Let $\alpha \geq 0$ be a given parameter and $(y_0, z_0) \in Z$, $(g_1, g_2) \in L^2(0,T;Z)$, $(q_1, q_2) \in L^2(0,T;\mathbb{R}^2)$ be given data. Then the system (5.22) has a unique solution $(y,z) \in C^0([0,T];Z)$ in the sense of transposition as given by Definition 5.2.1.*

Furthermore, $(y,z) \in L^2(0,T;\mathcal{H}) \cap H^1(0,T;\mathcal{H}^*)$ and it satisfies the natural energy estimate

$$\begin{aligned} &\|(y,z)\|_{C^0([0,T];Z)} + \|(y,z)\|_{L^2(0,T;\mathcal{H})} + \|(y_t,z_t)\|_{L^2(0,T;\mathcal{H}^*)} \\ &\quad \leq Ce^{CT}\left(\|(y_0,z_0)\|_Z + \|(g_1,g_2)\|_{L^2(0,T;Z)} + \|(q_1,q_2)\|_{L^2(0,T;\mathbb{R}^2)}\right), \end{aligned} \tag{5.24}$$

where the constant $C > 0$ does not depend on T.

The proof for the energy estimate can be done using a similar technique as implemented in the proof of Proposition 5.2.3. We skip the details.

Remark 5.2.1. *For the nonhomogeneous system (5.22), we can achieve the usual energy estimate (5.24) since the nonhomogeneous $L^2(0,T)$-boundary terms q_1, q_2 appear through the Neumann conditions. This phenomenon has been broadly studied in [Nit14] in the context of parabolic equations with nonhomogeneous Neumann data. We also refer [BB21, Proposition 2.4] where the usual energy estimate for parabolic equations with nonhomogeneous Robin condition (with L^2 boundary data) has been obtained.*

5.3 Controllability of the linearized system: The method of moments

This section is devoted to the proof of null-controllability for our linearized system (5.4), that is the Theorem 5.1.2. As mentioned earlier, the method of moments helps us to construct a boundary null-control for our system and as it is well-known, to deal with this method we first need to study the spectral analysis of the corresponding (adjoint) spatial operator. We discuss about this in the following section.

5.3.1 Spectral analysis of the operator A^*

The eigenvalue problem associated with the operator A^* is

$$A^*U = \lambda U, \quad \text{for } \lambda \in \mathbb{C},$$

with $U := (u, v)$, which explicitly looks like

$$\begin{cases} -u''(x) = \lambda u(x), & \text{for } x \in (0,1), \\ -v''(x) = \lambda v(x), & \text{for } x \in (0,1), \\ u'(0) = 0, \quad v'(0) = 0, \\ u'(1) = v'(1), \\ u(1) + v(1) + \alpha u'(1) = 0, & \alpha \geq 0. \end{cases} \tag{5.25}$$

We divide the analysis into several parts.

- Observe that the spatial operator (defined by (5.6)) is self-adjoint and thus, all eigenvalues are real.
- From the set of equations (5.25), it is clear that $u = 0 \Leftrightarrow v = 0$ for any $\lambda \in \mathbb{R}$.
- $\lambda = 0$ is an eigenvalue of the operator A^* associated with the eigenfunction $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

We denote this particular eigenfunction by $\Phi_{\lambda_{0,1}}$ associated with the eigenvalue $\lambda_{0,1} := 0$ just to be consistent with the notations introduced for the first set of eigenfunctions given by (5.27).

- Assume now that $\lambda \neq 0$ and denote $\mu = \sqrt{\lambda} \in \mathbb{R}^+$. Thanks to the boundary condition $u'(0) = v'(0)$, we expect the solutions to (5.25) as

$$u(x) = A_1 \cos(\mu x), \quad v(x) = A_2 \cos(\mu x), \quad \forall x \in [0,1].$$

Then, the boundary conditions $u'(1) = v'(1)$ and $u(1) + v(1) + \alpha u'(1) = 0$ respectively gives

$$A_1 \mu \sin\mu = A_2 \mu \sin\mu, \tag{5.26a}$$

$$A_1 \cos\mu + A_2 \cos\mu - \alpha A_1 \mu \sin\mu = 0. \tag{5.26b}$$

The case when $A_1 \neq A_2$, the equation (5.26a) yields $\mu = k\pi$ for any $k \geq 1$, since $\mu \neq 0$. Using this information in (5.26b), we deduce $A_1 = -A_2$. Therefore, the eigenfunctions of the first family, denote them as $\Phi_{\lambda_{k,1}}$, are given by

$$\Phi_{\lambda_{k,1}} := \begin{pmatrix} \cos(k\pi x) \\ -\cos(k\pi x) \end{pmatrix}, \tag{5.27}$$

associated with the eigenvalues $\lambda_{k,1} := k^2\pi^2$ for all $k \geq 1$.

In the case when $\sin\mu \neq 0$, that is $A_1 = A_2$ ($\neq 0$ since we seek for non-trivial μ), we have from (5.26b) that

$$h(\mu) := 2\cos\mu - \alpha\mu \sin\mu = 0, \quad \alpha \geq 0. \tag{5.28}$$

(i) The case $\alpha = 0$ is straightforward; we have the eigenfunctions $\Phi_{\lambda^0_{k,2}}$ as follows:

$$\Phi_{\lambda^0_{k,2}} := \begin{pmatrix} \cos((k+\frac{1}{2})\pi x) \\ \cos((k+\frac{1}{2})\pi x) \end{pmatrix}, \tag{5.29}$$

associated with the eigenvalues $\lambda^0_{k,2} := (k+\frac{1}{2})^2\pi^2$ for all $k \geq 0$.

(ii) The case when $\alpha \neq 0$, we compute that

$$h(k\pi) = (-1)^k 2 \quad \text{and} \quad h\left((k+\frac{1}{2})\pi\right) = (-1)^{k+1}\alpha\left((k+\frac{1}{2})\pi\right)$$

have different signs which ensures the existence of at least one root of h in the interval $\big(k\pi, (k+\frac{1}{2})\pi\big)$ for all $k \geq 0$.

To prove the uniqueness, we compute

$$h'(\mu) = -(\alpha+2)\sin\mu - \alpha\mu\cos\mu$$

which has the same sign throughout the interval $\big(k\pi, (k+\frac{1}{2})\pi\big)$ for any $k \geq 0$ and thus the required claim follows.

We denote this unique root by $\mu^\alpha_{k,2}$ and the eigenvalues by $\lambda^\alpha_{k,2} := (\mu^\alpha_{k,2})^2 \in \big(k^2\pi^2, (k+\frac{1}{2})^2\pi^2\big)$ for any $k \geq 0$. The associated eigenfunctions will be then

$$\Phi_{\lambda^\alpha_{k,2}} := \begin{pmatrix} \cos(\sqrt{\lambda^\alpha_{k,2}}x) \\ \cos(\sqrt{\lambda^\alpha_{k,2}}x) \end{pmatrix}, \quad \forall k \geq 0. \tag{5.30}$$

We now write the following lemma concerning the eigen-elements of A^*.

Lemma 5.3.1. *Let any $\alpha \geq 0$ be given. Then, we have the following.*

1. *The spectrum of the operator A^* consists of only real simple eigenvalues and it is given by*

$$\Lambda^\alpha := \left\{\lambda_{k,1}, \lambda^\alpha_{k,2}\right\}_{k\geq 0}, \tag{5.31}$$

where

$$\lambda_{k,1} = k^2\pi^2 \quad \textit{and} \quad \lambda^\alpha_{k,2} \begin{cases} = (k+\frac{1}{2})^2\pi^2, & \textit{when} \quad \alpha = 0, \\ \in \left(k^2\pi^2, (k+\frac{1}{2})^2\pi^2\right), & \textit{when} \quad \alpha > 0. \end{cases} \tag{5.32}$$

The associated eigenfunctions are

$$\Phi_{\lambda_{k,1}}(x) = \begin{pmatrix} \cos(k\pi x) \\ -\cos(k\pi x) \end{pmatrix} \quad \textit{and} \quad \Phi_{\lambda^\alpha_{k,2}}(x) = \begin{pmatrix} \cos(\sqrt{\lambda^\alpha_{k,2}}x) \\ \cos(\sqrt{\lambda^\alpha_{k,2}}x) \end{pmatrix}, \tag{5.33}$$

for the eigenvalues $\lambda_{k,1}$ and $\lambda^\alpha_{k,2}$ respectively for all $k \geq 0$.

2. *Moreover, the set of eigenfunctions $\left\{\Phi_{\lambda_{k,1}}, \Phi_{\lambda^\alpha_{k,2}}\right\}_{k\geq 0}$ forms an orthogonal basis in $Z = [L^2(0,1)]^2$.*

The formal proof of part 1 has been already discussed before the statement of Lemma 5.3.1. Further, we note that the operator A^* is self-adjoint and it can be proved that A^* has compact resolvent. Consequently, the result of part 2 follows.

Lemma 5.3.2 (Asymptotics of the eigenvalues for $\alpha > 0$)**.** *For each $\alpha > 0$, the asymptotic of the second set of eigenvalues $\lambda^\alpha_{k,2}$ are*

$$\lambda^\alpha_{k,2} = k^2\pi^2 + \frac{4}{\alpha} + O\left(\frac{1}{k^2}\right), \quad \textit{for large enough} \quad k \in \mathbb{N}^*. \tag{5.34}$$

Proof. Recall that $\mu^\alpha_{k,2} \in \left(k\pi, (k+\frac{1}{2})\pi\right)$ which uniquely satisfies the equation

$$2\cos\mu^\alpha_{k,2} - \alpha\mu^\alpha_{k,2}\sin\mu^\alpha_{k,2} = 0, \quad \text{for each } k \geq 0. \tag{5.35}$$

We set $\mu^\alpha_{k,2} = k\pi + \delta^\alpha_k$ with $\delta^\alpha_k \in [0, \frac{\pi}{2}]$. Then, from (5.35) we have

$$(-1)^k 2\cos\delta^\alpha_k - (-1)^k\alpha\left(k\pi + \delta^\alpha_k\right)\sin\delta^\alpha_k = 0, \tag{5.36}$$

$$\Rightarrow \tan\delta^\alpha_k = \frac{2}{\alpha(k\pi + \delta^\alpha_k)} \to 0 \quad \text{as} \quad k \to +\infty$$

$$\Rightarrow \delta^\alpha_k \to 0 \quad \text{as} \quad k \to +\infty. \tag{5.37}$$

Using the fact (5.37) in (5.36), one has

$$\delta^\alpha_k \sim_{+\infty} \frac{2}{\alpha k\pi},$$

and thus,

$$\mu^\alpha_{k,2} \sim_{+\infty} k\pi + \frac{2}{\alpha k\pi}.$$

Thereafter, expressing $\mu^\alpha_{k,2} = k\pi + \frac{2}{\alpha k\pi} + \tilde{\delta}^\alpha_k$ and substituting this in (5.35), one can obtain

$$\alpha\tilde{\delta}^\alpha_k k\pi = -\frac{4}{\alpha k^2\pi^2} - \alpha(\tilde{\delta}^\alpha_k)^2 - \frac{4\tilde{\delta}^\alpha_k}{k\pi},$$

which asymptotically gives $\tilde{\delta}^\alpha_k \sim_{+\infty} O(1/k^3)$. So, finally we have

$$\mu^\alpha_{k,2} = k\pi + \frac{2}{\alpha k\pi} + O\left(\frac{1}{k^3}\right), \quad \text{for large enough} \quad k \in \mathbb{N}^*,$$

and that the asymptotic expression (5.34) follows. □

5.3.2 Formulation of the control problem and approximate controllability

We first present an equivalent criterion for the null-controllability of the linear model (5.4).

Proposition 5.3.1 (Formulation of the control problem)**.** *Let any* $(y_0, z_0) \in Z$*, time* $T > 0$ *and parameter* $\alpha \geq 0$ *be given. Then a function* $q \in L^2(0,T)$ *is said to be a null-control for the system* (5.4) *if and only if it satisfies: for any* $(\zeta_T, \theta_T) \in Z$*,*

$$\big((y_0, z_0), e^{-TA^*}(\zeta_T, \theta_T)\big)_Z = \int_0^T q(t)\left\langle \begin{pmatrix}1\\0\end{pmatrix}, \big(e^{-(T-t)A^*}(\zeta_T, \theta_T)\big)(0)\right\rangle_{\mathbb{R}^2}. \tag{5.38}$$

We hereby introduce the observation operator

$$\mathcal{B}^* := \mathbb{1}_{\{x=0\}}\begin{pmatrix}1\\0\end{pmatrix} : \mathcal{H} \mapsto \mathbb{R} \tag{5.39}$$

(recall that $\mathcal{H} = [H^1(0,1)]^2$) and to this end, we have the following result.

Proposition 5.3.2 (Approximate controllability)**.** *Let* $\alpha \geq 0$ *be given. Then, the linearized system* (5.4) *is approximately controllable at any given time* $T > 0$ *in the space* Z*.*

Proof. Note that $\mathcal{B}^*\Phi_{\lambda_{k,1}} = \mathcal{B}^*\Phi_{\lambda^\alpha_{k,2}} = 1$ for all $\alpha \geq 0$ and $k \geq 0$. Then, by applying the Fattorini-Hautus criterion (see [Fat66, Oli14]), we conclude the proposition. □

5.3.3 The moments problem

Recall that for any parameter $\alpha \geq 0$, the set of eigenfunctions $\{\Phi_\lambda\}_{\lambda\in\Lambda^\alpha}$ of A^* forms an orthogonal basis in Z (see Lemma 5.3.1). Thus, it is enough to check the control problem (5.38) for all $\Phi_\lambda \in \{\Phi_\lambda\}_{\lambda\in\Lambda^\alpha}$. This gives us the following.

- For any $(y_0, z_0) \in Z$ and parameter $\alpha \geq 0$, a function $q \in L^2(0,T)$ is a null-control for the system (5.4) if and only if we have

$$\int_0^T e^{-\lambda(T-t)}q(t) = \frac{e^{-\lambda T}}{\mathcal{B}^*\Phi_\lambda}\big((y_0, z_0), \Phi_\lambda\big)_Z, \quad \text{for all } \lambda \in \Lambda^\alpha. \tag{5.40}$$

  Here, we have used the fact that

$$e^{-tA^*}\Phi_\lambda = e^{-t\lambda}\Phi_\lambda, \quad \forall \lambda \in \Lambda^\alpha.$$

We also recall that $\mathcal{B}^*\Phi_\lambda = 1$ for all $\lambda \in \Lambda^\alpha$ which ensures that the set of moment problems (5.40) is well-defined and we shall solve those in the next subsections.

5.3.4 Existence of bi-orthogonal family

In the framework of parabolic control theory, the existence of bi-orthogonal families to the family of exponential functions in $L^2(0,T)$ has been extensively studied from the pioneer work [FR75] up to the very recent developments. In this paper, we use [Boy23, Theorem V.4.26 & Corollary V.4.27] (which is similar to [BBGBO14, Theorem 1.5] but with a more general set of assumptions) to establish the following result.

Lemma 5.3.3. *For any* $\alpha \geq 0$ *recall the set* Λ^α *given by* (5.31)*. Then, there exists a family* $(p_\lambda)_{\lambda\in\Lambda^\alpha} \subset L^2(0,T)$ *bi-orthogonal to* $(e^{-\lambda(T-\cdot)})_{\lambda\in\Lambda^\alpha}$*, i.e.,*

$$\int_0^T p_\lambda(t)e^{-\widetilde{\lambda}(T-t)} = \delta_{\lambda,\widetilde{\lambda}}, \quad \textit{for any } \lambda, \widetilde{\lambda} \in \Lambda^\alpha. \tag{5.41}$$

In addition, they satisfy the following estimate

$$\|p_\lambda\|_{L^2(0,T)} \leq Ce^{\frac{C}{T}}e^{\frac{T}{2}\lambda + C\sqrt{\lambda}}, \quad \forall \lambda \in \Lambda^\alpha, \tag{5.42}$$

where the constant $C > 0$ *is independent in* T*.*

Remark 5.3.1. *Without loss of generality, we assume that all the eigenvalues are positive. In fact, we can choose some $c_0 > 0$ such that $\lambda + c_0 > 0$ for all $\lambda \in \Lambda^\alpha$. In what follows, an extra factor e^{Tc_0} will appear in the estimation of control cost, but without any consequences on our analysis.*

Now, as mentioned earlier, we shall use [Boy23, Theorem V.4.26] in order to prove Lemma 5.3.3, and for that we need to show that the set of eigenvalues Λ^α defined by (5.31), belongs to some sector of the complex half-plane, satisfies a uniform gap property and some asymptotic conditions on the counting function.

- *The sector condition.* For any $\nu > 0$, we define the sector

$$\mathcal{S}_\nu := \{z \in \mathbb{C} \mid \mathrm{Re} z > 0, \text{ and } |\mathrm{Im} z| < (\sinh \nu)\, \mathrm{Re} z\}.$$

  In our case, the set of eigenvalues Λ^α is real and so it is clear that there exists some $\nu > 0$ such that

$$\Lambda^\alpha \subset \mathcal{S}_\nu, \tag{5.43}$$

  for any $\alpha \geq 0$.

- *The gap condition.*

  Recall the set of eigenvalues given by (5.31) and the asymptotics of the eigenvalues $\lambda^\alpha_{k,2}$ for $\alpha > 0$ from Lemma 5.3.2. Then it can be seen that there exists some $c_1 > 0$ such that we have

$$\begin{aligned} &|\lambda_{k+1,1} - \lambda_{k,1}| \geq c_1 k, \quad \forall k \geq 1, \\ &|\lambda^\alpha_{k+1,2} - \lambda^\alpha_{k,2}| \geq c_1 k, \quad \forall k \geq 1 \text{ and } \alpha \geq 0, \end{aligned}$$

  and there is some $k_\alpha \in \mathbb{N}^*$ such that

$$\begin{aligned} &|\lambda^0_{k,2} - \lambda_{k,1}| \geq c_1 k, \quad \forall k \geq 1, \\ &|\lambda^\alpha_{k,2} - \lambda_{k,1}| \geq \frac{c_1}{\alpha}, \quad \forall k \geq k_\alpha. \end{aligned}$$

  Remark 5.3.2. *Unlike the case of $\alpha = 0$, we note that for $\alpha > 0$ the gap between $\lambda^\alpha_{k,2}$ and $\lambda_{k,1}$ tends to a finite positive number as k goes to infinity but does not tend to infinity like for the other cases. This is the reason why we needed to compute the precise asymptotic expansions of the eigenvalues $\lambda^\alpha_{k,2}$ for $\alpha > 0$.*

  Using the above lower bounds of the *differences of eigenvalues* and the fact that the spectrum is discrete, we can say that there is some $\rho > 0$ such that

$$|\lambda - \widetilde{\lambda}| \geq \rho, \quad \text{for any } \lambda, \widetilde{\lambda} \in \Lambda^\alpha \text{ with } \lambda \neq \widetilde{\lambda}, \tag{5.44}$$

  which is the uniform spectral gap property.

- *The condition on counting function.* Let $\mathcal{N}_\alpha$ be the counting function associated with the set of eigenvalues Λ^α (for any $\alpha \geq 0$) defined by

$$\mathcal{N}_\alpha(r) := \#\{\lambda \in \Lambda^\alpha, \text{ s.t. } |\lambda| \leq r\}, \quad \forall r > 0.$$

  Our goal is to show that there exists some $\kappa_0 > 0$ independent in the set of eigenvalues such that

$$\mathcal{N}_\alpha(r) \leq \kappa_0 r^{1/2}, \quad \forall r > 0, \tag{5.45a}$$

$$|\mathcal{N}_\alpha(r) - \mathcal{N}_\alpha(s)| \leq \kappa_0 \left(1 + |r-s|^{1/2}\right), \quad \forall r, s > 0. \tag{5.45b}$$

  From (5.31), we recall that

$$\Lambda^\alpha = \left\{\lambda_{k,1}, \lambda^\alpha_{k,2}\right\}_{k \geq 0}.$$

As it is shown for instance in [Boy23, Lemma V.4.20], it is enough to establish the required results (5.45) for each of the two sets $\{\lambda_{k,1}\}_{k\geq 0}$ and $\{\lambda^\alpha_{k,2}\}_{k\geq 0}$. We shall show this for $\{\lambda^\alpha_{k,2}\}_{k\geq 0}$ when $\alpha > 0$ since the same reasoning will be applicable for the set $\{\lambda^0_{k,2}\}_{k\geq 0}$ or $\{\lambda_{k,1}\}_{k\geq 0}$.

We denote the associated counting function by $\mathcal{N}_{\alpha,2}$.

- Let $r > 0$ be fixed. Then, $\mathcal{N}_{\alpha,2}(r) = k$ $(k \in \mathbb{N}^*)$ implies
$$\lambda^\alpha_{k-1,2} \leq r,$$
since $\{\lambda^\alpha_{k,2}\}_{k\geq 0}$ is increasing. But we have $\lambda^\alpha_{k-1,2} \in \big((k-1)^2\pi^2, (k-\frac{1}{2})^2\pi^2\big)$ for any $k \geq 1$, which gives
$$(k-1)^2\pi^2 \leq r, \quad \text{i.e.,} \quad k \leq 1 + \frac{1}{\pi}\sqrt{r},$$
and the first condition (5.45a) follows for the counting function.
- Let any $0 < s < r$ be given. Assume that $l = \mathcal{N}_{\alpha,2}(s)$ and $k = \mathcal{N}_{\alpha,2}(r)$ for some $l, k \in \mathbb{N}^*$ (certainly, $k > l$). Then, using the properties of the set $\{\lambda^\alpha_{k,2}\}_{k\geq 0}$, one has
$$(k-1)\pi \leq \sqrt{\lambda^\alpha_{k-1,2}} \leq \sqrt{r}, \quad (l + \frac{1}{2})\pi > \sqrt{\lambda^\alpha_{l,2}} > \sqrt{s},$$
which yields
$$k - l \leq \frac{3}{2} + \frac{1}{\pi}(\sqrt{r} - \sqrt{s}) \leq \frac{3}{2} + \frac{1}{\pi}\sqrt{r-s},$$
and that the second condition (5.45b) on the counting function is true.

Since the three conditions (5.43), (5.44) and (5.45) are now satisfied, by using [Boy23, Theorem V.4.16], we can ensure the existence of a bi-orthogonal family $(p_\lambda)_{\lambda\in\Lambda^\alpha} \subset L^2(0,T)$ to $(e^{-\lambda(T-\cdot)})_{\lambda_{\Lambda^\alpha}}$ satisfying the sharp estimate as mentioned in Lemma 5.3.3.

5.3.5 Existence of a boundary null-control

Now, we are in position to solve the set of moments problem (5.40) to find a control for the system (5.4).

Proof of Theorem 5.1.2. For any $\alpha \geq 0$ and initial data $(y_0, z_0) \in Z$, we consider
$$q(t) = \sum_{\lambda\in\Lambda^\alpha} q_\lambda(t), \quad \forall t \in [0,T], \tag{5.46a}$$
$$\text{with} \quad q_\lambda(t) = \frac{e^{-\lambda T}}{\mathcal{B}^*\Phi_\lambda}\big((y_0,z_0),\Phi_\lambda\big)_Z\, p_\lambda(t), \quad \forall t\in[0,T], \quad \forall \lambda \in \Lambda^\alpha \tag{5.46b}$$
where p_λ are given by Lemma 5.3.3. Observe that, the above choice of function q formally solves the set of moments problem (5.40), thanks to the property (5.41) verified by p_λ for each $\lambda \in \Lambda^\alpha$.

Now, recall that $\mathcal{B}^*\Phi_\lambda = 1$ for all $\lambda \in \Lambda^\alpha$ (see Proposition 5.3.2). Also, from the expressions of the eigenfunctions given by (5.27)–(5.29)–(5.30), we have $\|\Phi_\lambda\|_Z \leq C$ for any $\lambda \in \Lambda^\alpha$. Using these and the $L^2(0,T)$-estimates of bi-orthogonal family $(p_\lambda)_{\lambda\in\Lambda^\alpha}$ given by (5.42), we obtain $\lambda \in \Lambda^\alpha$, that
$$\|q_\lambda\|_{L^2(0,T)} \leq Ce^{-\lambda T}e^{\frac{C}{T}}e^{\frac{T}{2}\lambda + C\sqrt{\lambda}}\|(y_0,z_0)\|_Z \leq Ce^{\frac{C}{T}}e^{-\frac{T}{2}\lambda}e^{\frac{T}{4}\lambda+\frac{C^2}{T}}\|(y_0,z_0)\|_Z \leq Ce^{\frac{C}{T}}e^{-\frac{T}{4}\lambda}\|(y_0,z_0)\|_Z, \tag{5.47}$$
where we have used the Young's inequality
$$C\sqrt{\lambda} \leq \frac{T}{4}\lambda + \frac{C^2}{T}, \quad \forall \lambda \in \Lambda^\alpha.$$

Using (5.47) we have

$$\|q\|_{L^2(0,T)} \leq \sum_{\lambda\in\Lambda^\alpha} \|q_\lambda\|_{L^2(0,T)} \leq Ce^{\frac{C}{T}}\|(y_0,z_0)\|_Z \sum_{\lambda\in\Lambda^\alpha} e^{-\frac{T}{4}\lambda} \leq Me^{\frac{M}{T}}\|(y_0,z_0)\|_Z,$$

thanks to the fact that Λ^α is an increasing sequence of order k^2 (see (5.31)). Moreover, it is clear that the constant $M > 0$ does not depend on T or (y_0, z_0).

The proof is complete. □

5.4 Local null-controllability of the nonlinear system

This section is devoted to prove the local null-controllability result for the nonlinear system (5.1), i.e., Theorem 5.1.1. The proof will be based on the so-called source term method developed in [LTT13] followed by a Banach fixed point argument and to employ this we shall extensively use the control cost $Me^{\frac{M}{T}}\|(y_0,z_0)\|_Z$ for the linear system, given by Theorem 5.1.2.

5.4.1 The source term method

Let us discuss the source term method for our problem. We first consider the following system:

$$\begin{cases} y_t - y_{xx} = \xi, & \text{in } (0,T)\times(0,1),\\ z_t - z_{xx} = \eta, & \text{in } (0,T)\times(0,1),\\ y_x(t,0) = q(t), \quad z_x(t,0)=0, & \text{for } t\in(0,T),\\ y_x(t,1) = z_x(t,1), & \text{for } t\in(0,T),\\ y(t,1)+z(t,1)+\alpha y_x(t,1) = 0, & \text{for } t\in(0,T),\\ y(0,x)=y_0(x), \quad z(0,x)=z_0(x), & \text{in } (0,1). \end{cases} \tag{5.48}$$

Then, our goal is to establish the null-controllability of the above system for any given parameter $\alpha \geq 0$, initial data $(y_0, z_0) \in Z$ and source terms (ξ, η) which belong to some certain weighted $L^2(0,T;Z)$ space. Let us discuss it at length in the next couple of subsections.

5.4.1.1 Construction of weight functions and weighted spaces

Assume the constants $\beta > 0$, $\gamma > 1$ in such a way that

$$1 < \gamma < \sqrt{2}, \quad \text{and} \quad \beta > \frac{\gamma^2}{2-\gamma^2}. \tag{5.49}$$

We now define the weight functions

$$\begin{cases} \rho_0(t) = e^{-\frac{\beta M}{(\gamma-1)(T-t)}},\\ \rho_{\mathcal{S}}(t) = e^{-\frac{(1+\beta)\gamma^2 M}{(\gamma-1)(T-t)}}, \end{cases} \qquad \forall t \in \left[T\left(1-\frac{1}{\gamma^2}\right), T\right], \tag{5.50}$$

and extended them in a constant way in $\left[0, T\left(1-\frac{1}{\gamma^2}\right)\right]$ such that they are continuous and non-increasing in $[0,T]$. Note that $\rho_0(T) = \rho_{\mathcal{S}}(T) = 0$ and further, we compute that

$$\frac{\rho_0^2(t)}{\rho_{\mathcal{S}}(t)} = e^{\frac{\gamma^2 M + \beta M(\gamma^2-2)}{(\gamma-1)(T-t)}}, \qquad \forall t \in \left[T\left(1-\frac{1}{\gamma^2}\right), T\right].$$

Due to the choices of γ, β in (5.49), we have $M\big(\gamma^2 + \beta(\gamma^2-2)\big) < 0$, $(\gamma-1) > 0$ and therefore we conclude that

$$\frac{\rho_0^2(t)}{\rho_{\mathcal{S}}(t)} \leq 1, \quad \forall t \in [0,T]. \tag{5.51}$$

Let us now define the following weighted spaces:

$$\mathcal{S} := \left\{ \xi \in L^2(0,T;L^2(0,1)) \mid \frac{\xi}{\rho_{\mathcal{S}}} \in L^2(0,T;L^2(0,1)) \right\} \tag{5.52}$$

$$\mathcal{Y} := \left\{ (y,z) \in L^2(0,T;Z) \mid \left(\frac{y}{\rho_0}, \frac{z}{\rho_0} \right) \in L^2(0,T;Z) \right\} \tag{5.53}$$

$$\mathcal{Q} := \left\{ q \in L^2(0,T) \mid \frac{q}{\rho_0} \in L^2(0,T) \right\}, \tag{5.54}$$

where the functions ρ_0 and $\rho_{\mathcal{S}}$ are defined in (5.50). The inner product on the spaces $\mathcal{S}, \mathcal{Y}$ and $\mathcal{Q}$ are respectively given by

$$\left\langle \xi, \tilde{\xi} \right\rangle_{\mathcal{S}} := \int_0^T \frac{1}{\rho_{\mathcal{S}}^2(t)} \left\langle \xi(t), \tilde{\xi}(t) \right\rangle_{L^2(0,1)} dt, \quad \forall\, \xi, \tilde{\xi} \in \mathcal{S},$$

$$\langle (y,z), (\tilde{y}, \tilde{z}) \rangle_{\mathcal{Y}} := \int_0^T \frac{1}{\rho_0^2(t)} \langle (y(t), z(t)), (\tilde{y}(t), \tilde{z}(t)) \rangle_Z \, dt, \quad \forall\, (y,z), (\tilde{y}, \tilde{z}) \in \mathcal{Y},$$

$$\langle q, \tilde{q} \rangle_{\mathcal{Q}} := \int_0^T \frac{1}{\rho_0^2(t)} q(t) \tilde{q}(t) dt, \quad \forall\, q, \tilde{q} \in \mathcal{Q}.$$

Accordingly, the associated norms on the spaces $\mathcal{S}, \mathcal{Y}$ and $\mathcal{Q}$ are respectively

$$\|\xi\|_{\mathcal{S}}^2 := \int_0^T \frac{1}{\rho_{\mathcal{S}}^2(t)} \|\xi(t)\|_{L^2(0,1)}^2 \, dt, \quad \forall\, \xi \in \mathcal{S}, \tag{5.55}$$

$$\|(y,z)\|_{\mathcal{Y}}^2 := \int_0^T \frac{1}{\rho_0^2(t)} \|(y(t), z(t))\|_Z^2 \, dt, \quad \forall\, (y,z) \in \mathcal{Y}, \tag{5.56}$$

$$\|q\|_{\mathcal{Q}}^2 := \int_0^T \frac{1}{\rho_0^2(t)} |q(t)|^2 \, dt, \quad \forall\, q \in \mathcal{Q}. \tag{5.57}$$

5.4.1.2 Null-controllability of the linearized system with source terms

Our next result addresses the null-controllability of the inhomogeneous linear system (5.48) with given source terms ξ, η from the space $\mathcal{S}$ and by definition of $\mathcal{S}$, it is clear that the function ξ or η vanishes exponentially near $t = T$. With the above choice of source functions in hand, and then by utilizing the explicit *control cost* $Me^{\frac{M}{T}}$ for the homogeneous control system (see Section 5.3.5), we shall eventually show that there exists a solution-control pair $((y,z),q)$ in the space $\mathcal{Y} \times \mathcal{Q}$ to the system (5.48). Then, by definitions of the space $\mathcal{Y}$ and weight function ρ_0 (see (5.53) and (5.50) resp.), one can conclude that the solution (y,z) has to be "zero" at $t = T$. Precisely we prove the following proposition.

Proposition 5.4.1. *Let any parameter $\alpha \geq 0$ be given. Then, for any given initial state $(y_0, z_0) \in Z$ and source terms $(\xi, \eta) \in L^2(0,T;Z)$, there exists a linear map $\mathcal{T} : Z \times L^2(0,T;Z) \to \mathcal{Y} \times \mathcal{Q}$ such that $\mathcal{T}((y_0,z_0),(\xi,\eta)) := ((y,z),q)$ solves the system (5.48).*

In addition, we have the following estimate

$$\left\| \left(\frac{y}{\rho_0}, \frac{z}{\rho_0} \right) \right\|_{C^0([0,T];Z)} + \left\| \left(\frac{y}{\rho_0}, \frac{z}{\rho_0} \right) \right\|_{L^2(0,T;\mathcal{H})} + \left\| \frac{q}{\rho_0} \right\|_{L^2(0,T)} \leq C e^{CT + \frac{C}{T}} \left(\|(y_0, z_0)\|_Z + \left\| \left(\frac{\xi}{\rho_{\mathcal{S}}}, \frac{\eta}{\rho_{\mathcal{S}}} \right) \right\|_{L^2(0,T;Z)} \right), \tag{5.58}$$

for some constant $C > 0$ that is independent in T.

Proof. For the given time $T > 0$, let us define a sequence $(T_k)_{k \geq 0}$ given by

$$T_k := T - \frac{T}{\gamma^k}, \quad \forall k \geq 0, \tag{5.59}$$

where γ is introduced in (5.49), and it can be easily seen that

$$(0,T) = \cup_{k\geq 0}(T_k, T_{k+1}).$$

We also note that with this choice of T_k, one has

$$\rho_0(T_{k+2}) = e^{\frac{M}{T_{k+2}-T_{k+1}}}\rho_{\mathcal{S}}(T_k), \quad \forall k \geq 0, \tag{5.60}$$

where ρ_0 and $\rho_{\mathcal{S}}$ have been defined by (5.50) .

Now, our goal is to decompose (5.48) in (T_k, T_{k+1}) for each $k \geq 0$, into two parts: one is only with forcing terms and zero initial data, and the other one is a homogeneous control system along with the initial data.

• *Inhomogeneous system without control input.*

Let us define a sequence $(a_k)_{k\geq 0}$ such that

$$a_0 := (y_0, z_0) \in Z \quad \text{and} \quad a_{k+1} := \big(\tilde{y}(T_{k+1}^-), \tilde{z}(T_{k+1}^-)\big), \quad \forall k \geq 0, \tag{5.61}$$

where $(\tilde{y}, \tilde{z})$ is the unique weak solution to the system

$$\begin{cases} \tilde{y}_t - \tilde{y}_{xx} = \xi, & \text{in } (T_k, T_{k+1}) \times (0,1), \\ \tilde{z}_t - \tilde{z}_{xx} = \eta, & \text{in } (T_k, T_{k+1}) \times (0,1), \\ \tilde{y}_x(t,0) = 0, \quad \tilde{z}_x(t,0) = 0, & \text{for } t \in (T_k, T_{k+1}), \\ \tilde{y}_x(t,1) = \tilde{z}_x(t,1), & \text{for } t \in (T_k, T_{k+1}), \\ \tilde{y}(t,1) + \tilde{z}(t,1) + \alpha \tilde{y}_x(t,1) = 0, & \text{for } t \in (T_k, T_{k+1}), \\ \tilde{y}(T_k^+, \cdot) = 0, \quad \tilde{z}(T_k^+, \cdot) = 0, & \text{in } (0,1), \end{cases} \tag{5.62}$$

for all $k \geq 0$. Thanks to the estimate (5.18) in Proposition 5.2.3, we get

$$\|(\tilde{y}, \tilde{z})\|_{C^0([T_k, T_{k+1}];Z)} + \|(\tilde{y}, \tilde{z})\|_{L^2(T_k, T_{k+1};\mathcal{H})} \leq Ce^{CT}\|(\xi, \eta)\|_{L^2(T_k, T_{k+1};Z)}, \quad \forall k \geq 0. \tag{5.63}$$

In particular, by means of (5.61), we have

$$\|a_{k+1}\|_Z \leq Ce^{CT}\|(\xi, \eta)\|_{L^2(T_k, T_{k+1};Z)}, \quad \forall k \geq 0. \tag{5.64}$$

• *Control system without the source terms.* We now consider the following homogeneous control system:

$$\begin{cases} \hat{y}_t - \hat{y}_{xx} = 0, & \text{in } (T_k, T_{k+1}) \times (0,1), \\ \hat{z}_t - \hat{z}_{xx} = 0, & \text{in } (T_k, T_{k+1}) \times (0,1), \\ \hat{y}_x(t,0) = \hat{q}_k(t), \quad \hat{z}_x(t,0) = 0, & \text{for } t \in (T_k, T_{k+1}), \\ \hat{y}_x(t,1) = \hat{z}_x(t,1), & \text{for } t \in (T_k, T_{k+1}), \\ \hat{y}(t,1) + \hat{z}(t,1) + \alpha \hat{y}_x(t,1) = 0, & \text{for } t \in (T_k, T_{k+1}), \\ \big(\hat{y}(T_k^+, \cdot), \hat{z}(T_k^+, \cdot)\big) = a_k, & \text{in } (0,1), \end{cases} \tag{5.65}$$

for all $k \geq 0$. Using Theorem 5.1.2, we have the existence of a control $\hat{q}_k \in L^2(T_k, T_{k+1})$ with the estimate

$$\|\hat{q}_k\|_{L^2(T_k, T_{k+1})} \leq Me^{\frac{M}{T_{k+1}-T_k}}\|a_k\|_Z, \tag{5.66}$$

such that the associated solution $(\hat{y}, \hat{z})$ to (5.65) satisfies

$$\big(\hat{y}(T_{k+1}^-, x), \hat{z}(T_{k+1}^-, x)\big) = (0,0), \quad \forall x \in (0,1) \quad \text{and} \quad \forall k \geq 0.$$

Combining (5.66) with (5.64), we have

$$\|\hat{q}_{k+1}\|_{L^2(T_{k+1}, T_{k+2})} \leq Me^{\frac{M}{T_{k+2}-T_{k+1}}}\|a_{k+1}\|_Z \leq Ce^{CT}e^{\frac{M}{T_{k+2}-T_{k+1}}}\|(\xi, \eta)\|_{L^2(T_k, T_{k+1};Z)}, \quad \forall k \geq 0.$$

But $\rho_{\mathcal{S}}$ is a non-increasing function in $[T_k, T_{k+1}]$; in what follows we have

$$\|\hat{q}_{k+1}\|_{L^2(T_{k+1},T_{k+2})} \leq Ce^{CT} e^{\frac{M}{T_{k+2}-T_{k+1}}} \rho_{\mathcal{S}}(T_k) \left\| \left(\frac{\xi}{\rho_{\mathcal{S}}}, \frac{\eta}{\rho_{\mathcal{S}}} \right) \right\|_{L^2(T_k,T_{k+1};Z)}, \quad \forall k \geq 0.$$

Then, using the relation (5.60) between the weight functions ρ_0 and $\rho_{\mathcal{S}}$, we get

$$\|\hat{q}_{k+1}\|_{L^2(T_{k+1},T_{k+2})} \leq Ce^{CT} \rho_0(T_{k+2}) \left\| \left(\frac{\xi}{\rho_{\mathcal{S}}}, \frac{\eta}{\rho_{\mathcal{S}}} \right) \right\|_{L^2(T_k,T_{k+1};Z)}, \quad \forall k \geq 0. \tag{5.67}$$

Again, since ρ_0 is non-increasing, we deduce

$$\left\| \frac{\hat{q}_{k+1}}{\rho_0} \right\|_{L^2(T_{k+1},T_{k+2})} \leq \frac{1}{\rho_0(T_{k+2})} \|\hat{q}_{k+1}\|_{L^2(T_{k+1},T_{k+2})} \leq Ce^{CT} \left\| \left(\frac{\xi}{\rho_{\mathcal{S}}}, \frac{\eta}{\rho_{\mathcal{S}}} \right) \right\|_{L^2(T_k,T_{k+1};Z)}, \quad \forall k \geq 0. \tag{5.68}$$

We now define the control function q as follows:

$$q := \sum_{k \geq 0} \hat{q}_k \chi_{(T_k,T_{k+1})} \quad \text{in } (0,T). \tag{5.69}$$

Recall that we have already established the L^2-estimates of $\dfrac{\hat{q}_k}{\rho_0}$ for all $k \geq 1$ by (5.68). It only remains to find the L^2-estimate of $\dfrac{\hat{q}_0}{\rho_0}$. But from the bound (5.66), we get

$$\|\hat{q}_0\|_{L^2(0,T_1)} \leq Me^{\frac{M}{T_1}} \|a_0\|_Z = Me^{\frac{M}{T_1}} \|(y_0, z_0)\|_Z,$$

and then using the fact that ρ_0 is non-increasing, one has

$$\left\| \frac{\hat{q}_0}{\rho_0} \right\|_{L^2(0,T_1)} \leq \frac{1}{|\rho_0(T_1)|} \|\hat{q}_0\| \leq \frac{M}{\rho_0(T_1)} e^{\frac{M}{T_1}} \|(y_0, z_0)\|_Z = Me^{\frac{M\gamma(1+\beta\gamma)}{(\gamma-1)T}} \|(y_0, z_0)\|_Z, \tag{5.70}$$

where in the last inclusion, we have used the fact that $T_2 = T - \dfrac{T}{\gamma^2}$ and $\rho_0(T_1) = \rho_0(T_2) = e^{-\frac{\gamma^2 \beta M}{(\gamma-1)T}}$. Now, the quantity $\frac{M\gamma(1+\beta\gamma)}{(\gamma-1)}$ being positive, we eventually obtain (by combining (5.68) and (5.70))

$$\left\| \frac{q}{\rho_0} \right\|_{L^2(0,T)} \leq Ce^{CT + \frac{C}{T}} \left(\|(y_0, z_0)\|_Z + \left\| \left(\frac{\xi}{\rho_{\mathcal{S}}}, \frac{\eta}{\rho_{\mathcal{S}}} \right) \right\|_{L^2(0,T;Z)} \right), \tag{5.71}$$

where the constant $C > 0$ is independent in $T > 0$.

- *Control system with the source terms.* We now define

$$(y, z) = (\tilde{y}, \tilde{z}) + (\hat{y}, \hat{z}). \tag{5.72}$$

Then (y, z) satisfies the following system

$$\begin{cases} y_t - y_{xx} = \xi, & \text{in } (T_k, T_{k+1}) \times (0,1), \\ z_t - z_{xx} = \eta, & \text{in } (T_k, T_{k+1}) \times (0,1), \\ y_x(t,0) = \hat{q}_k(t), \quad z_x(t,0) = 0, & \text{for } t \in (T_k, T_{k+1}), \\ y_x(t,1) = z_x(t,1), & \text{for } t \in (T_k, T_{k+1}), \\ y(t,1) + z(t,1) + \alpha y_x(t,1) = 0, & \text{for } t \in (T_k, T_{k+1}), \\ (y(T_k, \cdot), z(T_k, \cdot)) = a_k, & \text{in } (0,1), \end{cases} \tag{5.73}$$

for all $k \geq 0$. Note that, the solution (y, z) satisfies

$$(y(T_0), z(T_0)) = a_0 = (y_0, z_0),$$

and, for all $k \geq 0$ we have

$$\left(y(T_{k+1}^-), z(T_{k+1}^-)\right) = \left(\tilde{y}(T_{k+1}^-), \tilde{z}(T_{k+1}^-)\right) + \left(\hat{y}(T_{k+1}^-), \hat{z}(T_{k+1}^-)\right) = a_{k+1},$$
$$\left(y(T_{k+1}^+), z(T_{k+1}^+)\right) = \left(\tilde{y}(T_{k+1}^+), \tilde{z}(T_{k+1}^+)\right) + \left(\hat{y}(T_{k+1}^+), \hat{z}(T_{k+1}^+)\right) = a_{k+1}.$$

Therefore (y, z) is continuous at T_k for all $k \geq 0$.

Now, applying the energy estimate (5.24) for the system (5.73), and using the estimations for a_{k+1} from (5.64) and $\hat{q}_{k+1}$ from (5.66), we have

$$\begin{aligned}
&\|(y,z)\|_{C^0([T_{k+1},T_{k+2}];Z)} + \|(y,z)\|_{L^2(T_{k+1},T_{k+2};\mathcal{H})} \\
&\leq Ce^{CT}\left(\|a_{k+1}\|_Z + \|(\xi,\eta)\|_{L^2(T_{k+1},T_{k+2};Z)} + \|\hat{q}_{k+1}\|_{L^2(T_{k+1},T_{k+2})}\right) \\
&\leq Ce^{CT}\left(\|a_{k+1}\|_Z + \|(\xi,\eta)\|_{L^2(T_{k+1},T_{k+2};Z)} + Me^{\frac{M}{T_{k+2}-T_{k+1}}}\|a_{k+1}\|_Z\right) \\
&\leq Ce^{CT}\|(\xi,\eta)\|_{L^2(T_k,T_{k+2};Z)} + Ce^{CT}e^{\frac{M}{T_{k+2}-T_{k+1}}}\|(\xi,\eta)\|_{L^2(T_k,T_{k+1};Z)} \\
&\leq Ce^{CT}e^{\frac{M}{T_{k+2}-T_{k+1}}}\|(\xi,\eta)\|_{L^2(T_k,T_{k+2};Z)},
\end{aligned}$$

for all $k \geq 0$.

Since $\rho_{\mathcal{S}}$ is non-increasing in $[T_k, T_{k+2}]$, we obtain from above,

$$\begin{aligned}
\|(y,z)\|_{C^0([T_{k+1},T_{k+2}];Z)} + \|(y,z)\|_{L^2(T_{k+1},T_{k+2};\mathcal{H})} &\leq Ce^{CT}e^{\frac{M}{T_{k+2}-T_{k+1}}}\rho_{\mathcal{S}}(T_k)\left\|\left(\frac{\xi}{\rho_{\mathcal{S}}},\frac{\eta}{\rho_{\mathcal{S}}}\right)\right\|_{L^2(T_k,T_{k+2};Z)} \\
&= Ce^{CT}\rho_0(T_{k+2})\left\|\left(\frac{\xi}{\rho_{\mathcal{S}}},\frac{\eta}{\rho_{\mathcal{S}}}\right)\right\|_{L^2(T_k,T_{k+2};Z)}, \qquad (5.74)
\end{aligned}$$

for all $k \geq 0$, since $\rho_0(T_{k+2}) = e^{\frac{M}{T_{k+2}-T_{k+1}}}\rho_{\mathcal{S}}(T_k)$ (see (5.60)).

Using the fact that ρ_0 is non-increasing on $[T_{k+1}, T_{k+2}]$, we further deduce from (5.74) that

$$\begin{aligned}
&\left\|\left(\frac{y}{\rho_0},\frac{z}{\rho_0}\right)\right\|_{C^0([T_{k+1},T_{k+2}];Z)} + \left\|\left(\frac{y}{\rho_0},\frac{z}{\rho_0}\right)\right\|_{L^2(T_{k+1},T_{k+2};\mathcal{H})} \\
&\leq \frac{1}{\rho_0(T_{k+2})}\left(\|(y,z)\|_{C^0([T_{k+1},T_{k+2}];Z)} + \|(y,z)\|_{L^2(T_{k+1},T_{k+2};\mathcal{H})}\right) \\
&\leq Ce^{CT}\left\|\left(\frac{\xi}{\rho_{\mathcal{S}}},\frac{\eta}{\rho_{\mathcal{S}}}\right)\right\|_{L^2(T_k,T_{k+2};Z)}, \qquad (5.75)
\end{aligned}$$

for all $k \geq 0$.

Now, it remains to find the estimates of (y, z) in $[0, T_1]$. Again, using the energy estimate (5.24) we find that (also having in mind $\rho_0(T_1) = e^{-\frac{\gamma\beta M}{(\gamma-1)T}}$)

$$\begin{aligned}
&\|(y,z)\|_{C^0([0,T_1];Z)} + \|(y,z)\|_{L^2(0,T_1;\mathcal{H})} \\
&\leq Ce^{CT}\left(\|a_0\|_Z + \|\hat{q}_0\|_{L^2(0,T_1)} + \|(\xi,\eta)\|_{L^2(0,T_1;Z)}\right) \\
&\leq Ce^{CT}\left(\|a_0\|_Z + Me^{\frac{M}{T_1}}\|a_0\|_Z + \|(\xi,\eta)\|_{L^2(0,T_1;Z)}\right) \\
&\leq Ce^{CT}e^{\frac{M\gamma(1+\beta)}{(\gamma-1)T}}\rho_0(T_1)\left(\|(y_0,z_0)\|_Z + Me^{\frac{M}{T_1}}\|a_0\|_Z + \|(\xi,\eta)\|_{L^2(0,T_1;Z)}\right).
\end{aligned}$$

But, ρ_0 and ρ_S are non-increasing functions in $[0, T_1]$ and thus the above estimate follows to:

$$\left\|\left(\frac{y}{\rho_0},\frac{z}{\rho_0}\right)\right\|_{C^0([0,T_1];Z)} + \left\|\left(\frac{y}{\rho_0},\frac{z}{\rho_0}\right)\right\|_{L^2(0,T_1;\mathcal{H})} \leq Ce^{CT+\frac{C}{T}}\left(\|(y_0,z_0)\|_Z + \left\|\left(\frac{\xi}{\rho_{\mathcal{S}}},\frac{\eta}{\rho_{\mathcal{S}}}\right)\right\|_{L^2(0,T_1;Z)}\right). \qquad (5.76)$$

Combining the estimates (5.75) and (5.76), we have

$$\left\|\left(\frac{y}{\rho_0},\frac{z}{\rho_0}\right)\right\|_{C^0([0,T];Z)} + \left\|\left(\frac{y}{\rho_0},\frac{z}{\rho_0}\right)\right\|_{L^2(0,T;\mathcal{H})} \leq Ce^{CT+\frac{C}{T}}\left(\|(y_0,z_0)\|_Z + \left\|\left(\frac{\xi}{\rho_{\mathcal{S}}},\frac{\eta}{\rho_{\mathcal{S}}}\right)\right\|_{L^2(0,T;Z)}\right), \qquad (5.77)$$

for some constant $C > 0$ independent in T.

The above bound (5.77) along with (5.71), we achieve the required estimate (5.58) of the proposition. This completes the proof. □

5.4.2 Application of Banach fixed point theorem

This section is devoted to prove the local null-controllability result of our nonlinear system (5.1), that is Theorem 5.1.1.

Let any parameter $\alpha \geq 0$ be given as earlier and assume any initial data $(y_0, z_0) \in Z$ such that $\|(y_0, z_0)\|_Z \leq \delta$, where $\delta > 0$ will be specified later. We now define the set

$$\mathfrak{S}_\delta := \left\{(\xi, \eta) \in \mathcal{S} \times \mathcal{S} \ : \ \|(\xi, \eta)\|_{\mathcal{S}\times\mathcal{S}} \leq \delta\right\},$$

where the space $\mathcal{S}$ is defined in (5.52).

By Proposition 5.4.1, we can say that for any given source term $(\xi, \eta) \in \mathcal{S}\times\mathcal{S}$, there exists a control $q \in L^2(0, T)$ such that the corresponding trajectory (y, z) of the system (5.48) satisfies the estimate (5.58). In what follows, we define the map $\mathfrak{F} : \mathfrak{S}_\delta \to L^2(0, T; Z)$ by

$$\mathfrak{F}(\xi, \eta) = \begin{pmatrix} f\big(y, z, \int_0^1 y, \int_0^1 z\big) \\ g\big(y, z, \int_0^1 y, \int_0^1 z\big) \end{pmatrix}, \tag{5.78}$$

for all $(\xi, \eta) \in \mathfrak{S}_\delta$, where we recall that the nonlinear functions f and g are givem by

$$\begin{cases} f\big(y, z, \int_0^1 y, \int_0^1 z\big) & = -yz + ay^2 + bz^2 + r_1(t)y, \\ g\big(y, z, \int_0^1 y, \int_0^1 z\big) & = yz + cy^2 + dz^2 + r_2(t)z, \end{cases} \tag{5.79}$$

where a, b, c, d are $L^\infty((0, T) \times (0, 1))$ functions and

$$\begin{cases} r_1(t) = \alpha_1 \displaystyle\int_0^1 \Big(\psi_{1,1}(x)y(t, x) + \psi_{2,1}(x)z(t, x)\Big)\, dx, \\ r_2(t) = \alpha_2 \displaystyle\int_0^1 \Big(\psi_{1,2}(x)y(t, x) + \psi_{2,2}(x)z(t, x)\Big)\, dx, \end{cases} \tag{5.80}$$

with $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\psi_{1,j}, \psi_{2,j} \in L^\infty(0, 1)$, $j = 1, 2$.

Our goal is to prove that there exists some $\delta > 0$ such that the map $\mathfrak{F}$ has a unique fixed point in the set $\mathfrak{S}_\delta$ and to do so, we shall apply the Banach fixed point theorem. We begin with the following lemma.

Lemma 5.4.1 (Stability)**.** *There exists some $\delta > 0$ such that $\mathfrak{S}_\delta \subset \mathcal{S} \times \mathcal{S}$ is stable under the map $\mathfrak{F}$.*

Proof. We have for $(\xi, \eta) \in \mathfrak{S}_\delta$,

$$\begin{aligned} \|\mathfrak{F}(\xi, \eta)\|^2_{\mathcal{S}\times\mathcal{S}} &= \left\| \begin{pmatrix} f\big(y, z, \int_0^1 y, \int_0^1 z\big) \\ g\big(y, z, \int_0^1 y, \int_0^1 z\big) \end{pmatrix} \right\|^2_{\mathcal{S}\times\mathcal{S}} \\ &\leq \left\|-yz + ay^2 + bz^2 + r_1(t)y\right\|^2_{\mathcal{S}} + \left\|yz + cy^2 + dz^2 + r_2(t)z\right\|^2_{\mathcal{S}}. \end{aligned}$$

Using the definition of norm in $\mathcal{S}$ (see (5.55)), we deduce from above that,

$$\begin{aligned} \|\mathfrak{F}(\xi, \eta)\|^2_{\mathcal{S}\times\mathcal{S}} \leq C \int_0^T \frac{1}{\rho^2_{\mathcal{S}}(t)} \Big(& \|y(t)z(t)\|^2_{L^2(0,1)} + \left\|y^2(t)\right\|^2_{L^2(0,1)} + \left\|z^2(t)\right\|^2_{L^2(0,1)} \\ & + \|r_1(t)y(t)\|^2_{L^2(0,1)} + \|r_2(t)z(t)\|^2_{L^2(0,1)}\Big)\, dt, \end{aligned} \tag{5.81}$$

where $C := C(\|a\|_{L^\infty}, \|b\|_{L^\infty}, \|c\|_{L^\infty}, \|d\|_{L^\infty}) > 0$. We now estimate the terms appearing in the r.h.s. of (5.81). Note that,

$$\|y(t)z(t)\|^2_{L^2(0,1)} = \int_0^1 |y(t,x)z(t,x)|^2\,dx \leq 2\int_0^1 \left(|y(t,x)|^4 + |z(t,x)|^4\right) dx. \tag{5.82}$$

and

$$\left\|y^2(t)\right\|^2_{L^2(0,1)} = \int_0^1 |y(t,x)|^4\,dx, \quad \left\|z^2(t)\right\|^2_{L^2(0,1)} = \int_0^1 |z(t,x)|^4\,dx. \tag{5.83}$$

We also have for $j = 1, 2$

$$\begin{aligned}\left\|r_j(t)y(t)\right\|^2_{L^2(0,1)} &= \left|\alpha_j\right|^2 \int_0^1 \left|y(t,x)\int_0^1 (\psi_{1,j}(x)y(t,x) + \psi_{2,j}(x)z(t,x))dx\right|^2 dx \\ &\leq C\left(\int_0^1 (|y(t,x)|^2 + |z(t,x)|^2)dx\right)\int_0^1 |y(t,x)|^2\,dx,\end{aligned} \tag{5.84}$$

where $C := C(|\alpha_1|, |\alpha_2|, \left\|\psi_{1,1}\right\|_{L^\infty}, \left\|\psi_{1,2}\right\|_{L^\infty}, \left\|\psi_{2,1}\right\|_{L^\infty}, \left\|\psi_{2,2}\right\|_{L^\infty}) > 0$.

Combining the above estimates (5.82), (5.83) and (5.84), we obtain from (5.81),

$$\begin{aligned}\|\mathfrak{F}(\xi,\eta)\|^2_{\mathcal{S}\times\mathcal{S}} &\leq C\int_0^T \frac{1}{\rho^2_{\mathcal{S}}(t)}\left(\int_0^1 \left(|y(t,x)|^4 + |z(t,x)|^4\right) dx\right) dt \\ &= C\int_0^T\int_0^1 \frac{\rho_0^4(t)}{\rho^2_{\mathcal{S}}(t)}\left|\frac{y(t,x)}{\rho_0(t)}\right|^4 dxdt + C\int_0^T\int_0^1 \frac{\rho_0^4(t)}{\rho^2_{\mathcal{S}}(t)}\left|\frac{z(t,x)}{\rho_0(t)}\right|^4 dxdt.\end{aligned} \tag{5.85}$$

Thanks to the fact (5.51), we get from (5.85) that

$$\begin{aligned}\|\mathfrak{F}(\xi,\eta)\|^2_{\mathcal{S}\times\mathcal{S}} &\leq C\int_0^T \left\|\frac{y(t)}{\rho_0(t)}\right\|^2_{L^\infty(0,1)}\left(\int_0^1 \left|\frac{y(t,x)}{\rho_0(t)}\right|^2 dx\right) dt + C\int_0^T \left\|\frac{z(t)}{\rho_0(t)}\right\|^2_{L^\infty(0,1)}\left(\int_0^1 \left|\frac{z(t,x)}{\rho_0(t)}\right|^2 dx\right) dt \\ &\leq C\int_0^T \left(\left\|\frac{y(t)}{\rho_0(t)}\right\|^2_{H^1(0,1)}\left\|\frac{y(t)}{\rho_0(t)}\right\|^2_{L^2(0,1)} + \left\|\frac{z(t)}{\rho_0(t)}\right\|^2_{H^1(0,1)}\left\|\frac{z(t)}{\rho_0(t)}\right\|^2_{L^2(0,1)}\right) dt \\ &\leq C\left\|\left(\frac{y}{\rho_0}, \frac{z}{\rho_0}\right)\right\|^2_{C^0([0,T];Z)}\left\|\left(\frac{y}{\rho_0}, \frac{z}{\rho_0}\right)\right\|^2_{L^2(0,T;\mathcal{H})}.\end{aligned}$$

Using the estimate (5.58) in above, we finally arrive to the following:

$$\begin{aligned}\|\mathfrak{F}(\xi,\eta)\|_{\mathcal{S}\times\mathcal{S}} &\leq C\left(\left\|\left(\frac{y}{\rho_0}, \frac{z}{\rho_0}\right)\right\|_{C^0([0,T];Z)} + \left\|\left(\frac{y}{\rho_0}, \frac{z}{\rho_0}\right)\right\|_{L^2(0,T;\mathcal{H})}\right)^2 \\ &\leq Ce^{CT+\frac{C}{T}}\left(\|(y_0,z_0)\|_Z + \|(\xi,\eta)\|_{\mathcal{S}\times\mathcal{S}}\right)^2 \\ &\leq Ce^{CT+\frac{C}{T}}\delta^2,\end{aligned} \tag{5.86}$$

due to our choices of initial data $\|(y_0,z_0)\|_Z \leq \delta$ and source terms $(\xi,\eta) \in \mathfrak{S}_\delta$.

Now, one can choose $\delta > 0$ small enough in (5.86) so that we have $\|\mathfrak{F}(\xi,\eta)\|_{\mathcal{S}\times\mathcal{S}} \leq \delta$ for all $(\xi,\eta) \in \mathfrak{S}_\delta$. This concludes our lemma. □

The following lemma shows that $\mathfrak{F} : \mathfrak{S}_\delta \to \mathfrak{S}_\delta$ is a contraction map.

Lemma 5.4.2 (Contraction)**.** *There exists a $\delta > 0$ such that the map $\mathfrak{F}$ defined by* (5.78) *is a contraction map on the closed ball $\mathfrak{S}_\delta$.*

Proof. Consider any two pairs $(\xi_i, \eta_i) \in \mathfrak{S}_\delta$ for $i = 1, 2$. Then, by means of Proposition 5.4.1, there exist control functions $q_i \in \mathcal{Q}$ for the system (5.48) with solutions $(y_i, z_i) \in \mathcal{Y}$ associated to $(\xi_i, \eta_i) \in \mathfrak{S}_\delta$ for $i = 1, 2$.

Accordingly, we use the notations f_i, g_i for the nonlinear functions (see (5.79)–(5.80)) where

$$\begin{cases} f_i\big(y_i, z_i, \int_0^1 y_i, \int_0^1 z_i\big) & = -y_i z_i + a y_i^2 + b z_i^2 + r_{i,1}(t) y_i, \\ g_i\big(y_i, z_i, \int_0^1 y_i, \int_0^1 z_i\big) & = y_i z_i + c y_i^2 + d z_i^2 + r_{i,2}(t) z_i, \end{cases}$$

with

$$\begin{cases} r_{i,1}(t) = \alpha_1 \displaystyle\int_0^1 \Big(\psi_{1,1}(x) y_i(t,x) + \psi_{2,1}(x) z_i(t,x)\Big)\, dx, \\ r_{i,2}(t) = \alpha_2 \displaystyle\int_0^1 \Big(\psi_{1,2}(x) y_i(t,x) + \psi_{2,2}(x) z_i(t,x)\Big)\, dx, \end{cases}$$

for $i = 1, 2$. Then, we compute

$$\begin{aligned} &\|\mathfrak{F}(\xi_1, \eta_1) - \mathfrak{F}(\xi_2, \eta_2)\|^2_{\mathcal{S}\times\mathcal{S}} \\ &= \left\| \begin{pmatrix} -y_1 z_1 + a y_1^2 + b z_1^2 + r_{1,1}(t) y_1 \\ y_1 z_1 + c y_1^2 + d z_1^2 + r_{1,2}(t) z_1 \end{pmatrix} - \begin{pmatrix} -y_2 z_2 + a y_2^2 + b z_2^2 + r_{2,1}(t) y_2 \\ y_2 z_2 + c y_2^2 + d z_2^2 + r_{2,2}(t) z_2 \end{pmatrix} \right\|^2_{\mathcal{S}\times\mathcal{S}} \\ &= \left\| \begin{pmatrix} -(y_1 z_1 - y_2 z_2) + a(y_1^2 - y_2^2) + b(z_1^2 - z_2^2) + r_{1,1}(t) y_1 - r_{2,1}(t) y_2 \\ y_1 z_1 - y_2 z_2 + c(y_1^2 - y_2^2) + d(z_1^2 - z_2^2) + r_{1,2}(t) z_1 - r_{2,2}(t) z_2 \end{pmatrix} \right\|^2_{\mathcal{S}\times\mathcal{S}} \\ &\le C \int_0^T \frac{1}{\rho^2_{\mathcal{S}}(t)} \Big(\|y_1(t) z_1(t) - y_2(t) z_2(t)\|^2_{L^2(0,1)} + \big\|y_1^2(t) - y_2^2(t)\big\|^2_{L^2(0,1)} \\ &\quad + \big\|z_1^2(t) - z_2^2(t)\big\|^2_{L^2(0,1)} + \big\|r_{1,1}(t) y_1(t) - r_{2,1}(t) y_2(t)\big\|^2_{L^2(0,1)} \\ &\quad + \big\|r_{1,2}(t) z_1(t) - r_{2,2}(t) z_2(t)\big\|^2_{L^2(0,1)} \Big) dt. \end{aligned} \tag{5.87}$$

To this end, we find

$$\begin{aligned} &\|y_1(t) z_1(t) - y_2(t) z_2(t)\|^2_{L^2(0,1)} \qquad (5.88) \\ &\le 2 \Big(\|y_1(t)(z_1(t) - z_2(t))\|^2_{L^2(0,1)} + \|(y_1(t) - y_2(t)) z_2(t)\|^2_{L^2(0,1)} \Big) \\ &\le C \|y_1(t)\|^2_{L^\infty(0,1)} \|z_1(t) - z_2(t)\|^2_{L^2(0,1)} + C \|z_2(t)\|^2_{L^\infty(0,1)} \|y_1(t) - y_2(t)\|^2_{L^2(0,1)} \\ &\le C \|y_1(t)\|^2_{H^1(0,1)} \|z_1(t) - z_2(t)\|^2_{L^2(0,1)} + C \|z_2(t)\|^2_{H^1(0,1)} \|y_1(t) - y_2(t)\|^2_{L^2(0,1)} . \end{aligned} \tag{5.89}$$

A straightforward computation also gives

$$\big\|y_1^2(t) - y_2^2(t)\big\|^2_{L^2(0,1)} \le \Big(\|y_1(t)\|^2_{H^1(0,1)} + \|y_2(t)\|^2_{H^1(0,1)} \Big) \|y_1(t) - y_2(t)\|^2_{L^2(0,1)} , \tag{5.90}$$

and

$$\big\|z_1^2(t) - z_2^2(t)\big\|^2_{L^2(0,1)} \le \Big(\|z_1(t)\|^2_{H^1(0,1)} + \|z_2(t)\|^2_{H^1(0,1)} \Big) \|z_1(t) - z_2(t)\|^2_{L^2(0,1)} . \tag{5.91}$$

Next we look at the remaining terms in (5.87), we compute

$$\begin{aligned} &\big\|r_{1,1}(t) y_1(t) - r_{2,1}(t) y_2(t)\big\|^2_{L^2(0,1)} \\ &\le 2 \int_0^1 \big|r_{1,1}(t)(y_1(t,x) - y_2(t,x))\big|^2 dx + 2 \int_0^1 \big|(r_{1,1}(t) - r_{2,1}(t)) y_2(t,x)\big|^2 dx \\ &\le 2 |\alpha_1|^2 \left| \int_0^1 (\psi_{1,1}(x) y_1(t,x) + \psi_{2,1}(x) z_1(t,x)) dx \right|^2 \int_0^1 |y_1(t,x) - y_2(t,x)|^2 dx \\ &\quad + 2 \int_0^1 |y_2(t,x)|^2 dx \left| \alpha_1 \int_0^1 (\psi_{1,1}(x) y_1(t,x) + \psi_{2,1}(x) z_1(t,x)) dx \right. \end{aligned}$$

$$
\begin{aligned}
&\qquad - \alpha_1 \int_0^1 (\psi_{1,1}(x) y_2(t,x) + \psi_{2,1}(x) z_2(t,x)) dx \Big|^2 \\
&\le C \left\| y_1(t) - y_2(t) \right\|^2_{L^2(0,1)} \int_0^1 (|y_1(t,x)|^2 + |z_1(t,x)|^2) dx \\
&\quad + C \left\| y_2(t) \right\|^2_{L^2(0,1)} \int_0^1 \left(\left|\psi_{1,1}(x)\right|^2 |y_1(t,x) - y_2(t,x)|^2 + \left|\psi_{2,1}(x)\right|^2 |z_1(t,x) - z_2(t,x)|^2 \right) \\
&\le C \left(\|y_1(t)\|^2_{L^2(0,1)} + \|y_2(t)\|^2_{L^2(0,1)} + \|z_1(t)\|^2_{L^2(0,1)} \right) \\
&\qquad \times \left(\|y_1(t) - y_2(t)\|^2_{L^2(0,1)} + \|z_1(t) - z_2(t)\|^2_{L^2(0,1)} \right). \qquad (5.92)
\end{aligned}
$$

We similarly obtain

$$
\begin{aligned}
\left\| r_{1,2}(t) z_1(t) - r_{2,2}(t) z_2(t) \right\|^2_{L^2(0,1)} &\le C \left(\|y_1(t)\|^2_{L^2(0,1)} + \|z_1(t)\|^2_{L^2(0,1)} + \|z_2(t)\|^2_{L^2(0,1)} \right) \\
&\qquad \times \left(\|y_1(t) - y_2(t)\|^2_{L^2(0,1)} + \|z_1(t) - z_2(t)\|^2_{L^2(0,1)} \right). \qquad (5.93)
\end{aligned}
$$

Combining the estimates (5.88), (5.90), (5.91), (5.92) and (5.93), we obtain from (5.87), that

$$
\begin{aligned}
&\|\mathfrak{F}(\xi_1, \eta_1) - \mathfrak{F}(\xi_2, \eta_2)\|^2_{\mathcal{S}\times\mathcal{S}} \\
&\le C \int_0^T \frac{1}{\rho^2_{\mathcal{S}}(t)} \left[\|(y_1(t), z_1(t))\|^2_{\mathcal{H}} + \|(y_2(t), z_2(t))\|^2_{\mathcal{H}} \right] \times \left\| (y_1(t) - y_2(t), z_1(t) - z_2(t)) \right\|^2_Z dt \\
&\le C \int_0^T \frac{\rho_0^4(t)}{\rho^2_{\mathcal{S}}(t)} \left[\left\| \left(\frac{y_1(t)}{\rho_0(t)}, \frac{z_1(t)}{\rho_0(t)} \right) \right\|^2_{\mathcal{H}} + \left\| \left(\frac{y_2(t)}{\rho_0(t)}, \frac{z_2(t)}{\rho_0(t)} \right) \right\|^2_{\mathcal{H}} \right] \times \left\| \left(\frac{y_1(t) - y_2(t)}{\rho_0(t)}, \frac{z_1(t) - z_2(t)}{\rho_0(t)} \right) \right\|^2_Z dt \\
&\le C \left\| \left(\frac{y_1}{\rho_0}, \frac{z_1}{\rho_0} \right) - \left(\frac{y_2}{\rho_0}, \frac{z_2}{\rho_0} \right) \right\|^2_{C^0([0,T];Z)} \times \left[\left\| \left(\frac{y_1}{\rho_0}, \frac{z_1}{\rho_0} \right) \right\|^2_{L^2(0,T;\mathcal{H})} + \left\| \left(\frac{y_2}{\rho_0}, \frac{z_2}{\rho_0} \right) \right\|^2_{L^2(0,T;\mathcal{H})} \right], \qquad (5.94)
\end{aligned}
$$

where we have used the fact that $\frac{\rho_0^2(t)}{\rho_{\mathcal{S}}(t)} \le 1$ (see (5.51)).

But, due to the linearity of the solution map (see Proposition 5.4.1), we have the following estimate (by means of (5.58))

$$
\left\| \left(\frac{y_1}{\rho_0}, \frac{z_1}{\rho_0} \right) - \left(\frac{y_2}{\rho_0}, \frac{z_2}{\rho_0} \right) \right\|_{C^0([0,T];Z)} + \left\| \left(\frac{y_1}{\rho_0}, \frac{z_1}{\rho_0} \right) - \left(\frac{y_2}{\rho_0}, \frac{z_2}{\rho_0} \right) \right\|_{L^2(0,T;\mathcal{H})} \le Ce^{CT + \frac{C}{T}} \|(\xi_1, \eta_1) - (\xi_2, \eta_2)\|_{\mathcal{S}\times\mathcal{S}}.
$$

Using the above bound and the estimate (5.58) in (5.94), we get

$$
\begin{aligned}
&\|\mathfrak{F}(\xi_1, \eta_1) - \mathfrak{F}(\xi_2, \eta_2)\|_{\mathcal{S}\times\mathcal{S}} \\
&\quad \le Ce^{CT + \frac{C}{T}} \|(\xi_1, \eta_1) - (\xi_2, \eta_2)\|_{\mathcal{S}\times\mathcal{S}} \times \left[\|(y_0, z_0)\|_Z + \|(\xi_1, \eta_1)\|_{\mathcal{S}\times\mathcal{S}} + \|(\xi_2, \eta_2)\|_{\mathcal{S}\times\mathcal{S}} \right] \\
&\quad \le Ce^{CT + \frac{C}{T}} \delta \|(\xi_1, \eta_1) - (\xi_2, \eta_2)\|_{\mathcal{S}\times\mathcal{S}} \\
&\quad \le \frac{1}{2} \|(\xi_1, \eta_1) - (\xi_2, \eta_2)\|_{\mathcal{S}\times\mathcal{S}},
\end{aligned}
$$

for chosen $0 < \delta \le \frac{1}{2Ce^{CT + C/T}}$.

This proves the contraction property of the map $\mathfrak{F}$ in the closed ball $\mathfrak{S}_\delta$ provided we start with initial data $\|(y_0, z_0)\|_Z \le \delta$ and source terms in $\mathfrak{S}_\delta$. □

We now conclude the proof of the main result of our work.

Proof of Theorem 5.1.1. Let any boundary parameter $\alpha \ge 0$ and time $T > 0$ be given. According to Lemma 5.4.1 and Lemma 5.4.2, there exists some $\delta > 0$ small enough such that if we choose the initial data $(y_0, z_0) \in Z$ with $\|(y_0, z_0)\|_Z \le \delta$, then by using Banach fixed point theorem we can ensure that the map $\mathfrak{F} : \mathfrak{S}_\delta \to \mathfrak{S}_\delta$ (defined by (5.78)) has a unique fixed point $(\hat{\xi}, \hat{\eta}) \in \mathfrak{S}_\delta$.

At this point, by means of Proposition 5.4.1, there exists a solution-control pair $((y,z),q) \in \mathcal{Y} \times Q$ to the system (5.48) associated with the above source term $(\hat{\xi}, \hat{\eta}) \in \mathfrak{S}_\delta$, which in addition satisfy the estimate (5.58). Then, by construction of the space $\mathcal{Y}$ (see (5.53)) and the property $\lim_{t \to T^-} \rho_0(t) = 0$ force the solution (y,z) to satisfy

$$y(T,x) = 0, \quad z(T,x) = 0, \quad \forall x \in (0,1),$$

which is the required boundary local null-controllability result of our nonlinear system (5.1). □

5.5 Concluding remarks

In the present paper, we study the controllability property of a parabolic system where the boundary couplings are posed in terms of the δ'-type condition. The linear model of our work (see (5.4)) simply consists of the aforementioned boundary couplings, and no internal coupling appears. It would be interesting if one could impose internal coupling(s) as well, for instance let us consider the following linear system,

$$\begin{cases} y_t - y_{xx} + k_1 z = 0, & \text{in } (0,T) \times (0,1), \\ z_t - z_{xx} + k_2 y = 0, & \text{in } (0,T) \times (0,1), \\ y_x(t,0) = q(t), \quad z_x(t,0) = 0, & \text{for } t \in (0,T), \\ y_x(t,1) = z_x(t,1), & \text{for } t \in (0,T), \\ y(t,1) + z(t,1) + \alpha y_x(t,1) = 0, & \text{for } t \in (0,T), \\ y(0,x) = y_0(x), \quad z(0,x) = z_0(x), & \text{in } (0,1), \end{cases} \tag{5.95}$$

with some constants $(k_1, k_2) \neq (0,0)$. In this regard, we mention the work [BBHS21], where the presence of a zeroth order internal coupling in a parabolic system with Kirchhoff boundary condition leads to different controllability results depending on the position of the boundary control (i.e., on y or, on z). To study the controllability of system (5.95), the main work will be to investigate the spectral properties of the associated adjoint operator, which is not so obvious in the case of δ'-type boundary condition. Thus, this needs further care and it is an interesting open problem in the viewpoint of controllability of coupled parabolic systems.

Chapter 6

Conclusion and Future Directions

In this chapter, we will summarize the contents of this thesis and mention some of the open questions and future directions that can be pursued based on the work in this thesis.

We have first considered the compressible Navier-Stokes systems linearized around some constant steady states (Q_0, V_0) (with $Q_0, V_0 > 0$) for barotropic fluids and around (Q_0, V_0, ψ_0) (with $Q_0, V_0, \psi_0 > 0$) for non-barotropic fluids. In the barotropic case, we have considered three types of boundary conditions onto the system, namely Periodic, Dirichlet and mixed (Periodic-Dirichlet) type, and studied the null controllability using only one boundary control acting either in density or velocity. We summarize the results that we have obtained for the barotropic system in the following table (NC=null controllable); see Theorems 1.1.1–1.1.5.

Barotropic Case		
Boundary conditions	Controls acting in	
	density	velocity
Periodic	• NC at $T > \frac{2\pi}{V_0}$ in $\dot{L}^2 \times \dot{L}^2$ iff $\frac{2\sqrt{bQ_0 - V_0^2}}{\mu_0} \notin \mathbb{N}$. • Not NC at $0 < T < \frac{2\pi}{V_0}$ in $\dot{L}^2 \times \dot{L}^2$.	• NC at $T > \frac{2\pi}{V_0}$ in $\dot{H}^1 \times \dot{L}^2$ iff $\frac{2\sqrt{bQ_0 - V_0^2}}{\mu_0} \notin \mathbb{N}$. • Not NC at any $T > 0$ in $\dot{H}^s \times \dot{L}^2$ for $0 \leq s < 1$.
Dirichlet	• NC at $T > 1$ in $\dot{L}^2 \times L^2$ if $c^4 + 8c^2 + 5 < 4\pi^2$. • Not NC at $0 < T < 1$ in $L^2 \times L^2$.	Unknown
Mixed-type	• NC at $T > 1$ in $\dot{L}^2 \times L^2$ if $c^4 + 8c^2 + 5 < 4\pi^2$. • Not NC at $0 < T < 1$ in $L^2 \times L^2$.	• NC at $T > 1$ in $\dot{H}_\sharp^{\frac{1}{2}} \times L^2$ if $c^4 + 8c^2 + 5 < 4\pi^2$. • Not NC at any $T > 0$ in $\dot{H}_\sharp^s \times L^2$ for $0 \leq s < \frac{1}{2}$.

As a consequence of the above null controllability results, we obtained approximate controllability of the barotropic system at large time T in respective spaces.

On the other hand, for non-barotropic fluids, we have only considered the periodic setup and studied boundary null controllability properties of the linearized system using only one control acting either in density, velocity or temperature. Using a boundary control acting only in the density part, we have proved null controllability of the system at time $T > \frac{2\pi}{V_0}$ in $(\dot{L}^2(0, 2\pi))^3$ under two assumptions; (i) the eigenvalues of A^* have geometric multiplicity 1 and (ii) the coefficients λ_0, κ_0 satisfies $\sqrt{\frac{\lambda_0}{\kappa_0}} \notin \mathbb{Q}$ and $\left|\sqrt{\frac{\lambda_0}{\kappa_0}} - \frac{a}{b}\right| > \frac{1}{b^M}$ for all rationals $\frac{a}{b}$ and some $M > 0$ (see Theorem 1.1.6-Part (i)). Further, in this case, we have proved that null controllability fails when the time is small, that is $0 < T < \frac{2\pi}{V_0}$, in the space $(L^2(0, 2\pi))^3$ (Proposition 1.1.1-Part (i)). When a boundary control acts either in velocity or temperature, we have proved null controllability of the linearized system at time $T > \frac{2\pi}{V_0}$ in $\dot{H}^1_{\text{per}}(0, 2\pi) \times (\dot{L}^2(0, 2\pi))^2$ under the same two hypotheses mentioned above (see Theorem 1.1.6-Part (ii)) and that null controllability fails in the space $\dot{H}^s_{\text{per}}(0, 2\pi) \times (\dot{L}^2(0, 2\pi))^2$ with $0 \leq s < 1$ at any $T > 0$ (Proposition 1.1.1-Part (ii)). In addition, we have proved that the condition $\sqrt{\frac{\lambda_0}{\kappa_0}} \notin \mathbb{Q}$ on the coefficients is not sufficient to conclude null controllability of the linearized system (see Proposition 1.1.2).

Finally, we have considered a coupled system consisting of two nonlinear parabolic equations with square, product and non-local nonlinearities. In the system, a Neumann boundary control is applied to only one state while the other satisfies homogeneous Neumann boundary condition at $x = 0$. On the other hand, the states are coupled at $x = 1$ in terms of "equality condition of their normal derivatives" and a combined Robin-type condition. In this setup, we have proved small-time local null controllability of the system in the space $(L^2(0,1))^2$ by applying the so called "source term method".

In view of all the above discussions, we now make some comments and give future directions on controllability of the systems considered/ related to this thesis, which will be addressed soon.

6.1 Linearized compressible Navier-Stokes system (barotropic case)

6.1.1 Optimal in time

We have seen that the system (1.9) is null controllable at time $T > \frac{2\pi}{V_0}$ and is not null controllable at $0 < T < \frac{2\pi}{V_0}$. The question of null controllability of (1.9) at the optimal time $T = \frac{2\pi}{V_0}$ is still open. We mention here that, to prove null controllability of (1.9) at time $T = T_0 := \frac{2\pi}{V_0}$, it is sufficient to solve the following sets of moments problem

$$\int_0^{T_0} e^{\nu_n^h (T_0 - t)} p(t)dt = c_n,$$
$$\int_0^{T_0} e^{\nu_n^p (T_0 - t)} p(t)dt = d_n,$$

for all $n \in \mathbb{Z}$ and for some sequences $(c_n)_{n\in\mathbb{Z}}, (d_n)_{n\in\mathbb{Z}}$, where $(\nu_n^h)_{n\in\mathbb{Z}}$ and $(\nu_n^p)_{n\in\mathbb{Z}}$ are eigenvalues of A^*, given by (3.22)–(3.23) respectively. To solve these moments problem, one might adapt the approach of Martin, Rosier and Rouchon [MRR13] to find a suitable biorthogonal sequence of the family $\{\nu_n^h,\ \nu_n^p\ ;\ n \in \mathbb{Z}\}$, see for instance [CM15]. Since we have explicit expressions of these eigenvalues, this method might be useful here. However, one might need to take more regular initial states (possibly) due to the bounds on the biorthogonal family corresponding to $\left(e^{\nu_n^h t}\right)_{n\in\mathbb{Z}}$; see [MRR13, Proposition 2.2] or [CM15, Proposition 3.2]. The same question can be asked for the systems (1.12) and (1.13).

6.1.2 Controllability under Dirichlet boundary conditions

Let $T, L > 0$ be given. We consider the following system

$$\begin{cases} \rho_t + V_0\rho_x + Q_0 u_x = 0, & \text{in } (0,T)\times(0,L),\\ u_t - \mu_0 u_{xx} + V_0 u_x + b\rho_x = 0, & \text{in } (0,T)\times(0,L),\\ \rho(t,0) = 0, \quad u(t,0) = 0, \quad u(t,L) = q(t), & \text{for } t \in (0,T),\\ \rho(0,x) = \rho_0(x), \quad u(0,x) = u_0(x), & \text{in } (0,L), \end{cases} \tag{6.1}$$

where $q \in L^2(0,T)$ is a boundary control. In this setup, no controllability result is known for the system (6.1) and in fact, it is a very challenging and interesting open problem. We mention here that the associated linearized operator has compact resolvent and so we have the existence of spectrum of the linearized operator. However, finding explicit (or even asymptotic) expression of the eigenfunctions of the linearized operator is a very challenging task. This difficulty arises due to the fact that the operator $\frac{d}{dx}$ on $H^1_{\{0\}}(0,L)$ do not have any non-trivial spectrum. Thus, to solve this problem, one need to apply different methods that do not require the explicit spectrum of the associated linearized operator.

6.1.3 The vanishing viscosity method

As mentioned earlier, controllability results for the system (6.1) at time T is unknown. To study the controllability of (6.1), one may apply the vanishing viscosity method, introduced by Coron and Guerrero [CG05] in the following way.

Let $T, L > 0$. For given $\varepsilon > 0$, we consider the following problem

$$\begin{cases} \rho_t - \varepsilon\rho_{xx} + V_0\rho_x + Q_0 u_x = 0, & \text{in } (0,T)\times(0,L), \\ u_t - \mu_0 u_{xx} + V_0 u_x + b\rho_x = 0, & \text{in } (0,T)\times(0,L), \\ \rho(t,0) = 0, \quad u(t,0) = 0, \quad u(t,L) = q(t), & \text{for } t \in (0,T), \\ \rho(0,x) = \rho_0(x), \quad u(0,x) = u_0(x), & \text{in } (0,L), \end{cases} \tag{6.2}$$

where $q \in L^2(0,T)$ is the control input. Note that, this system is an one parameter family of parabolic equations having first order coupling. There are several methods to deal with the controllability of this system, but here we are interested on the explicit dependence of the observability constant/ control cost in terms of this ε. Then, by passing ε tends to 0, one may conclude some controllability results for the Navier-Stokes system (6.1). We note here that the method of moments or some suitable Carleman estimates might be helpful to find the explicit dependence of the control cost with respect to this ε.

Note that, in the system (6.2), we still have the difficulty related to the (Dirichlet) boundary conditions. To avoid this, one may consider the following system:

$$\begin{cases} \rho_t + V_0\rho_x + Q_0 u_x = 0, & \text{in } (0,T)\times(0,L), \\ u_t - \mu_0 u_{xx} + V_0 u_x + b\rho_x = 0, & \text{in } (0,T)\times(0,L), \\ \rho(t,0) = \varepsilon\rho(t,L), & \text{for } t \in (0,T), \\ u(t,0) = 0, \quad u(t,L) = q(t), & \text{for } t \in (0,T), \\ \rho(0,x) = \rho_0(x), \quad u(0,x) = u_0(x), & \text{in } (0,L), \end{cases} \tag{6.3}$$

for some $\varepsilon > 0$ small enough. In the particular case when $\varepsilon = 1$, we have already studied controllability of this system using a boundary control acting on velocity, see Chapter 4. The same analysis can be done by taking this small parameter ε. The only thing one require is a uniform estimate of the control with respect to this ε. Then, by passing ε tends to 0, one may conclude some controllability results for the Dirichlet system (6.1).

6.1.4 Distributed controllability under Dirichlet conditions

Let $T, L > 0$. We consider the following problem

$$\begin{cases} \rho_t + V_0\rho_x + Q_0 u_x = 0, & \text{in } (0,T)\times(0,L), \\ u_t - \mu_0 u_{xx} + V_0 u_x + b\rho_x = f\chi_\omega, & \text{in } (0,T)\times(0,L), \\ \rho(t,0) = 0, \quad u(t,0) = 0, \quad u(t,L) = 0, & \text{for } t \in (0,T), \\ \rho(0,x) = \rho_0(x), \quad u(0,x) = u_0(x), & \text{in } (0,L), \end{cases} \tag{6.4}$$

where $f \in L^2(0,T; L^2(0,L))$ is a distributed control acting in the velocity equation and supported on $\omega := (0,\ell) \subset (0,L)$. It is known in [AMM22] that this system is not null controllable in $(L^2(0,L))^2$ when the time is small, that is $0 < T < \frac{2\pi}{V_0}$. Moreover, it is also known in [Cho15] that this system (6.4) is approximately controllable at large time T in the space $L^2(0,L)\times L^2(0,L)$. However, there is no known null controllability results available in the literature for large time T. We give below an useful method that might be applicable here to prove null controllability of the system (6.4) at a large time.

Step 1. We first consider the cascade system

$$\begin{cases} \rho_t + V_0\rho_x + Q_0 u_x = 0, & \text{in } (0,T)\times(0,L), \\ u_t - \mu_0 u_{xx} + V_0 u_x = f\chi_\omega, & \text{in } (0,T)\times(0,L), \\ \rho(t,0) = 0, \quad u(t,0) = 0, \quad u(t,L) = 0, & \text{for } t \in (0,T), \\ \rho(0,x) = \rho_0(x), \quad u(0,x) = u_0(x), & \text{in } (0,L), \end{cases} \tag{6.5}$$

The method addressed in [FCdT04] might be helpful here to prove null controllability of this system (6.5), where one can utilize the exact controllability of the transport equation together with a suitable Carleman estimate for the parabolic equation to prove the required observability inequality. We mention here that one may not able to find a basis of the associated adjoint operator. In fact, $(0, \xi_n)$, for $n \in \mathbb{N}$, are the only eigenfunctions of the associated adjoint operator, where ξ_n is the eigenfunction of the (Dirichlet) operator $\mu_0 \partial_{xx} + V_0 \partial_x$. Note that, first component of the eigenfunctions is zero because the operator ∂_x on $H^1_{\{L\}}(0, L)$ do not have any non-trivial spectrum, as mentioned before in Section 6.1.2.

Step 2. We then consider the following system

$$\begin{cases} \rho_t + V_0 \rho_x + Q_0 u_x = 0, & \text{in } (0,T)\times(0,L), \\ u_t - \mu_0 u_{xx} + V_0 u_x + \xi = f\chi_\omega, & \text{in } (0,T)\times(0,L), \\ \rho(t,0) = 0, \quad u(t,0) = 0, \quad u(t,L) = 0, & \text{for } t \in (0,T), \\ \rho(0,x) = \rho_0(x), \quad u(0,x) = u_0(x), & \text{in } (0,L), \end{cases} \tag{6.6}$$

for some function ξ. Once the null controllability of the above system (6.5) is proved, one may apply some fixed-point method (Banach or Schauder) by defining a map $\xi \mapsto b\rho_x$ in some suitable spaces to conclude null controllability results for the main system (6.4).

This result has an importance in the context of boundary controllability of the Dirichlet system (6.1). More precisely, for given $\varepsilon > 0$, if we choose $\omega = (L - \varepsilon, L)$ in the system (6.4), then boundary controllability of the system (6.1) can be achieved from distributed controllability of (6.4) by taking ε tends to 0. In fact, the distributed control supported in the interval $(L - \varepsilon, L)$ will converge to the boundary control at $x = L$ as $\varepsilon \to 0$. This kind of technique has been applied in may works, see for instance [CSPS20, Fab92].

6.2 Linearized compressible Navier-Stokes system (non-barotropic case)

Like the barotropic case, one can ask the similar questions for the non-barotropic case also, which we listed below.

6.2.1 Optimal in time

The question of proving null controllability of the system (1.17) at the optimal time $T = \frac{2\pi}{V_0}$ will not be similar to the barotropic case mentioned above (see Section 6.1.1). This is because, in this case, we don't have explicit expression of the eigenvalues, and therefore it will not be straightforward to apply the idea of Martin, Rosier and Rouchon [MRR13] directly. However, one may adapt some modified techniques to solve the corresponding moments problem to conclude the null controllability at $T = \frac{2\pi}{V_0}$ in this case.

6.2.2 Controllability under Dirichlet boundary conditions

Let $T, L > 0$ be given. We consider the following system

$$\begin{cases} \rho_t + V_0 \rho_x + Q_0 u_x = f\chi_{\mathcal{O}_1}, & \text{in } (0,T)\times(0,L), \\ u_t - \lambda_0 u_{xx} + \dfrac{R\psi_0}{Q_0}\rho_x + V_0 u_x + R\theta_x = g\chi_{\mathcal{O}_2}, & \text{in } (0,T)\times(0,L), \\ \theta_t - \kappa_0 \theta_{xx} + \dfrac{R\psi_0}{c_v} u_x + V_0 \theta_x = h\chi_{\mathcal{O}_3}, & \text{in } (0,T)\times(0,L), \\ \rho(t,0) = p(t), \quad u(t,0) = 0, \quad u(t,L) = q(t), & \text{for } t \in (0,T), \\ \theta(t,0) = 0, \quad \theta(t,L) = r(t), & \text{for } t \in (0,T), \\ \rho(0,x) = \rho_0(x),\ u(0,x) = u_0(x),\ \theta(0,x) = \theta_0(x), & \text{in } (0,L), \end{cases} \tag{6.7}$$

where $p, q, r \in L^2(0,T)$ are boundary controls and $f, g, h \in L^2(0,T; L^2(0,L))$ are distributed controls supported in the subsets $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3 \subset (0,L)$ respectively. In this setup, one can study similar controllability problems using only one control (distributed/ boundary) acting either in density, velocity or temperature; as mentioned before in Sections 6.1.2–6.1.4. We note here that, in the case of boundary controllability using only one control $p \in L^2(0,T)$ acting on the density part, one can introduce the mixed-type boundary conditions as mentioned in the barotropic case, that is, $\rho(t,0) = \rho(t,L) + h(t)$ for $t \in (0,T)$, and by defining $p(t) = \rho(t,L) = h(t)$ for $t \in (0,T)$, one may obtain the desired null controllability result for the Dirichlet system (6.7).

6.3 Nonlinear compressible Navier-Stokes system

Let $T, L > 0$ be given. We recall the following nonlinear Navier-Stokes system for compressible barotropic fluids

$$\begin{cases} \rho_t + (\rho u)_x = 0, & \text{in } (0,T)\times(0,L), \\ \rho(u_t + uu_x) + a\gamma\rho^{\gamma-1}\rho_x - (\lambda+2\mu)u_{xx} = 0, & \text{in } (0,T)\times(0,L). \end{cases} \tag{6.8}$$

In this case, we wish to study small-time local (or global) controllability of the compressible Navier-Stokes equation (1.1) around a constant steady state using a distributed/ boundary control acting either in density or velocity. There are some known results available regarding the local exact controllability of the nonlinear system (1.1) around C^2-trajectory using two boundary controls, see for instance the works [EGGP12, ES18]; see also the work [EGG16] in higher dimensional case. However, no (local) controllability results are known for the system (1.1) using one boundary control (acting either on density or velocity).

We wish to study the local null controllability of this system at time T using only one boundary control acting either on density or velocity. Recall that, in the periodic setup, we have proved null controllability of the associated linearized system (using a boundary condition acting either on density or velocity), provided the coefficients satisfy a necessary and sufficient condition $\frac{2\sqrt{bQ_0 - V_0^2}}{\mu_0} \notin \mathbb{N}$. Whereas, when a boundary control acts on the velocity component through the mixed-type conditions, we have obtained null controllability at large time provided the coefficient c lies outside a countable set $\mathcal{N}$. Thus, we can ask local null controllability of the nonlinear system (6.8) when the coefficient satisfy $\frac{2\sqrt{bQ_0 - V_0^2}}{\mu_0} \in \mathbb{N}$ in the periodic setup and when c belong to the critical set $\mathcal{N}$ in the mixed-type boundary conditions. However, due to the complicated nonlinearities $\rho u u_x$ and $(\rho u)_x$, this problem is very difficult to tackle and so we first want to study the problem in a simplified setup. The system is given by

$$\begin{cases} \rho_t + V_0\rho_x + Q_0 u_x = 0, & \text{in } (0,T)\times(0,2\pi), \\ u_t - \mu_0 u_{xx} + V_0 u_x + b\rho_x = uu_x, & \text{in } (0,T)\times(0,2\pi), \\ \rho(t,0) = \rho(t,2\pi) + p(t), & \text{for } t \in (0,T), \\ u(t,0) = u(t,2\pi) + q(t), \quad u_x(t,0) = u_x(t,2\pi), & \text{for } t \in (0,T), \\ \rho(0,x) = \rho_0(x), \quad u(0,x) = u_0(x), & \text{in } (0,L), \end{cases} \tag{6.9}$$

where $p, q \in L^2(0,T)$ are boundary controls. We have considered the periodic boundary conditions to avoid any unnecessary difficulty and also because we have obtained optimal controllability results (with respect to time, space and coefficients) for the corresponding linearized system in Chapter 3. Using these known results of the linearized system, some fixed-point argument might be implemented to get a local controllability result for this system when only one control is acting on the system (that is, either $p = 0$ or $q = 0$). Further, note that, if $\frac{2\sqrt{bQ_0 - V_0^2}}{\mu_0} \in \mathbb{N}$, the corresponding linearized system is not null controllable at any time T in $(L^2(0,2\pi))^2$ in either cases. Thus, proving some local controllability results for the nonlinear system (6.9) under this restriction on the coefficients will be a very interesting problem. In this context, we refer to the works [CMZ20, CC09b, Cer07], where local controllability of the nonlinear KdV equation is obtained via power series expansion method, when the linearized system is not controllable, see also the book [Cor07] and the survey articles [RZ09, Cer14].

Similarly, we can study the local null controllability of the above simplified nonlinear model but with Dirichlet/ mixed-type boundary conditions when the coefficient belong to the critical set $\mathcal{N}$. We note here that, full characterization of the set $\mathcal{N}$ is still not known and hence the problem is more difficult to handle. In this context, we refer to the work [Ros97] where similar characterization of the critical set is known for the linear KdV equation.

Further, the same question can be asked for the non-barotropic case also, that is, for the nonlinear system (1.16).

On the other hand, there is no global controllability results known for the nonlinear compressible Navier-stokes systems (barotropic and non-barotropic). In this context, we refer to the work [FI95], where the authors proved that the viscous Burgers equation in the interval $(0, L)$ is not globally approximately controllable in $L^2(0, L)$ using a localized distributed control; see also the lecture note [FI96, Chapter 1, Section 6, Page 53].

6.4 Nonlinear coupled parabolic equations

6.4.1 Nonlinear system with space dependent coefficients

Let $T > 0$ be given. We consider the following system:

$$\begin{cases} y_t - (\gamma_1 y_x)_x + \alpha_1 y + \beta_1 z = f_1(y, z), & \text{in } (0,T) \times (0,1), \\ z_t - (\gamma_2 z_x)_x + \alpha_2 y + \beta_2 z = f_2(y, z), & \text{in } (0,T) \times (0,1), \\ y_x(t, 0) = q(t), \quad z_x(t, 0) = r(t), & \text{for } t \in (0,T), \\ y_x(t, 1) = z_x(t, 1), & \text{for } t \in (0,T), \\ \gamma_1(1) y(t, 1) + \gamma_2(1) z(t, 1) + \alpha y_x(t, 1) = 0, & \text{for } t \in (0,T), \\ y(0, x) = y_0(x), \quad z(0, x) = z_0(x), & \text{in } (0,1), \end{cases} \tag{6.10}$$

where $q, r \in L^2(0, T)$ are boundary controls. Here $\alpha_i, \beta_i, \gamma_i$ for $i = 1, 2$ are functions of x and f_1, f_2 are nonlinear functions which are given by (5.2)–(5.3) (as mentioned in Chapter 5). In this setup, we can study the small-time local null controllability using only one boundary control q or r by the source term method. We note here that a suitable Carleman estimate is needed to prove null controllability of the associated linearized system. Then, with the help of the control cost $Ce^{\frac{C}{T}}$, one may prove small-time local null controllability of the nonlinear system (6.10) using Banach fixed point. In this context, we refer to the articles [BBHS21, BB21] (and the references therein) for a detail study of null controllability of similar linear systems with different boundary conditions.

6.4.2 A nonlinear 3-parabolic system

Let $T > 0$. We consider the following system

$$\begin{cases} u_t - \varepsilon_1 u_{xx} = \alpha(v - uv + u - \beta u^2), & \text{in } (0,T) \times (0,1), \\ v_t - \varepsilon_2 v_{xx} = \dfrac{1}{\alpha}(\gamma w - v - uv), & \text{in } (0,T) \times (0,1), \\ w_t - \varepsilon_3 w_{xx} = \delta(u - w), & \text{in } (0,T) \times (0,1). \end{cases} \tag{6.11}$$

This kind of system is called the Field-Noyes model and serves as a model for Belousov-Zhabotinsky reactions in chemical kinetics, see for instance [Smo83, Example 4, Page 210]. The functions u, v, w denote the chemical concentrations, $\varepsilon_i > 0$ for $i = 1, 2, 3$ and $\alpha, \beta, \gamma, \delta > 0$ are constants. For this system also, one may apply the source term method to study the small-time local controllability for this system (6.11) using one boundary control. This problem is very interesting in the application point of view.

6.5 Controllability in higher dimension

All the above controllability questions can be posed for higher dimensional models also, as many interesting physical fluid models appear in several space dimensions. To ease the difficulty, one can start with a rectangle in $\mathbb{R}^2$ and with suitable boundary conditions so that the knowledge of the spectrum of the corresponding linearized models helps us understand the control aspects of the linearized systems and of the total nonlinear system.

We conclude this chapter with the comment that all of the above questions are only a few of the open problems. The lessons and knowledge learned in this thesis can be kept in mind to progress in these above directions in future.

"*We have not succeeded in answering all our problems. The answers we have found only serve to raise a whole set of new questions. In some ways we feel we are as confused as ever, but we believe we are confused on a higher level and about more important things.*"

- A quote mentioned in the book [Ok03].

APPENDIX **A**

Proof of the Well-Posedness Results

In this chapter, we prove the well-posedness results for the linearized compressible Navier-Stokes system (both barotropic and non-barotropic). We first write the related results for the barotropic case to simplify the presentation.

A.0.1 Existence of semigroup: proof of Lemma 4.2.1

The proof is divided into several parts. Recall the operator $(A, D(A))$ given by (4.7)–(4.8) and denote $\mathrm{Z} = L^2(0,1) \times L^2(0,1)$ over the field $\mathbb{C}$.

Part 1. *The operator A is dissipative.* We check that, all $\mathrm{U} = (\rho, u) \in D(A)$

$$
\begin{aligned}
&\mathrm{Re}\,\langle A\mathrm{U}, \mathrm{U}\rangle_{\mathrm{Z}} = \mathrm{Re}\left\langle \begin{pmatrix} -\rho_x - bu_x \\ -b\rho_x + u_{xx} - u_x \end{pmatrix}, \begin{pmatrix} \rho \\ u \end{pmatrix} \right\rangle_{\mathrm{Z}} \\
&= \mathrm{Re}\left(-\int_0^1 \bar{\rho}\rho_x dx - b\int_0^1 \bar{\rho} u_x dx - b\int_0^1 \rho_x \bar{u} dx + \int_0^1 \bar{u} u_{xx} dx - \int_0^1 \bar{u} u_x dx \right) \\
&= -\frac{1}{2}\int_0^1 \frac{d}{dx}(|\rho|^2) dx - \int_0^1 \bar{u}_x u_x dx - \frac{1}{2}\int_0^1 \frac{d}{dx}(|u|^2) dx \\
&= -\int_0^1 |u_x|^2\, dx \leq 0,
\end{aligned}
$$

Part 2. *The operator A is maximal.* This is equivalent to the following. For any $\lambda > 0$ and any $\begin{pmatrix} f \\ g \end{pmatrix} \in \mathrm{Z}$ we can find a $\begin{pmatrix} \rho \\ u \end{pmatrix} \in D(A)$ such that

$$
(\lambda I - A)\begin{pmatrix} \rho \\ u \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \tag{A.1}
$$

that is

$$
\begin{aligned}
\lambda\rho + \rho_x + bu_x &= f, \\
\lambda u + b\rho_x - u_{xx} + u_x &= g.
\end{aligned}
$$

Let $\epsilon > 0$. Instead of solving the above problem, we will solve the following regularized problem

$$
\begin{aligned}
\lambda\rho + \rho_x + bu_x - \epsilon\rho_{xx} &= f, \\
\lambda u + b\rho_x + u_x - u_{xx} &= g,
\end{aligned} \tag{A.2}
$$

with the following boundary conditions

$$
\rho(0) = \rho(1), \quad \rho_x(0) = \rho_x(1), \quad u(0) = 0, \quad u(1) = 0.
$$

We now proceed through the following steps.

Step 1. We consider the space V, given by

$$V = \left\{(\rho, u) \in H^1(0,1) \times H^1(0,1) \;:\; \rho(0) = \rho(1), \quad u(0) = 0, \quad u(1) = 0\right\}.$$

Using Lax-Milgram theorem, we first prove that the system (A.2) has a unique solution in V. Define the operator $B : V \times V \to \mathbb{C}$ by

$$\begin{aligned} B\left(\begin{pmatrix}\rho\\u\end{pmatrix}, \begin{pmatrix}\sigma\\v\end{pmatrix}\right) &= \epsilon \int_0^1 \rho_x \bar{\sigma}_x dx + b\int_0^1 u_x\bar{\sigma}dx + \int_0^1 \rho_x\bar{\sigma}dx + \lambda\int_0^1 \rho\bar{\sigma}dx \\ &+ \int_0^1 u_x\bar{v}_x dx + \int_0^1 u_x\bar{v}dx + b\int_0^1 \rho_x\bar{v}dx + \lambda\int_0^1 u\bar{v}dx, \end{aligned}$$

for all $\begin{pmatrix}\rho\\u\end{pmatrix}, \begin{pmatrix}\sigma\\v\end{pmatrix} \in V$. Then, one can show that B is continuous and coercive. Thus, by Lax-Milgram theorem, for every $\epsilon > 0$, there exists a unique solution $(\rho^\epsilon, u^\epsilon) \in V$ such that

$$B\left(\begin{pmatrix}\rho^\epsilon\\u^\epsilon\end{pmatrix}, \begin{pmatrix}\sigma\\v\end{pmatrix}\right) = F\left(\begin{pmatrix}\sigma\\v\end{pmatrix}\right), \quad \forall \begin{pmatrix}\sigma\\v\end{pmatrix} \in V,$$

where $F : V \to \mathbb{C}$ is the linear functional given by

$$F\left(\begin{pmatrix}\sigma\\v\end{pmatrix}\right) := \int_0^1 f\bar{\sigma}dx + \int_0^1 g\bar{v}dx.$$

Step 2. Now, observe that

$$\mathrm{Re}\left(B\left(\begin{pmatrix}\rho^\epsilon\\u^\epsilon\end{pmatrix}, \begin{pmatrix}\rho^\epsilon\\u^\epsilon\end{pmatrix}\right)\right) \le \int_0^1 \left|f\overline{\rho^\epsilon}\right| + \int_0^1 \left|g\overline{u^\epsilon}\right| \le \frac{1}{2}\int_0^1 \left(|f|^2 + \left|\overline{\rho^\epsilon}\right|^2\right) + \frac{1}{2}\int_0^1 \left(|g|^2 + \left|\overline{u^\epsilon}\right|^2\right),$$

which yields

$$\epsilon\int_0^1 |\rho_x^\epsilon|^2 + \frac{\lambda}{2}\int_0^1 |\rho^\epsilon|^2 + \int_0^1 |u_x^\epsilon|^2 + \frac{\lambda}{2}\int_0^1 |u^\epsilon|^2 \le \frac{1}{2}\int_0^1 |f|^2 + \frac{1}{2}\int_0^1 |g|^2$$

This shows that $(u^\epsilon)_{\epsilon\ge0}$ is bounded in $H^1(0,1)$, $(\rho^\epsilon)_{\epsilon\ge0}$ is bounded in $L^2(0,1)$ and $(\sqrt{\epsilon}\rho_x^\epsilon)_{\epsilon\ge0}$ is bounded in $L^2(0,1)$. Since the spaces $H^1(0,1)$ and $L^2(0,1)$ are reflexive, there exist subsequences, still denoted by $(u^\epsilon)_{\epsilon\ge0}$, $(\rho^\epsilon)_{\epsilon\ge0}$, and functions $\rho \in L^2(0,1)$ and $u \in H^1(0,1)$, such that

$$u^\epsilon \rightharpoonup u \text{ in } H^1(0,1), \text{ and } \rho^\epsilon \rightharpoonup \rho \text{ in } L^2(0,1).$$

Furthermore, we have

$$\int_0^1 \left|\epsilon\rho_x^\epsilon\right|^2 = \epsilon\int_0^1 \left|\sqrt{\epsilon}\rho^\epsilon\right|^2 \to 0, \text{ as } \epsilon \to 0.$$

Now, since $B\left(\begin{pmatrix}\rho^\epsilon\\u^\epsilon\end{pmatrix}, \begin{pmatrix}\sigma\\v\end{pmatrix}\right) = F\left(\begin{pmatrix}\sigma\\v\end{pmatrix}\right)$, for all $\begin{pmatrix}\sigma\\v\end{pmatrix} \in V$, we may take $\begin{pmatrix}\sigma\\0\end{pmatrix} \in V$, so that we obtain

$$\epsilon\int_0^1 \rho_x^\epsilon\bar{\sigma}_x + b\int_0^1 u_x^\epsilon\bar{\sigma} + \int_0^1 \rho_x^\epsilon\bar{\sigma} + \lambda\int_0^1 \rho^\epsilon\bar{\sigma} = \int_0^1 f\bar{\sigma}. \tag{A.3}$$

Similarly, by taking $\begin{pmatrix}0\\v\end{pmatrix} \in V$, we get

$$\int_0^1 u_x^\epsilon\bar{v}_x + \int_0^1 u_x^\epsilon\bar{v} + b\int_0^1 \rho_x^\epsilon\bar{v} + \lambda\int_0^1 u^\epsilon\bar{v} = \int_0^1 g\bar{v}. \tag{A.4}$$

Integrating by parts, we get from equation (A.3) that,

$$\epsilon \int_0^1 \rho_x^\epsilon \bar{\sigma}_x + b \int_0^1 u_x^\epsilon \bar{\sigma} - \int_0^1 \rho^\epsilon \bar{\sigma}_x + \lambda \int_0^1 \rho^\epsilon \bar{\sigma} = \int_0^1 f \bar{\sigma}.$$

Then, passing to the limit $\epsilon \to 0$, we obtain

$$b \int_0^1 u_x \bar{\sigma} - \int_0^1 \rho \bar{\sigma}_x + \lambda \int_0^1 \rho \bar{\sigma} = \int_0^1 f \bar{\sigma},$$

and the above relation is true $\forall \sigma \in C_c^\infty(0,1)$. As a consequence,

$$bu_x + \rho_x + \lambda \rho = f, \tag{A.5}$$

in the sense of distribution and therefore $\rho_x = f - bu_x - \lambda\rho \in L^2(0,1)$; in other words, $\rho \in H^1(0,1)$.

Step 3. We now show $u(0) = u(1) = 0$. Since the inclusion map $i : H^1(0,1) \to C^0([0,1])$ is compact and $u^\epsilon \rightharpoonup u$ in $H^1(0,1)$, we obtain

$$u^\epsilon \to u \quad \text{in } C^0([0,1]).$$

Thus, $(u^\epsilon(0), u^\epsilon(1)) \to (u(0), u(1))$. Since $u^\epsilon(0) = u^\epsilon(1) = 0$ for all $\epsilon > 0$, we have

$$u(0) = u(1) = 0.$$

Similarly from the identity (A.4), one can deduce that

$$-u_{xx} + u_x + b\rho_x + \lambda u = g, \tag{A.6}$$

in the sense of distribution and therefore $u_{xx} \in L^2(0,1)$, that is $u \in H^2(0,1)$.

We now show $\rho(0) = \rho(1)$. Recall that, $bu_x + \rho_x + \lambda\rho = f$ and therefore

$$b \int_0^1 u_x \bar{\sigma} + \int_0^1 \rho_x \bar{\sigma} + \lambda \int_0^1 \rho \bar{\sigma} = \int_0^1 f \bar{\sigma}.$$

Integrating by parts, we get

$$b \int_0^1 u_x \bar{\sigma} - \int_0^1 \rho \bar{\sigma}_x + \rho\bar{\sigma}|_0^1 + \lambda \int_0^1 \rho \bar{\sigma} = \int_0^1 f \bar{\sigma}. \tag{A.7}$$

From (A.3), we deduce

$$\epsilon \int_0^1 \rho_x^\epsilon \bar{\sigma}_x + b \int_0^1 u_x^\epsilon \bar{\sigma} - \int_0^1 \rho^\epsilon \bar{\sigma}_x + \lambda \int_0^1 \rho^\epsilon \bar{\sigma} = \int_0^1 f \bar{\sigma}. \tag{A.8}$$

Taking $\epsilon \to 0$, we get

$$b \int_0^1 u_x \bar{\sigma} - \int_0^1 \rho \bar{\sigma}_x + \lambda \int_0^1 \rho \bar{\sigma} = \int_0^1 f \bar{\sigma}. \tag{A.9}$$

Comparing (A.7) and (A.9), one has $\rho(0)\bar{\sigma}(0) = \rho(1)\bar{\sigma}(1)$. But $\sigma(0) = \sigma(1)$, and thus

$$\rho(0) = \rho(1).$$

So, we get $\begin{pmatrix} \rho \\ u \end{pmatrix} \in D(A)$. Hence, the operator A is maximal.

A.0.2 Solution by transposition: proof of Theorem 4.2.2

In this section, we are going to proof the existence of solution to our control problem (4.5), more precisely Theorem 4.2.2. We omit the proof for Theorem 4.2.1, when a control acts on the velocity part.

Step 1. We first consider system (4.5) with zero initial data and nonhomogeneous boundary conditions, that is,

$$\begin{cases} \rho_t + \rho_x + bu_x = 0 & \text{in } (0,T)\times(0,1), \\ u_t - u_{xx} + u_x + b\rho_x = 0 & \text{in } (0,T)\times(0,1), \\ \rho(t,0) = \rho(t,1) + p(t) & \text{for } t\in(0,T), \\ u(t,0) = 0, \quad u(t,1) = 0 & \text{for } t\in(0,T), \\ \rho(0,x) = u(0,x) = 0 & \text{for } x\in(0,1), \end{cases} \tag{A.10}$$

with $p \in L^2(0,T)$.

We now prove the existence of solution to the new system (A.10).

Theorem A.0.1. *For a given $p \in L^2(0,T)$, the system (A.10) has a unique solution $(\tilde\rho,\tilde u)$ belonging to the space $L^2(0,T;L^2(0,1))\times L^2(0,T;L^2(0,1))$ in the sense of transposition. Moreover, the operator:*

$$p \mapsto (\tilde\rho,\tilde u),$$

is linear and continuous from $L^2(0,T)$ into $L^2(0,T;L^2(0,1))\times L^2(0,T;L^2(0,1))$.

Proof. Existence: Let us define a map $\Lambda_1 : L^2(0,T;L^2(0,1))\times L^2(0,T;L^2(0,1)) \to L^2(0,T)$,

$$\Lambda_1(f,g) = \sigma(t,1), \tag{A.11}$$

where (σ,v) is the unique solution to the adjoint system (4.14) with given source term (f,g). The map Λ_1 is well-defined because of the hidden regularity as mentioned in Appendix A.1, Corollary A.1.1.

Now, thanks to Proposition 4.2.1, the map

$$(f,g) \mapsto (\sigma,v)$$

is linear and continuous from $L^2(0,T;L^2(0,1))\times L^2(0,T;L^2(0,1))$ to $L^2(0,T;L^2(0,1))\times L^2(0,T;H^1_0(0,1))$, which implies that the map Λ_1 given by (A.11) is linear and continuous (Corollary A.1.1).

So, we can define the adjoint to Λ_1 as follows

$$\Lambda_1^* : L^2(0,T) \to L^2(0,T;L^2(0,1))\times L^2(0,T;L^2(0,1)), \tag{A.12}$$

which is also linear and continuous.

Let us denote $\Lambda_1^*(p) = (\tilde\rho,\tilde u)$. Then, for $(\tilde\rho,\tilde u)$, we have

$$\begin{aligned} &\int_0^T\int_0^1 \tilde\rho(t,x)f(t,x)dxdt + \int_0^T\int_0^1 \tilde u(t,x)g(t,x)dxdt \\ &\qquad = \left\langle \Lambda_1^* p, (f,g)\right\rangle = \langle p, \Lambda_1(f,g)\rangle = \int_0^T p(t)\sigma(t,1)dt, \end{aligned}$$

for every (f,g) in $L^2(0,T;L^2(0,1))\times L^2(0,T;L^2(0,1))$. Hence for any $p\in L^2(0,T)$, $(\tilde\rho,\tilde u)$ is the solution to the system (A.10) in the sense of transposition and

$$\|(\tilde\rho,\tilde u)\|_{L^2(L^2)\times L^2(L^2)} = \|\Lambda_1^*(p)\|_{L^2(L^2)\times L^2(L^2)} \le \|\Lambda_1^*\|\ \|p\|_{L^2(0,T)}. \tag{A.13}$$

Uniqueness: If $p = 0$ on $(0,T)$, we have

$$\int_0^T\int_0^1 \rho(t,x)f(t,x)dxdt + \int_0^T\int_0^1 u(t,x)g(t,x)dxdt = 0,$$

for all $(f,g)\in L^2(0,T;L^2(0,1))\times L^2(0,T;L^2(0,1))$, which gives $(\rho,u) = (0,0)$ and therefore the solution to the system (A.10) is unique. □

Step 2. We now consider the system (4.5) with non-zero initial data and homogeneous boundary conditions and check the existence, uniqueness of solution. The system reads as

$$
\begin{cases}
\rho_t + \rho_x + b u_x = 0 & \text{in } (0,T)\times(0,1),\\
u_t - u_{xx} + u_x + b\rho_x = 0 & \text{in } (0,T)\times(0,1),\\
\rho(t,0) = \rho(t,1) & \text{for } t\in(0,T),\\
u(t,0)=0, \quad u(t,1)=0 & \text{for } t\in(0,T),\\
\rho(0,x)=\rho_0(x),\ u(0,x)=u_0(x) & \text{in } (0,1),
\end{cases}
\tag{A.14}
$$

with $(\rho_0,u_0)\in L^2(0,1)\times L^2(0,1)$.

Theorem A.0.2. *For any $(\rho_0,u_0)\in L^2(0,1)\times L^2(0,1)$, the system (A.14) has a unique solution $(\check{\rho},\check{u})$ belonging to the space $L^2(0,T;L^2(0,1))\times L^2(0,T;L^2(0,1))$ in the sense of transposition. Moreover, the operator:*

$$(\rho_0,u_0)\mapsto(\check{\rho},\check{u}),$$

is linear and continuous from $L^2(0,1)\times L^2(0,1)$ into $L^2(0,T;L^2(0,1))\times L^2(0,T;L^2(0,1))$.

Proof. Existence: Let us define a map $\Lambda_2 : L^2(0,T;L^2(0,1))\times L^2(0,T;L^2(0,1))\to L^2(0,1)\times L^2(0,1)$,

$$\Lambda_2(f,g) = (\sigma(0,\cdot), v(0,\cdot)), \tag{A.15}$$

where (σ,v) is the unique solution to the adjoint system (4.14) with given source term (f,g).

Now, thanks to Proposition 4.2.1, the map

$$(f,g)\mapsto(\sigma,v)$$

is linear and continuous from $L^2(0,T;L^2(0,1))\times L^2(0,T;L^2(0,1))$ to the space $C([0,T];L^2(0,1))\times [C([0,T];L^2(0,1))\cap L^2(0,T;H^1_0(0,1))]$, which implies that the map Λ_2 given by (A.15) is linear and continuous.

So, we can define the adjoint to Λ_2 as follows

$$\Lambda_2^* : L^2(0,1)\times L^2(0,1)\to L^2(0,T;L^2(0,1))\times L^2(0,T;L^2(0,1)), \tag{A.16}$$

which is also linear and continuous.

Let us denote $\Lambda_2^*(\rho_0,u_0)=(\check{\rho},\check{u})$. Then, for $(\check{\rho},\check{u})$, we have

$$
\begin{aligned}
&\int_0^T\int_0^1 \check{\rho}(t,x)f(t,x)dxdt + \int_0^T\int_0^1 \check{u}(t,x)g(t,x)dxdt\\
&\quad = \left\langle \Lambda_2^*(\rho_0,u_0),(f,g)\right\rangle = \langle(\rho_0,u_0),\Lambda_2(f,g)\rangle = \langle(\rho_0,u_0),(\sigma(0,\cdot),v(0,\cdot))\rangle,
\end{aligned}
$$

for every (f,g) in $L^2(0,T;L^2(0,1))\times L^2(0,T;L^2(0,1))$. Hence for any $(\rho_0,u_0)\in L^2(0,1)\times L^2(0,1)$, $(\check{\rho},\check{u})$ is the solution to the system (A.10) and

$$\|(\check{\rho},\check{u})\|_{L^2(L^2)\times L^2(L^2)} = \|\Lambda_2^*(\rho_0,u_0)\|_{L^2(L^2)\times L^2(L^2)} \leq \|\Lambda_2^*\|\ \|(\rho_0,u_0)\|_{L^2(0,1)\times L^2(0,1)}. \tag{A.17}$$

Uniqueness: Let the system (A.14) has two solutions (ρ_1,u_1) and (ρ_2,u_2). Introduce

$$(\rho,u) = (\rho_1,u_1)-(\rho_2,u_2).$$

Then one can show that the only possibility is $(\rho,u)=(0,0)$, using the initial and boundary conditions: $\rho(0,x)=u(0,x)=0$ for all $x\in(0,1)$ and $\rho(t,0)=\rho(t,1)$, $u(t,0)=u(t,1)=0$ for all $t\in(0,T)$. ☐

Proof of Theorem 4.2.2. We now recall the system (4.5) with given boundary data $p \in L^2(0,T)$ and initial data $(\rho_0, u_0) \in L^2(0,1) \times L^2(0,1)$. Then, thanks to Theorem A.0.1 & A.0.2,

$$(\rho, u) := (\tilde{\rho}, \tilde{u}) + (\check{\rho}, \check{u}),$$

is the unique solution to (4.5).

It remains to prove the continuity estimate of the solution (ρ, u). Let $H : L^2(0,1) \times L^2(0,1) \times L^2(0,T) \to L^2(0,T; L^2(0,1) \times L^2(0,1))$ be defined by

$$H(\rho_0, u_0, p) = (\rho, u). \tag{A.18}$$

Then H is linear. Furthermore, using (A.13) and (A.17), we get

$$\begin{aligned}
\|H(\rho_0, u_0, p)\|_{L^2(0,T;L^2(0,1))\times L^2(0,T;L^2(0,1))} &= \|(\tilde{\rho}, \tilde{u}) + (\check{\rho}, \check{u})\|_{L^2(0,T;L^2(0,1))\times L^2(0,T;L^2(0,1))} \\
&\leq \left\|\Lambda_1^*\right\| \|p\|_{L^2(0,T)} + \left\|\Lambda_2^*\right\| \|(\rho_0, u_0)\|_{L^2(0,1)\times L^2(0,1)} \\
&\leq C\left(\|p\|_{L^2(0,T)} + \|\rho_0\|_{L^2(0,1)} + \|u_0\|_{L^2(0,1)}\right).
\end{aligned}$$

Finally, the required regularity result (4.16)–(4.17) can be obtained by applying the usual regularity of parabolic equation (with homogeneous boundary data) and then using that, the regularity of transport part follows immediately.

The proof is complete. □

A.1 A hidden regularity result

Consider the following system

$$\begin{cases}
\rho_t + \rho_x + b u_x = 0 & \text{in } (0,T) \times (0,1), \\
u_t - u_{xx} + u_x + b\rho_x = 0 & \text{in } (0,T) \times (0,1), \\
\rho(t,0) = \rho(t,1) + p(t) & \text{for } t \in (0,T), \\
u(t,0) = 0, \ u(t,1) = 0 & \text{for } t \in (0,T), \\
\rho(0,x) = \rho_0(x), \ u(0,x) = u_0(x) & \text{in } (0,1),
\end{cases} \tag{A.19}$$

where $(\rho_0, u_0) \in L^2(0,1) \times L^2(0,1)$ and $p \in L^2(0,T)$ are given data. Then, one has the following result.

Lemma A.1.1. *For any $(\rho_0, u_0) \in L^2(0,1) \times L^2(0,1)$ and $p \in L^2(0,T)$, the density component ρ to the system (A.19) satisfies $\rho(\cdot, 1) \in L^2(0,T)$.*

Proof. The proof is split into two steps. First, recall Theorem 4.2.2 so that one has

$$(\rho, u) \in C^0([0,T]; L^2(0,1)) \times [C^0([0,T]; L^2(0,1)) \cap L^2(0,T; H_0^1(0,1))].$$

Step 1. Let us take the initial state $\rho_0 \in H^1_\sharp(0,1)$ (i.e., $\rho_0 \in H^1(0,1)$ with $\rho_0(0) = \rho_0(1)$), $u_0 \in H_0^1(0,1)$ and the boundary data $p \in H^1_{\{0\}}(0,T)$. Then one can prove that the solution (ρ, u) to system (A.19) lies in the space $[H^1(0,T; L^2(0,1)) \cap L^2(0,T; H^1(0,1))] \times [L^2(0,T; H^2(0,1) \cap H_0^1(0,1)) \cap H^1(0,T; L^2(0,1))]$, see for instance [CR13]. Therefore, $u_x \in L^2(0,T; H^1(0,1))$ and so the integration by parts are justified. Multiplying the first equation of (A.19) by $x\rho$, we get

$$\int_0^T \int_0^1 x\rho\rho_t dxdt + \int_0^T \int_0^1 x\rho\rho_x dxdt + b\int_0^T \int_0^1 x\rho u_x dxdt = 0.$$

Integrating by parts and using the boundary conditions, we obtain

$$\frac{1}{2}\int_0^1 x(\rho^2(T,x) - \rho_0^2(x))dx + \frac{1}{2}\int_0^T \rho^2(t,1)dt - \frac{1}{2}\int_0^T \int_0^1 \rho^2 dxdt + b\int_0^T \int_0^1 x\rho u_x dxdt = 0. \tag{A.20}$$

Therefore

$$\int_0^T \rho^2(t,1)dt = -\int_0^1 x(\rho^2(T,x) - \rho_0^2(x))dx + \int_0^T\int_0^1 \rho^2 dxdt - 2b\int_0^T\int_0^1 x\rho u_x dxdt$$
$$\leq (1+b)\int_0^T\int_0^1 \rho^2 dxdt + b\int_0^T\int_0^1 u_x^2 dxdt + \int_0^1 \rho_0^2(x)dx.$$

Using the continuity estimate (4.17), we obtain

$$\int_0^T \rho^2(t,1)dt \leq C\left(\int_0^1 \rho_0^2(x)dx + \int_0^1 u_0^2(x)dx + \int_0^T p^2(t)dt\right). \tag{A.21}$$

Step 2. Let $(\rho_0, u_0) \in L^2(0,1)\times L^2(0,1)$ and $p \in L^2(0,T)$. By density, there exists sequences $\rho_0^n \in H^1_\sharp(0,1)$, $u_0^n \in H^1_0(0,1)$ and $p^n \in H^1_{\{0\}}(0,T)$ such that $\rho_0^n \to \rho$, $u_0^n \to u_0$ in $L^2(0,1)$ and $p^n \to p$ in $L^2(0,T)$. Let (ρ^n, u^n) be the solution to (A.19) corresponding to the initial state (ρ_0^n, u_0^n) and boundary data p^n. Using (A.21) from Step 1, we have

$$\int_0^T (\rho^n)^2(t,1)dt \leq C\left(\int_0^1 (\rho_0^n)^2(x)dx + \int_0^1 (u_0^n)^2(x)dx + \int_0^T (p^n)^2(t)dt\right).$$

We first observe that

$$\int_0^1 (\rho_0^n)^2(x)dx + \int_0^1 (u_0^n)^2(x)dx + \int_0^T (p^n)^2(t)dt \to \int_0^1 \rho_0^2(x)dx + \int_0^1 u_0^2(x)dx + \int_0^T p^2(t)dt,$$

as $n \to +\infty$. Therefore, the sequence $\left(\int_0^T (\rho^n)^2(t,1)dt\right)_n$ is indeed a Cauchy sequence and hence convergent. Then, by the uniqueness of solution to (A.19), we have $\lim\limits_{n\to+\infty}\int_0^T (\rho^n)^2(t,1)dt = \int_0^T \rho^2(t,1)dt$, which yields

$$\int_0^T \rho^2(t,1)dt \leq C\left(\int_0^1 \rho_0^2(x)dx + \int_0^1 u_0^2(x)dx + \int_0^T p^2(t)dt\right).$$

This concludes the proof of the lemma. □

Let us now consider the following system

$$\begin{cases} -\sigma_t - \sigma_x - bv_x = f & \text{in } (0,T)\times(0,1),\\ -v_t - v_{xx} - v_x - b\sigma_x = g & \text{in } (0,T)\times(0,1),\\ \sigma(t,0) = \sigma(t,1) & \text{for } t \in (0,T),\\ v(t,0) = v(t,1) = 0 & \text{for } t\in(0,T),\\ \sigma(T,x) = 0, \quad v(T,x) = 0 & \text{in } (0,1), \end{cases} \tag{A.22}$$

with $f, g \in L^2(0,T;L^2(0,1))$. We can similarly conclude the following result.

Corollary A.1.1. *For any $f, g \in L^2(0,T;L^2(0,1))$, the solution component σ to the adjoint system (A.22) satisfies the following estimate.*

$$\|\sigma(\cdot,1)\|_{L^2(0,T)} \leq C\left(\|f\|_{L^2(0,T;L^2(0,1))} + \|g\|_{L^2(0,T;L^2(0,1))}\right). \tag{A.23}$$

A.1.1 Existence of semigroup: proof of Lemma 3.3.1

The proof is divided into several parts.

Part 1. *The operator A is dissipative.* Indeed, for all $(\xi,\eta,\zeta)^\dagger \in \mathcal{D}(A)$

$$\mathrm{Re}\,\langle A\mathrm{U}, \mathrm{U}\rangle_{(L^2(0,2\pi))^3} = \mathrm{Re}\left\langle \begin{pmatrix} -\bar{u}\xi_x - \bar{\rho}\eta_x \\ -\frac{R\bar{\theta}}{\bar{\rho}}\xi_x + \lambda_0\eta_{xx} - \bar{u}\eta_x - R\zeta_x \\ -\frac{R\bar{\theta}}{c_0}\eta_x + \kappa_0\zeta_{xx} - \bar{u}\zeta_x \end{pmatrix}, \begin{pmatrix}\xi\\ \eta\\ \zeta\end{pmatrix}\right\rangle_{(L^2(0,2\pi))^3}$$

$$= \mathrm{Re}\Bigg(-R\bar{\theta}\bar{u}\int_0^{2\pi}\bar{\xi}\xi_x dx - R\bar{\theta}\bar{\rho}\int_0^{2\pi}\bar{\xi}\eta_x dx - R\bar{\theta}\bar{\rho}\int_0^{2\pi}\xi_x\bar{\eta}dx + \lambda_0\bar{\rho}^2\int_0^{2\pi}\bar{\eta}\eta_{xx}dx - \bar{\rho}^2\bar{u}\int_0^{2\pi}\bar{\eta}\eta_x dx$$
$$-R\bar{\rho}^2\int_0^{2\pi}\bar{\eta}\zeta_x dx - R\bar{\rho}^2\int_0^{2\pi}\eta_x\bar{\zeta}dx + \kappa_0\frac{\bar{\rho}^2c_0}{\bar{\theta}}\int_0^{2\pi}\bar{\zeta}\zeta_{xx}dx - \bar{u}\frac{\bar{\rho}^2c_0}{\bar{\theta}}\int_0^{2\pi}\bar{\zeta}\zeta_x dx\Bigg)$$
$$= -\frac{R\bar{\theta}\bar{u}}{2}\int_0^{2\pi}\frac{d}{dx}(|\xi|^2)dx - \lambda_0\bar{\rho}^2\int_0^{2\pi}\bar{\eta}_x\eta_x dx - \frac{\bar{\rho}^2\bar{u}}{2}\int_0^{2\pi}\frac{d}{dx}(|\eta|^2)dx - \kappa_0\frac{\bar{\rho}^2c_0}{\bar{\theta}}\int_0^{2\pi}\bar{\zeta}_x\zeta_x dx$$
$$-\frac{\bar{u}}{2}\frac{\bar{\rho}^2c_0}{\bar{\theta}}\int_0^{2\pi}\frac{d}{dx}(|\zeta|^2)dx - \lambda_0\bar{\rho}^2\int_0^{2\pi}|u_x|^2\,dx - \kappa_0\frac{\bar{\rho}^2c_0}{\bar{\theta}}\int_0^{2\pi}|\zeta_x|^2\,dx \le 0.$$

Part 2. *The operator A is maximal.* This is equivalent to the following. For any $\nu > 0$ and any $\begin{pmatrix} f \\ g \\ h \end{pmatrix} \in (L^2(0,2\pi))^3$, we can find a $\begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} \in \mathcal{D}(A)$ such that

$$(\nu I - A)\begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = \begin{pmatrix} f \\ g \\ h \end{pmatrix},$$

that is,

$$\nu\xi + \bar{u}\xi_x + \bar{\rho}\eta_x = f,$$
$$\nu\eta + \frac{R\bar{\theta}}{\bar{\rho}}\xi_x - \lambda_0\eta_{xx} + \bar{u}\eta_x + R\zeta_x = g,$$
$$\nu\zeta + \frac{R\bar{\theta}}{c_0}\eta_x - \kappa_0\zeta_{xx} + \bar{u}\zeta_x = h.$$

Let $\epsilon > 0$. Instead of solving the above problem, we will solve the following regularized problem

$$\begin{cases} \nu\xi + \bar{u}\xi_x - \epsilon\xi_{xx} + \bar{\rho}\eta_x = f, \\ \nu\eta + \frac{R\bar{\theta}}{\bar{\rho}}\xi_x - \lambda_0\eta_{xx} + \bar{u}\eta_x + R\zeta_x = g, \\ \nu\zeta + \frac{R\bar{\theta}}{c_0}\eta_x - \kappa_0\zeta_{xx} + \bar{u}\zeta_x = h. \end{cases} \tag{A.24}$$

with the following boundary conditions

$$\xi(0) = \xi(2\pi),\quad \xi_x(0) = \xi_x(2\pi),\quad \eta(0) = \eta(2\pi),\quad \eta_x(0) = \eta_x(2\pi),\quad \zeta(0) = \zeta(2\pi),\quad \zeta_x(0) = \zeta_x(2\pi).$$

We now proceed through the following steps.

Step 1. Using Lax-Milgram theorem, we first prove that the system (A.24) has a unique solution in $(H^1_{\text{per}}(0,2\pi))^3$. Define the operator $B : (H^1_{\text{per}}(0,2\pi))^3 \times (H^1_{\text{per}}(0,2\pi))^3 \to \mathbb{C}$ by

$$B\left(\begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix},\begin{pmatrix} \xi_1 \\ \eta_1 \\ \zeta_1 \end{pmatrix}\right) = \epsilon\int_0^{2\pi}\xi_x(\bar{\xi}_1)_x dx + \bar{\rho}\int_0^{2\pi}\eta_x\bar{\xi}_1 dx + \bar{u}\int_0^{2\pi}\xi_x\bar{\xi}_1 dx + \nu\int_0^{2\pi}\xi\bar{\xi}_1 dx + \lambda_0\int_0^{2\pi}\eta_x(\bar{\eta}_1)_x dx$$
$$+\bar{u}\int_0^{2\pi}\eta_x\bar{\eta}_1 dx + \frac{R\bar{\theta}}{\bar{\rho}}\int_0^{1}\xi_x\bar{\eta}_1 dx + R\int_0^{2\pi}\zeta_x\bar{\eta}_1 dx + \nu\int_0^{2\pi}\eta\bar{\eta}_1 dx$$
$$+\kappa_0\int_0^{2\pi}\zeta_x(\bar{\zeta}_1)_x dx + \bar{u}\int_0^{2\pi}\zeta_x\bar{\zeta}_1 dx + \frac{R\bar{\theta}}{c_0}\int_0^{2\pi}\eta_x\bar{\zeta}_1 dx + \nu\int_0^{2\pi}\zeta\bar{\zeta}_1 dx,$$

for all $\begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix},\begin{pmatrix} \xi_1 \\ \eta_1 \\ \zeta_1 \end{pmatrix} \in (H^1_{\text{per}}(0,2\pi))^3$. Then, one can show that B is continuous and coercive. Thus, by Lax-Milgram theorem, for every $\epsilon > 0$, there exists a unique solution $(\xi^\epsilon, \eta^\epsilon, \zeta^\epsilon)^\dagger \in (H^1_{\text{per}}(0,2\pi))^3$ such that

$$B\left(\begin{pmatrix}\xi^\epsilon\\ \eta^\epsilon\\ \zeta^\epsilon\end{pmatrix},\begin{pmatrix}\xi\\ \eta\\ \zeta\end{pmatrix}\right)=F\left(\begin{pmatrix}\xi\\ \eta\\ \zeta\end{pmatrix}\right),\quad \forall\begin{pmatrix}\xi\\ \eta\\ \zeta\end{pmatrix}\in (H^1_{\text{per}}(0,2\pi))^3,$$

where $F:(H^1_{\text{per}}(0,2\pi))^3\to\mathbb{C}$ is the linear functional given by

$$F\left(\begin{pmatrix}\xi\\ \eta\\ \zeta\end{pmatrix}\right):=\int_0^{2\pi}f\bar{\xi}dx+\int_0^{2\pi}g\bar{\eta}dx+\int_0^{2\pi}h\bar{\zeta}dx.$$

Step 2. Observe that

$$\begin{aligned}\mathrm{Re}\left(B\left(\begin{pmatrix}\xi^\epsilon\\ \eta^\epsilon\\ \zeta^\epsilon\end{pmatrix},\begin{pmatrix}\xi^\epsilon\\ \eta^\epsilon\\ \zeta^\epsilon\end{pmatrix}\right)\right)&\le\int_0^{2\pi}\left|f\overline{\xi^\epsilon}\right|dx+\int_0^{2\pi}\left|g\overline{\eta^\epsilon}\right|dx+\int_0^{2\pi}\left|h\overline{\zeta^\epsilon}\right|dx\\ &\le\frac{1}{2}\int_0^{2\pi}\left(|f|^2+|g|^2+|h|^2\right)dx+\frac{1}{2}\int_0^{2\pi}\left(\left|\overline{\xi^\epsilon}\right|^2+\left|\overline{\eta^\epsilon}\right|^2+\left|\overline{\zeta^\epsilon}\right|^2\right)dx,\end{aligned}$$

which yields

$$\begin{aligned}\epsilon\int_0^{2\pi}\left|\xi^\epsilon_x\right|^2+\frac{\nu}{2}\int_0^{2\pi}|\xi^\epsilon|^2+\lambda_0\int_0^{2\pi}\left|\eta^\epsilon_x\right|^2+\frac{\nu}{2}\int_0^{2\pi}|\eta^\epsilon|^2&\\ +\kappa_0\int_0^{2\pi}\left|\zeta^\epsilon_x\right|^2+\frac{\nu}{2}\int_0^{2\pi}|\zeta^\epsilon|^2\le\frac{1}{2}\int_0^{2\pi}(|f|^2+|g|^2+|h|^2)&\end{aligned}$$

This shows that the sequences (η^ϵ) and (ζ^ϵ) are bounded in $H^1(0,2\pi)$ and the sequences (ξ^ϵ) and $(\sqrt{\epsilon}\xi^\epsilon_x)$ are bounded in $L^2(0,2\pi)$. Since the spaces $H^1(0,2\pi)$ and $L^2(0,2\pi)$ are reflexive, there exist subsequences, still denoted by $(\eta^\epsilon),(\zeta^\epsilon)$, (ξ^ϵ), and functions $\xi\in L^2(0,2\pi)$ and $\eta\in H^1(0,2\pi)$ such that

$$\eta^\epsilon\rightharpoonup\eta \text{ in } H^1(0,2\pi),\ \text{and } \xi^\epsilon\rightharpoonup\xi \text{ in } L^2(0,2\pi).$$

Furthermore, we have

$$\int_0^{2\pi}\left|\epsilon\xi^\epsilon_x\right|^2=\epsilon\int_0^1\left|\sqrt{\epsilon}\xi^\epsilon_x\right|^2\to 0,\ \text{as } \epsilon\to 0.$$

Now, since $B\left(\begin{pmatrix}\xi^\epsilon\\ \eta^\epsilon\\ \zeta^\epsilon\end{pmatrix},\begin{pmatrix}\xi\\ \eta\\ \zeta\end{pmatrix}\right)=F\left(\begin{pmatrix}\xi\\ \eta\\ \zeta\end{pmatrix}\right)$, for all $\begin{pmatrix}\xi\\ \eta\\ \zeta\end{pmatrix}\in(H^1_{\text{per}}(0,2\pi))^3$, we may take $\begin{pmatrix}\xi_1\\ 0\\ 0\end{pmatrix}\in(H^1_{\text{per}}(0,2\pi))^3$, so that we obtain

$$\epsilon\int_0^{2\pi}\xi^\epsilon_x(\bar{\xi}_1)_xdx+\bar{\rho}\int_0^{2\pi}\eta^\epsilon_x\bar{\xi}_1dx+\bar{u}\int_0^{2\pi}\xi^\epsilon_x\bar{\xi}_1dx+\nu\int_0^{2\pi}\xi^\epsilon\bar{\xi}_1dx=\int_0^{2\pi}f\bar{\xi}_1dx. \tag{A.25}$$

Similarly, by taking $\begin{pmatrix}0\\ \eta_1\\ 0\end{pmatrix},\begin{pmatrix}0\\ 0\\ \zeta_1\end{pmatrix}\in(H^1_{\text{per}}(0,2\pi))^3$, we get

$$\lambda_0\int_0^{2\pi}\eta^\epsilon_x(\bar{\eta}_1)_xdx+\bar{u}\int_0^{2\pi}\eta^\epsilon_x\bar{\eta}_1dx+\frac{R\bar{\theta}}{\bar{\rho}}\int_0^1\xi^\epsilon_x\bar{\eta}_1dx+R\int_0^{2\pi}\zeta^\epsilon_x\bar{\eta}_1dx+\nu\int_0^{2\pi}\eta^\epsilon\bar{\eta}_1dx=\int_0^{2\pi}g\bar{\eta}_1dx, \tag{A.26}$$

and

$$\kappa_0\int_0^{2\pi}\zeta^\epsilon_x(\bar{\zeta}_1)_xdx+\bar{u}\int_0^{2\pi}\zeta^\epsilon_x\bar{\zeta}_1dx+\frac{R\bar{\theta}}{c_0}\int_0^{2\pi}\eta^\epsilon_x\bar{\zeta}_1dx+\nu\int_0^{2\pi}\zeta^\epsilon\bar{\zeta}_1dx=\int_0^{2\pi}h\bar{\zeta}_1dx \tag{A.27}$$

Integrating by parts, we get from equation (A.25) that,

$$\epsilon \int_0^{2\pi} \xi_x^\epsilon (\bar{\xi}_1)_x dx + \bar{\rho} \int_0^{2\pi} \eta_x^\epsilon \bar{\xi}_1 dx - \bar{u} \int_0^{2\pi} \xi^\epsilon (\bar{\xi}_1)_x dx + \nu \int_0^{2\pi} \xi^\epsilon \bar{\xi}_1 dx = \int_0^{2\pi} f \bar{\xi}_1 dx.$$

Then, passing to the limit $\epsilon \to 0$, we obtain

$$\bar{\rho} \int_0^{2\pi} \eta_x \bar{\xi}_1 dx + \bar{u} \int_0^{2\pi} \xi_x \bar{\xi}_1 dx + \nu \int_0^{2\pi} \xi \bar{\xi}_1 dx = \int_0^{2\pi} f \bar{\xi}_1 dx,$$

and the above relation is true $\forall \xi_1 \in C_c^\infty(0, 2\pi)$. As a consequence,

$$\bar{\rho}\eta_x + \bar{u}\xi_x + \nu\xi = f,$$

in the sense of distribution and therefore $\bar{u}\xi_x = f - \bar{\rho}\eta_x - \nu\xi \in L^2(0, 2\pi)$; in other words, $\xi \in H^1(0, 2\pi)$. We similarly have from identities (A.26) and (A.27)

$$\nu\eta + \frac{R\bar{\theta}}{\bar{\rho}}\xi_x - \lambda_0 \eta_{xx} + \bar{u}\eta_x + R\zeta_x = g,$$

$$\nu\zeta + \frac{R\bar{\theta}}{c_0}\eta_x - \kappa_0 \zeta_{xx} + \bar{u}\zeta_x = h,$$

in the sense of distribution and therefore $\eta, \zeta \in H^2(0, 2\pi)$.

Step 3. We now show $\eta(0) = \eta(2\pi)$ and $\eta_x(0) = \eta_x(2\pi)$. Since the inclusion map $i : H^1(0, 2\pi) \to C^0(0, \bar{2}\pi)$ is compact and $\eta^\epsilon \rightharpoonup \eta$ in $H^1(0, 2\pi)$, we obtain

$$\eta^\epsilon \to \eta \quad \text{in } C^0[0, 2\pi].$$

Thus, $(\eta^\epsilon(0), \eta^\epsilon(2\pi)) \to (\eta(0), \eta(2\pi))$. Since $\eta^\epsilon(0) = \eta^\epsilon(2\pi)$ for all $\epsilon > 0$, we have

$$\eta(0) = \eta(2\pi).$$

From (A.26), we have after passing the limit as $\epsilon \to 0$

$$\lambda_0 \int_0^{2\pi} \eta_x (\bar{\eta}_1)_x dx + \bar{u} \int_0^{2\pi} \eta_x \bar{\eta}_1 dx + \frac{R\bar{\theta}}{\bar{\rho}} \int_0^1 \xi_x \bar{\eta}_1 dx + R \int_0^{2\pi} \zeta_x \bar{\eta}_1 dx + \nu \int_0^{2\pi} \eta \bar{\eta}_1 dx = \int_0^{2\pi} g \bar{\eta}_1 dx.$$

Integrating by parts, we get

$$-\lambda_0 \int_0^{2\pi} \eta_{xx} \bar{\eta}_1 dx + \lambda_0(\eta_x(2\pi)\bar{\eta}_1(2\pi) - \eta_x(0)\bar{\eta}_1(0)) + \bar{u} \int_0^{2\pi} \eta_x \bar{\eta}_1 dx + \frac{R\bar{\theta}}{\bar{\rho}} \int_0^1 \xi_x \bar{\eta}_1 dx$$
$$+ R \int_0^{2\pi} \zeta_x \bar{\eta}_1 dx + \nu \int_0^{2\pi} \eta \bar{\eta}_1 dx = \int_0^{2\pi} g \bar{\eta}_1 dx,$$

and therefore

$$\eta_x(2\pi)\bar{\eta}_1(2\pi) - \eta_x(0)\bar{\eta}_1(0) = 0$$

that is $\eta_x(0) = \eta_x(2\pi)$. In a similar way, we can obtain $\zeta(0) = \zeta(2\pi)$ and $\zeta_x(0) = \zeta_x(2\pi)$.

We now show $\xi(0) = \xi(2\pi)$. Recall that we have after taking limit as $\epsilon \to 0$

$$\bar{\rho} \int_0^{2\pi} \eta_x \bar{\xi}_1 dx - \bar{u} \int_0^{2\pi} \xi (\bar{\xi}_1)_x dx + \nu \int_0^{2\pi} \xi \bar{\xi}_1 dx = \int_0^{2\pi} f \bar{\xi}_1 dx.$$

Integrating by parts, we get

$$\bar{\rho} \int_0^{2\pi} \eta_x \bar{\xi}_1 dx + \bar{u} \int_0^{2\pi} \xi_x \bar{\xi}_1 dx - \bar{u}(\xi(2\pi)\xi_1(2\pi) - \xi(0)\xi_1(0)) + \nu \int_0^{2\pi} \xi \bar{\xi}_1 dx = \int_0^{2\pi} f \bar{\xi}_1 dx, \tag{A.28}$$

and therefore

$$\xi(0) = \xi(2\pi).$$

So, we get $\begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} \in \mathcal{D}(A)$. Hence, the operator A is maximal.

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